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Optimal impulsive harvesting policy for single population☆

Xiaoying Zhang, Zhisheng Shuai, Ke Wang*

Department of Mathematics, Key Laboratory for Vegetation Ecology of Education Ministry P.R. China, Northeast Normal University, Changchun 130024, People's Republic of China

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Abstract

In this paper, we established the exploitation of impulsive harvesting single autonomous population model by Logistic equation. By some special methods, we analysis the impulsive harvesting population equation and obtain existence, the explicit expression and global attractiveness of impulsive periodic solutions for constant yield harvest and proportional harvest. Then, we choose the maximum sustainable yield as management objective, and investigate the optimal impulsive harvesting policies respectively. The optimal harvest effort that maximizes the sustainable yield per unit time, the corresponding optimal population levels are determined. At last, we point out that the continuous harvesting policy is superior to the impulsive harvesting policy, however, the latter is more beneficial in realistic operation.

Keywords: Logistic equation; Impulsive differential equation; Impulsive periodic solution; Global attractive; Optimal impulsive harvesting policy

1. Introduction

The optimal management of renewable resources, which has a direct relationship to sustainable development, has been studied extensively by many authors [3,9]. Clark has studied the optimal harvesting of the logistic equation [3]: this is logistic equation

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* Corresponding author. Tel.: +86-431-5624048.
E-mail address: shuaizs@hotmail.com (Ke Wang).

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without exploitation as follows

$$\dot{x} = rx\left(1 - \frac{x}{K}\right), \quad x(t_0) = x_0, \tag{1.1}$$

where $x(t)$ is the density of the resource population X at time t, r, assumed to be positive constant, is called the *intrinsic growth rate*, and the positive constant K is usually referred to as the environmental *carrying capacity*, or saturation level. Suppose that the population described by the Logistic equation (1.1) is subject to harvesting at a rate $h(t) = constant$ or under the *catch-per-unit-effort hypothesis* $h = Ex$. Then the equations of the harvested population reads respectively

$$\dot{x} = rx\left(1 - \frac{x}{K}\right) - h, \tag{1.2}$$

or

$$\dot{x} = rx\left(1 - \frac{x}{K}\right) - Ex, \tag{1.3}$$

where E denotes the harvesting effort. Many authors have discussed the optimal harvesting policy. For instance, in 1995, A.W.Lcung studied optimal harvesting-coefficient control of prey-predator diffusive Volterra-Lotka systems [7]; in 1998, Meng Fan and Ke Wang gained some results about optimal harvesting policy for single population with periodic coefficients [4]; Alvarez and Shepp have learned optimal harvesting of stochastically fluctuating populations in 1998 [1]. It is considered that in the real world fisherman cannot fish the whole day or for 24 hours. In fact, it is obvious that they only fish for some time every day and they fish net by net. Besides, seasons will also decide the fishing period. These processes are submitted to short temporary impulse. So the problem of impulsive time harvest is more significance. In fact, some authors have studied impulsive differential equations for the short temporary perturbations [5,6,8]. Further, Angelora and Dishlier considered the optimization problems for one-impulsive models from population dynamics [2]. However, there have been no results on impulsive harvest for renewable resources. In this paper, we will discuss the impulsive harvesting problems of population resources. We should emphasize that our method is not different to usual methods in other literature and is only analytic technique while practical.

The organization of the paper is as follows: in the followed section, we establish the mathematical model of impulse time harvest for famous logistic equation which is not only different to the other impulsive differential equation in expression but also is much cleaner than others in other literatures; and the maximum of increasing density of population per unit time is obtained. In Section 3, the impulsive harvesting equation with constant yield is considered and existence and global attractiveness of the impulsive periodic solution are obtained, further we conclude the optimal harvesting policy. In Section 4, we learned the impulsive harvesting policy for proportion harvest or constant effort harvest and some similar results are obtained; besides, we respectively discuss the optimal impulsive harvesting policies for harvest effort and impulse time. At last, by theoretical proof, we explain the fact: the optimal impulsive harvesting policy is just continuous harvest without practical operation.

2. Model and fundamental results

Considering the practical meaning, we suppose that we harvest once every time T for the population X, which obeys the logistic growth law. We shall establish the mathematical model of impulse time harvest for famous logistic equation:

$$\begin{cases} \dfrac{\mathrm{d}N}{\mathrm{d}t} = rN\left(1-\dfrac{N}{K}\right) - \delta(s(t))Eh(N(t)), \\ N(t_0) = N_0. \end{cases} \tag{2.1}$$

Here, the meaning of N, r, K and E is similar to the (1.1), (1.2) and (1.3); $h(N(t))$ is the function of general harvesting; δ is the Dirac impulse function, which satisfies $\delta(0)=\infty$ and $\delta(s)=0$ for $s \neq 0$ and $\int_{-\infty}^{\infty} \delta(s)\,\mathrm{d}s = 1$; $s(t)$ is defined as follows:

$$s(t) = \begin{cases} 0, & t = nT, \quad n \in N, \\ -1, & t \neq nT, \quad n \in N. \end{cases}$$

By our explanation, it is obvious that the population X will increase according to logistic increasing curve without exploitation and the management of the resource will harvest $Eh(N(t))$ every T. For explaining the latter, we discuss the impulse function δ deliberately. As is known to all, the Heaviside function $\theta(t)$, which satisfies

$$\theta(t) = \begin{cases} 1, & \text{if } t \geqslant 0, \\ 0, & \text{if } t < 0. \end{cases}$$

Besides, according to generalized derivatives, $\theta' = \delta$. Thus, if $t \neq nT$, $s(t) = -1$ and $\theta(s(t))=0$, namely, the management does not harvest; if $t=nT$, $s(t)=0$ and $\theta(s(t))=1$, namely, in nT, the management harvests $Q(nT)$, which satisfies

$$\begin{aligned} Q(nT) &= \int_{-\infty}^{nT} \delta(s(t))Eh(N(t))\,\mathrm{d}t - \int_{-\infty}^{(n-1)T} \delta(s(t))Eh(N(t))\,\mathrm{d}t \\ &= Eh(N(nT)). \end{aligned}$$

The solution of (1.1) is denoted by $x(t, t_0, x_0)$. Clearly, when the harvesting does not exist, (1.1) is easily solved by separation of variables. The solution may be written in the form

$$x(t, x_0) := x(t, 0, x_0) = \frac{K}{1 + C\mathrm{e}^{-rt}}, \quad \text{where } C = \frac{K - x_0}{x_0}, \quad x_0 = x(0). \tag{2.2}$$

For biological considerations, we are only interested in positive solutions. In this paper, we always need $x_0 = x(t_0) > 0$. After time T, the increasing density of population of (1.1) is $x(T, y) - y =: f(y)$, then

$$f(y) =: x(T, y) - y = \frac{(\mathrm{e}^{rT} - 1)y(K - y)}{(\mathrm{e}^{rT} - 1)y + K}. \tag{2.3}$$

In the following, our object is to find a y such that $f(y)$ reaches its maximum at $y=\bar{y}$. The equation $f'(y)=0$ have two roots,

$$\bar{y}=\frac{(-1+\sqrt{e^{rT}})K}{e^{rT}-1},\qquad \bar{Y}=\frac{(-1-\sqrt{e^{rT}})K}{e^{rT}-1}<0,$$

and $f'(y)>0$ for $0<y<\bar{y}$ and $f'(y)<0$ for $\bar{y}<y$. Therefore, the maximum of increasing density of population is

$$\omega := \max f(y) = f(\bar{y}) = \frac{(e^{rT/2}-1)^2K}{e^{rT}-1}, \tag{2.4}$$

and the maximum of increasing density of population per unit time is

$$\max\frac{f(y)}{T}=\frac{f(\bar{y})}{T}=\frac{(e^{rT/2}-1)^2K}{(e^{rT}-1)T}. \tag{2.5}$$

3. Optimal impulsive harvesting policy for constant yield harvest

Now, we consider single population X of size $N(t)$, which obeys the logistic growth law, is impulsively harvested by means of a constant yield, $h(N)\equiv 1$, namely, every T, the management harvest constant is E. Equation of the impulsively harvested population reads

$$\begin{cases} \dfrac{dN}{dt}=rN\left(1-\dfrac{N}{K}\right)-\delta(s(t))E, \\ N(t_0)=N_0. \end{cases} \tag{3.1}$$

The solution of (3.1) is denoted by $N(t,t_0,N_0)$, while $x(t,t_0,x_0)$ represents the solution of (1.1).

Theorem 3.1. *If $0<E<\omega$, there exist two positive impulsive periodic solutions $\xi_1(t)$ and $\xi_2(t)$ of (3.1) with*

$$\xi_1(nT)=\frac{1}{2}\left(K-E-\sqrt{K^2-2EK+E^2-\frac{4EK}{e^{rT}-1}}\right),\quad \forall n\in N,$$

$$\xi_2(nT)=\frac{1}{2}\left(K-E+\sqrt{K^2-2EK+E^2-\frac{4EK}{e^{rT}-1}}\right),\quad \forall n\in N.$$

If $E=\omega$, there exists a unique positive impulsive periodic solution $\xi(t)$ of (3.1) with $\xi(nT)=\frac{1}{2}(K-E), \forall n\in N$.

Proof. Let

$$F(y):=f(y)-E=\frac{(e^{rT}-1)y(K-y)}{(e^{rT}-1)y+K}-E,$$

where $f(y)$ is defined by (2.3). If $0 < E < \omega$, we know that

$$K^2 - 2EK + E^2 - \frac{4EK}{e^{rT} - 1} > 0.$$

It is easy to see that the equation $F(y) = 0$ has two and only two roots

$$y_1 = \frac{1}{2}\left(K - E - \sqrt{K^2 - 2EK + E^2 - \frac{4EK}{e^{rT} - 1}}\right),$$

$$y_2 = \frac{1}{2}\left(K - E + \sqrt{K^2 - 2EK + E^2 - \frac{4EK}{e^{rT} - 1}}\right).$$

We have

$$N(T,0,y_1) = x(T,y_1) - E = x(T,y_1) - y_1 - E + y_1$$
$$= f(y_1) - E + y_1 = F(y_1) + y_1 = y_1 = N(0,0,y_1),$$

and

$$N(2T,0,y_1) = N(2T,T,N(T,y_1)) = N(2T,T,y_1)$$
$$= x(2T,T,y_1) - E = x(T,y_1) - E = y_1,$$

then we get $N(nT,0,y_1) = y_1$ for $\forall n \in N$. We also have $N(nT,0,y_2) = y_2 = N(0,0,y_2)$ for $\forall n \in N$. Therefore, $\xi_1(t) = N(t,0,y_1)$ and $\xi_2(t) = N(t,0,y_2)$ are impulse periodic solutions of (3.1) with $\xi_1(nT) = y_1$ and $\xi_2(nT) = y_2$ for $\forall n \in N$. If $E = \omega$, the equation $F(y) = 0$ has one and only one root $y_1 = y_2 = \frac{1}{2}(K - E)$, so (3.1) has only one impulsive periodic solution $\xi(t)$ with $\xi(nT) = \frac{1}{2}(K - E)$ for $\forall n \in N$. The proof is complete. □

Theorem 3.2. *If $E < \omega$, then $N(t,0,N_0) \to \xi_2(t)$ as $t \to +\infty$ for $N_0 > y_1$ and $N(t,0,N_0) \to 0$ for $N_0 < y_1$. If $E = \omega$, then $N(t,0,N_0) \to \xi(t)$ as $t \to +\infty$ for $N_0 > \frac{1}{2}(K - E)$ and $N(t,0,N_0) \to 0$ as $t \to +\infty$ for $N_0 < \frac{1}{2}(K - E)$. If $E > \omega$, then $N(t,0,N_0) \to 0$ as $t \to +\infty$ for $\forall N_0 \geqslant 0$.*

Proof. If $y < \bar{y}$, $dF(y)/dy = f'(y) > 0$; and if $y > \bar{y}$, $dF(y)/dy = f'(y) < 0$. It follows that $y_2 > \bar{y} > y_1$. So we have

$$K > K - E > y_2 > \bar{y} > y_1 > 0,$$

and $F(y) < 0$ for $0 < y < y_1$ and $y > y_2$ and $F(y) > 0$ for $y_1 < y < y_2$. Suppose that $E < \omega$. For any given $N_0 > y_2$, let $N_n = N(nT,0,N_0)$. We have

$$N_1 = x(T,N_0) - E = f(N_0) + N_0 - E = F(N_0) + N_0 < N_0.$$

On the other hand, $N_0 > y_2$ implies that

$$N_1 = x(T,N_0) - E > x(T,y_2) - E = N(T,0,y_2) = \xi_2(T) = y_2.$$

It follows that

$$N_2 = N(2T,0,N_0) = N(2T,T,N_1) = x(T,N_1) - E = F(N_1) + N_1 < N_1$$

and $N_2 = x(T,N_1) - E > x(T,y_2) - E = \xi_2(T) = y_2$. By the same arguments we can prove that the sequence $\{N_n\}$ is decreasing with a lower bound y_2. Thus, the sequence $\{N_n\}$ has a limit supposed by β. It is obvious that $\beta \geqslant y_2$. If $\beta > y_2$, from

$$N_{n+1} - N_n = N((n+1)T, nT, N_n) - N_n = x(T,N_n) - E - N_n = F(N_n),$$

as $n \to \infty$ we have $0 = F(\beta)$. This is a contradiction, because the equation $F(y) = 0$ has only two roots, and it follows that $\beta = y_2$.

For any given $\varepsilon > 0$ there is a $\delta \in (0,\varepsilon)$ such that $|x_0 - y_2| < \delta$ implies $|x(t,x_0) - x(t,y_2)| < \varepsilon$ for $t \in [0,T)$. There is a natural $\bar{N}$ such that $n \geqslant \bar{N}$ implies $0 < N_n - y_2 < \delta$, and then, for any $n \geqslant \bar{N}$ and $t \in [nT,(n+1)T)$, we have

$$\begin{aligned}|N(t,0,N_0) - \xi_2(t)| &= |N(t,nT,N_n) - N(t,nT,y_2)| \\ &= |(x(t,nT,N_n) - x(t,nT,y_2)| < |x(t-nT,0,N_n) \\ &\quad - x(t-nT,0,y_2)| < \varepsilon.\end{aligned}$$

Thus, we have

$$|N(t,0,N_0) - \xi_2(t)| \leqslant \varepsilon \quad \text{for } t \geqslant \bar{N}T,$$

that is, if $E < \omega$, $N(t,0,N_0) \to \xi_2(t)$ as $t \to \infty$ for $N_0 > y_2$. The other conclusions of Theorem 3.2 can be proved by similar methods, so we omit them here, and complete the proof. □

From Theorem 3.1 and Theorem 3.2 if $0 < E < \omega = \max f(y)$, there exist two positive impulsive periodic solutions $\xi_1(t)$ and $\xi_2(t)$ of (3.1); and $\xi_2(t)$ is stable but $\xi_1(t)$ is unstable. If the initial population is less than $\xi_1(t)$, $N(t)$ will approach 0 in a finite time, otherwise $N(t)$ will approach $\xi_2(t)$.

If $E > \omega$, the population approaches 0 for any initial level N_0 in a finite time.

If $E = \omega$, there exists a unique positive impulse periodic solution $\xi(t)$ with $\xi(nT) = \frac{1}{2}(K - E)$ of (3.1), which is "semistable" in the sense that $N(t) \to \xi(t)$ if $N_0 > \xi(T)$, but $N(t) \to 0$ if $N_0 < \xi(T)$. At the same time, we will obtain the optimal harvest $Y^* = \omega$ every time T, namely, per unit time we can get the harvest ω/T. But the harvesting policy with constant yield is difficult to be controlled like the harvesting policy with continuous time harvesting by Clark [3].

4. Optimal impulsive harvesting policy for proportional harvest

The assumption in Section 3 that the harvesting yield is a constant leads that we cannot control exploitation for dangerous region. In this section, we will use the phrase *catch-per-unit-effort hypothesis* to describe an assumption that *catch-per-unit-effort* is proportional to the stock level, namely, that $h(N) = EN$, where E denotes effort and satisfies $0 \leqslant E < 1$. In other word, the management harvests $Q(nT) = EN(nT)$ at the moment $t = nT$. Equation of the impulsively harvested population reads

$$\begin{cases} \dfrac{dN}{dt} = rN\left(1 - \dfrac{N}{K}\right) - \delta(s(t))EN, \\ N(t_0) = N_0. \end{cases} \tag{4.1}$$

In this section, the solution of (4.1) is denoted by $N(t, t_0, N_0)$.

Now we investigate the optimal impulsive harvesting policy, namely, the optimal harvesting effort, the maximum sustainable yield and the corresponding optimal population level.

Definition 4.1 (Yoshizawa [10]). A solution of (4.1) $\xi(t)$ is said to be globally attractive for positive initial value if $N(t, 0, N_0)$ with $N_0 > 0$ satisfies condition as follows:

$$\lim_{t \to +\infty} |N(t, 0, N_0) - \xi(t)| = 0.$$

Theorem 4.1. *If* $0 < E < 1 - e^{-rT}$, *there exists a unique positive impulsive periodic solution* $\xi(t)$ *of* (4.1) *with*

$$\xi(nT) = \tilde{y} = \frac{(e^{rT}(1 - E) - 1)K}{e^{rT} - 1}.$$

In addition, $\xi(t)$ *is global attractive for positive initial value.*

Proof. Let

$$\begin{aligned} G(y) &= x(T, y) - Ex(T, y) - y = (1 - E)(f(y) + y) - y = (1 - E)f(y) - Ey \\ &= \frac{(1 - E)(e^{rT} - 1)(K - y)}{(e^{rT} - 1)y + K} - Ey. \end{aligned}$$

When $0 < E < 1 - e^{-rT}$, the equation $G(y) = 0$ has a unique positive root

$$\tilde{y} = \frac{(e^{rT}(1 - E) - 1)K}{e^{rT} - 1}, \tag{4.2}$$

and $G(y) > 0$ for $0 < y < \tilde{y}$ and $G(y) < 0$ for $y > \tilde{y}$. We have

$$N(T, 0, \tilde{y}) = x(T, \tilde{y}) - Ex(T, \tilde{y}) = G(\tilde{y}) + \tilde{y} = \tilde{y},$$

and

$$\begin{aligned} N(2T, 0, \tilde{y}) &= N(2T, T, N(T, \tilde{y})) = N(2T, T, \tilde{y}) \\ &= (1 - E)x(2T, T, \tilde{y}) = (1 - E)x(T, \tilde{y}) = G(\tilde{y}) + \tilde{y} = \tilde{y}. \end{aligned}$$

Inductively, we can prove that

$$N(nT, 0, \tilde{y}) = \tilde{y}, \quad \text{for } \forall n \in N.$$

Therefore, (4.1) has unique impulse periodic solution $\xi(t) := N(t,0,\tilde{y})$ with $\xi(nT) = \tilde{y}$ for $\forall n \in N$.

For any given $N_0 > \tilde{y} > 0$, let $N_n := N(nT,0,N_0), n \in N$. We have

$$N_1 = N(T,0,N_0) = x(T,N_0) - Ex(T,N_0) = G(N_0) + N_0 < N_0.$$

And $N_0 > \tilde{y}$ implies

$$N_1 = N(T,0,N_0) = (1-E)x(T,N_0) > (1-E)x(T,\tilde{y}) = N(T,0,\tilde{y}) = \tilde{y}.$$

Similarly, we can prove that $\tilde{y} < N_2 < N_1$. Inductively, we get that the sequence $\{N_n\}$ is decreasing with a lower bound $\tilde{y}$. So the sequence $\{N_n\}$ has a limit $\gamma \geqslant \tilde{y}$.

If $\gamma > \tilde{y}$, then

$$\begin{aligned} N_{n+1} - N_n &= x((n+1)T, nT, N_n) - Ex((n+1)T, nT, N_n) - N_n \\ &= x(T,N_n) - Ex(T,N_n) - N_n = G(N_n), \end{aligned}$$

which implies that $G(\gamma) = 0$, but this is a contradiction because the equation $G(y) = 0$ has a unique positive root. Thus, we have

$$\lim_{n\to\infty} N_n = \tilde{y}.$$

For any given $\varepsilon > 0$, there is a $\delta \in (0,\varepsilon)$ such that $|x(t,x_0) - x(t,\tilde{y})| < \varepsilon$ for $t \in [0,T)$ whenever $|x_0 - \tilde{y}| < \delta$, and there is a natural $\bar{N}$ such that $n \geqslant \bar{N}$ implies $0 < N_n - \tilde{y} < \delta$. If $n \geqslant \bar{N}$ and $t \in [nT,(n+1)T)$, then

$$\begin{aligned} |N(t,0,N_0) - \xi(t)| &= |(1-E)(x(t,nT,N_n) - x(t,nT,\tilde{y}))| \\ &= |1-E| \cdot |x(t-nT,N_n) - x(t-nT,\tilde{y})| < (1-E)\varepsilon. \end{aligned}$$

That is,

$$\lim_{t\to+\infty} |N(t,0,N_0) - \xi(t)| = 0, \quad \text{for } N_0 > \tilde{y}.$$

By the same argument, we can prove

$$\lim_{t\to+\infty} |N(t,0,N_0) - \xi(t)| = 0, \quad \text{for } 0 < N_0 < \tilde{y}.$$

Thus, the impulsive periodic solution $\xi(t)$ is global attractive for positive initial value. So we complete the proof. □

Note that if $E = 1 - e^{-rT}$, we can get

$$\xi(nT) = \frac{(e^{rT}(1-E) - 1)K}{e^{rT} - 1} = 0.$$

So we have the conclusion as follows:

Theorem 4.2. *If $E \geqslant 1 - e^{-rT}$, the size of population X will tend to extinguish.*

Fisherman would make a decision how to fish every day. From Theorem 4.1, when T is a constant, the sustainable yield per unit time is

$$Y(E) = \frac{EK(e^{rT} - 1 - e^{rT}E)}{T(e^{rT} - 1)(1 - E)}. \tag{4.3}$$

Our object is to find an E^* such that $Y(E)$ reaches its maximum at $E = E^*$. This is an optimization of a function. We have

$$Y'(E) = \frac{K(E^2 e^{rT} - 2Ee^{rT} + e^{rT} - 1)}{T(e^{rT} - 1)(1 - E)^2} = 0,$$

then $E^2 - 2E - e^{-rT} + 1 = 0$, so $E = 1 \pm e^{-rT/2}$. From Theorem 4.1, we have

$$E^* = 1 - e^{-rT/2}. \tag{4.4}$$

Besides, we can obtain

$$Y''(E) = \frac{-2e^{-rT}k}{T(-1+E)^2(1 - E - e^{-rT} + e^{-rT}E)} < 0, \quad \forall 0 < E < 1 - e^{-rT}.$$

Then $Y(E)$ reaches its maximum at $E = E^*$. Substituting Eq. (4.4) into Eq. (4.2), we have

$$x^*(T) = \frac{K}{e^{rT/2}} + 1. \tag{4.5}$$

Substituting Eq. (4.4) into Eq. (4.3), we can get the maximum sustainable yield per unit time $Y(E^*)$:

$$Y(E^*) = \frac{E^*K(e^{rT} - 1 - e^{rT}E^*)}{T(e^{rT} - 1)(1 - E^*)} = \frac{K(e^{rT/2} - 1)^2}{T(e^{rT} - 1)}. \tag{4.6}$$

So we obtain the optimal harvest effort E^* that maximizes the sustainable yield per unit time $Y(E^*)$, the corresponding optimal population level $x^*(T)$. Note that the optimal harvest is equal to the result in Section 3 but the former is superior to the latter in operation, which is compatible with classic results by Clark [3].

As we know, sometimes fisherman do not change the harvesting effort but fish in different periods. So we will discuss the optimal harvesting policy when E is a fixed value.

By (4.3), we obtain

$$\begin{aligned} g(T) &= \frac{EK(e^{rT} - 1 - e^{rT}E)}{T(e^{rT} - 1)(1 - E)} \\ &= \frac{KE}{1 - E} \, \frac{1 - \frac{E}{1 - e^{-rT}}}{T}. \end{aligned} \tag{4.7}$$

Then

$$g'(T) = -\frac{(E - e^{-rT}E + re^{-rT}TE + 2e^{-rT} - 1 - e^{-2rT})KE}{T^2(-1 + e^{-rT})(1 - E - e^{-rT} + e^{-rT}E)}.$$

Let $g'(T) = 0$, then

$$H(T) := E - e^{-rT}E + re^{-rT}TE + 2e^{-rT} - 1 - e^{-2rT} = 0. \quad (4.8)$$

It is easy to see that when T_0 satisfy $e^{-rT} = 1 - E$,

$$H(T_0) = -E(1 - E)\ln(1 - E) > 0. \quad (4.9)$$

Besides, we find

$$\lim_{T\to\infty} H(T) = E - 1 < 0. \quad (4.10)$$

By (4.9) and (4.10), there exists at least $T^* > T_0 = -\ln(1 - E)/r$ such that $H(T^*) = 0$. Moreover,

$$H'(T) = -2re^{-rT} + 2re^{-2rT} + 2rEe^{-rT} - r^2ETe^{-rT},$$

and

$$H'(T) = re^{-rT}(-2 + 2e^{-rT} + 2E - ErT) < 0, \quad \text{if } T > T_0. \quad (4.11)$$

Then there exists a unique $T^* > T_0 > 0$. By (4.11), $H'(T^*) = g''(T^*) < 0$, so we prove that we can obtain the maximum sustainable yield per unit time when $T = T^* > -\ln(1 - E)/r$. Moreover, more is constant E, more is T^*. In reality, we know that if we reduce effort and fish a little, we can harvest fish frequently.

5. Conclusion

In the section, we will point out that continuous harvest is better than impulsive harvesting, though the latter is convenient to be used in reality and the former is difficult to be applied.

First, we get to the conclusion that $Y(E^*)$ in (4.6) is decreasing function of T. Note

$$Y_{E^*}(T) = \frac{K(e^{rT/2} - 1)^2}{T(e^{rT} - 1)},$$

then

$$Y'_{E^*}(T) = \frac{rTe^{rT/2} - e^{rT} + 1}{T^2}.$$

It is easy to prove that if $T > 0$, $rTe^{rT/2} - e^{rT} + 1 < 0$. So the less impulse period is, the more our harvest is.

Now our object is to look for the maximum of (4.3). (4.3) is written again as follows:

$$Y(E,T)=\frac{EK(e^{rT}-1-e^{rT}E)}{T(e^{rT}-1)(1-E)}. \tag{5.1}$$

Let

$$\begin{cases}\dfrac{\partial Y}{\partial E}=\dfrac{K(E^2e^{rT}-2Ee^{rT}+e^{rT}-1)}{T(e^{rT}-1)(1-E)^2}=0,\\[2ex] \dfrac{\partial Y}{\partial T}=-\dfrac{(E-e^{-rT}E+re^{-rT}TE+2e^{-rT}-1-e^{-2rT})KE}{T^2(-1+e^{-rT})(1-E-e^{-rT}+e^{-rT}E)}=0,\end{cases} \tag{5.2}$$

that is

$$\begin{cases}E=1-e^{-rT/2},\\ E-e^{-rT}E+re^{-rT}TE+2e^{-rT}-1-e^{-2rT}=0.\end{cases} \tag{5.3}$$

From (5.3), we obtain

$$(1-e^{-rT})^2-(1-e^{-rT/2})(1-e^{-rT}+rTe^{-rT})=0. \tag{5.4}$$

However, if $T>0$, we have

$$(1-e^{-rT})^2-(1-e^{-rT/2})(1-e^{-rT}+rTe^{-rT})>0. \tag{5.5}$$

So Equation (5.4) does not exist any positive real roots, namely, function $Y(E,T)$ does not exist maximal value or minimal value when $T>0$. In fact, from (4.6), we learn when T tends to 0, $Y(E^*)$ will reach the maximum, $rK/4$. On the other hand, it is compatible to the conclusion by Clark; on the other hand, we conclude that (E,T) is nearer to $(0,0)$, the more the harvest is. So the best impulsive harvesting policy is continuous harvesting without considering realistic operation.

At last, we recommend our results:

If we use the impulsive constant yield harvesting policy, under some conditions, there are two impulsive periodic solutions; under the critical condition, there is a unique impulsive periodic solution and as the same time we get the maximum harvest. But the policy cannot keep stationary for perturbations.

If we use the impulsive constant effort harvesting policy, under some conditions, there exists a global attractive impulsive periodic solutions and we can get the same optimal harvest when T is fixed, however, the optimal harvest is stable. Besides, we also get the optimal harvesting policy when E is fixed.

Through studying the optimal harvesting policy when E and T can be also controlled, we conclude when (E,T) tends to $(0,0)$, the harvest is optimal. Namely, the optimal impulsive harvesting policy is continuous harvest.

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Appendix A.

Proof of (4.16): If $T > 0$, we have

$$
\begin{aligned}
& (1-\mathrm{e}^{-rT})^2 - (1-\mathrm{e}^{-rT/2})(1-\mathrm{e}^{-rT}+rT\mathrm{e}^{-rT}) > 0 \\
\Leftrightarrow \quad & \sqrt{c} - c\sqrt{c} + c\ln c > 0, \quad \text{where } 0 < c = \mathrm{e}^{-rT} < 1 \\
\Leftrightarrow \quad & 1 - c + \sqrt{c}\ln c > 0.
\end{aligned}
$$

Let

$$L(c) = 1 - c + \sqrt{c}\ln c,$$

we obtain

$$
\begin{aligned}
L'(c) &= -1 + \frac{1}{2}\frac{\ln c}{\sqrt{c}} + \frac{1}{\sqrt{c}} \\
&= -1 + \frac{\ln c + 2}{2\sqrt{c}}.
\end{aligned}
$$

Let

$$J(c) = 2\sqrt{c} - \ln c - 2,$$

then

$$J'(c) = \frac{1}{\sqrt{c}} - \frac{1}{c} < 0, \quad \text{if } 0 < c < 1.$$

So

$$J(c) > J(1) = 0,$$

then

$$L'(c) = -1 + \frac{\ln c + 2}{2\sqrt{c}} < 0.$$

We can get

$$L(c) > L(1) = 0.$$

The proof of (4.16) is completed. □

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