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GLOBAL STABILITY OF AN EPIDEMIC MODEL IN A PATCHY ENVIRONMENT

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ABSTRACT. We investigate an SIR compartmental epidemic model in a patchy environment where individuals in each compartment can travel among n patches. We derive the basic reproduction number R_0 and prove that, if $R_0 \leq 1$, the disease-free equilibrium is globally asymptotically stable. In the case of $R_0 > 1$, we derive sufficient conditions under which the endemic equilibrium is unique and globally asymptotically stable.

1 Introduction In the literature of population dynamics, both continuous reaction-diffusion systems and discrete patchy models are used to study the spatial heterogeneity [**15**]. While reaction-diffusion systems are suitable for random spatial dispersal, patchy models are often used to describe directed movement among patches. When modeling the spread of infectious diseases in spatially heterogeneous host populations, directed movement can be migration among countries and regions or travel among cities.

Discrete spatial epidemic models in patch environments give rise to large systems of nonlinear differential equations, and establishing their global dynamics can be a mathematical challenge. Arino and van den Driessche [**2**] formulated n-city epidemic models to investigate the effects of inter-city travel on the spatial spread of infectious diseases among cities. The basic reproduction number R_0 was derived and numerical simulations were carried out to show that R_0 determines whether the disease dies out ($R_0 < 1$) or becomes endemic ($R_0 > 1$). Wang and Zhao [**21**] studied an n-patch SIS model with bilinear incidence. In the case that both suspectable and infectious individuals on each patch have the same dispersal rates, they proved that the disease-free equilibrium is

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globally asymptotically stable if $R_0 < 1$. They also proved that the system is uniformly persistent and admits an endemic equilibrium if $R_0 > 1$. Under the same assumption that the dispersal rates of susceptible and infectious individuals are the same, Jin and Wang [**10**] showed that the n-patch SIS model can be reduced to a monotone system. Using the theory of monotone dynamical systems, they proved the uniqueness and global stability of the endemic equilibrium when $R_0 > 1$. Salmani and van den Driessche [**16**] studied an SEIRS model with standard incidence in a patchy environment and proved that, if $R_0 < 1$, the disease-free equilibrium is globally asymptotically stable, regardless of travel rates. Uniqueness and global stability of endemic equilibria when $R_0 > 1$ is often unresolved for many patchy epidemic models. The method of global Lyapunov functions has seen little success for epidemic models in patchy environments.

Recently, a graph-theoretical approach is developed in [**8, 9, 13**] and that systemizes the construction of global Lyapunov functions for large-scale coupled systems. The approach has been successfully applied to resolve global-stability problems for the endemic equilibrium of multi-group epidemic models [**8, 9, 13**], and for the positive equilibrium of predator-prey models in a patchy environment [**13**]. In this paper, we utilize this new approach to investigate the global stability of the endemic equilibrium of epidemic models with travel among n patches. We consider the following SIR epidemic model with bilinear incidence in a patchy environment,

$$
\begin{aligned}
S_i' &= \Lambda_i - \beta_i S_i I_i - d_i^S S_i + \sum_{j=1}^n a_{ij} S_j - \sum_{j=1}^n a_{ji} S_i,\\
I_i' &= \beta_i S_i I_i - (d_i^I + \gamma_i) I_i + \sum_{j=1}^n b_{ij} I_j - \sum_{j=1}^n b_{ji} I_i,\\
R_i' &= \gamma_i I_i - d_i^R R_i + \sum_{j=1}^n c_{ij} R_j - \sum_{j=1}^n c_{ji} R_i,\\
&\qquad i = 1, 2, \ldots, n.
\end{aligned} \tag{1.1}
$$

Here, S_i, I_i and R_i represent the susceptible, infectious and removed populations in the i-th patch, respectively, Λ_i is the influx of individuals into the i-th patch, β_i is the transmission coefficient between susceptible and infectious individuals in the i-th patch, d_i^S, d_i^I and d_i^R represent death rates of S, I and R populations in the i-th patch, respectively, and

γ_i represents the recovery rate of infectious individuals in the i-th patch. The travel rates of susceptible, infectious, and removed individuals from the j-th patch to the i-th patch are given by a_{ij}, b_{ij} and c_{ij}, respectively. All parameter values are assumed to be nonnegative and $\Lambda_i, \beta_i, d_i^S, d_i^I > 0$ for all i. The travel matrices $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ are not required to be symmetric, namely, the travel rate from the i-th patch to the j-th patch may not be the same as that from the j-th to the i-th. A typical assumption we impose on these matrices is that they are irreducible. In biological terms, this means individuals in each compartment can travel between any two patches directly or indirectly. For detailed discussions of epidemic model with patches, we refer the reader to articles [**3, 20**] and references therein. Model (1.1) includes as special cases several earlier models in the literature. A two-patch SIS model [**19**] and a two-patch SIRS model [**6**] become special cases of model (1.1) if we assume that the disease has permanent immunity. An n-patch model similar to (1.1) was proposed in [**14**] without global-stability analysis. Model (1.1) differs from those in [**16**] in that bilinear incidence are used in (1.1) while standard incidences are assumed in [**16**].

Since the variable R_i does not appear in the first two equations of (1.1), we can first study the reduced system

$$
\begin{aligned}
S_i' &= \Lambda_i - \beta_i S_i I_i - d_i^S S_i + \sum_{j=1}^{n} a_{ij} S_j - \sum_{j=1}^{n} a_{ji} S_i,\\
I_i' &= \beta_i S_i I_i - (d_i^I + \gamma_i) I_i + \sum_{j=1}^{n} b_{ij} I_j - \sum_{j=1}^{n} b_{ji} I_i,\\
&\qquad i = 1, 2, \ldots, n,
\end{aligned}
\tag{1.2}
$$

with initial conditions $S_i(0) \geq 0$ and $I_i(0) \geq 0$. Behaviors of R_i can then be determined from the last equation of (1.1). Our results in this paper will be stated for system (1.2) and can be translated straightforwardly to system (1.1).

In this paper, we establish the global dynamics of system (1.2) with any finite number of patches. We prove that, if $R_0 \leq 1$, the disease-free equilibrium is globally asymptotically stable. In the case of $R_0 > 1$, we derive sufficient conditions under which, whenever an endemic equilibrium exists, it is unique and globally asymptotically stable. Our results can be readily applied to the n-patch epidemic model in [**14**] and yield global-stability analysis. When the disease has permanent immunity, the global stability of the endemic equilibrium for two-patch epidemic models in [**6, 19**] is resolved as special cases of our results. Our proof demonstrates that the graph-theoretical approach developed in [**13**] is applicable to the global-stability problem in patchy models.

The paper is organized as follows. In the next section, we prove some preliminary results for system (1.2). In Section 3, the global stability of the disease-free equilibrium is proved. The global stability of the endemic equilibrium is proved in Section 4. We include in the Appendix a combinatorial identity that is needed for our proof.

2 Preliminaries

2.1 Disease-free equilibrium To find the disease-free equilibrium of (1.2), we consider the following linear system

$$\Lambda_i - d_i^S S_i + \sum_{j=1}^{n} a_{ij} S_j - \sum_{j=1}^{n} a_{ji} S_i = 0, \qquad i = 1, 2, \ldots, n, \tag{2.1}$$

or in the form of matrix system

$$DS = \Lambda,$$

where

$$D = \begin{pmatrix} d_1^S + \sum_{j \neq 1} a_{j1} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & d_2^S + \sum_{j \neq 2} a_{j2} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & d_n^S + \sum_{j \neq n} a_{jn} \end{pmatrix}, \tag{2.2}$$

$S = (S_1, S_2, \cdots, S_n)^T$, and $\Lambda = (\Lambda_1, \Lambda_2, \cdots, \Lambda_n)^T$. Since all off-diagonal entries of D are nonpositive and the sum of the entries in each column of D is positive, D is a nonsingular M-matrix and $D^{-1} \geq 0$ [**4,** p. 137]. Hence, linear system (2.1) has a unique positive solution $S^0 = (S_1^0, S_2^0, \cdots, S_n^0)^T = D^{-1}\Lambda$, $S_i^0 > 0$ for all i. As a consequence, system (1.2) has a unique disease-free equilibrium

$$P_0 = (S_1^0, 0, S_2^0, 0, \cdots, S_n^0, 0).$$

We thus have the following result.

Proposition 2.1. *System* (1.2) *always has a unique disease-free equilibrium* P_0.

2.2 Feasible region Let $\bar{\Lambda} = \sum_{i=1}^n \Lambda_i$, $d^* = \min\{d_i^S, d_i^I + \gamma_i \mid i = 1, 2, \dots, n\}$, and $N = \sum_{i=1}^n (S_i + I_i)$. Adding all equations of (1.2) gives $N' \le \bar{\Lambda} - d^* N$, which implies that $\limsup_{t\to\infty} N \le \bar{\Lambda}/d^*$. Since all off-diagonal entries of D are nonpositive, it follows from the first equation of (1.2) that

$$S_i' \le \Lambda_i - d_i^S S_i + \sum_{j=1}^n a_{ij} S_j - \sum_{j=1}^n a_{ji} S_i = (DS^0 - DS)_i \le 0,$$

when $S_i = S_i^0$ and $S_j \le S_j^0$ for $j \ne i$. Thus the feasible region of (1.2) can be chosen as

$$\Gamma = \left\{ (S_1, I_1, \cdots, S_n, I_n) \in \mathbb{R}_+^{2n} \;\middle|\; N = \sum_{i=1}^n (S_i + I_i) \le \frac{\bar{\Lambda}}{d^*}, \right.$$
$$\left. S_i \le S_i^0,\ 1 \le i \le n \right\}.$$

It can be verified that Γ is positively invariant with respect to (1.2). Let $\mathring{\Gamma}$ denote the interior of Γ, and $\partial\Gamma$ the boundary of Γ.

2.3 The basic reproduction number Define

$$F = \begin{pmatrix} \beta_1 S_1^0 & 0 & \cdots & 0 \\ 0 & \beta_2 S_2^0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_n S_n^0 \end{pmatrix} \tag{2.3}$$

and

$$V = \begin{pmatrix} d_1^I + \gamma_1 + \sum_{j\ne i} b_{j1} & -b_{12} & \cdots & -b_{1n} \\ -b_{21} & d_2^I + \gamma_2 + \sum_{j\ne 2} b_{j2} & \cdots & -b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{n1} & -b_{n2} & \cdots & d_n^I + \gamma_n + \sum_{j\ne n} b_{jn} \end{pmatrix}. \tag{2.4}$$

Using the method of van den Driessche and Watmough [**18**], the basic reproduction number can be calculated as

$$R_0 = \rho(FV^{-1}),$$

where ρ represents the spectral radius of the matrix and FV^{-1} is the next generation matrix. The following result follows from Theorem 2 of [**18**].

Proposition 2.2. *The disease-free equilibrium P_0 is locally asymptotically stable if $R_0 < 1$, and unstable if $R_0 > 1$.*

2.4 Other boundary equilibria

Theorem 2.3. *Suppose that $B = (b_{ij})$ is irreducible. Then there exist no other equilibria besides P_0 in $\partial\Gamma$.*

Proof. We show that $I_i = 0$ for some i implies that $I_j = 0$ for all j. If $I_i = 0$, from the second equation of (1.2) we obtain

$$\sum_{j \neq i} b_{ij} I_j = 0.$$

As a consequence, we know that $I_j = 0$ if $b_{ij} > 0$. Namely, for any $1 \leq i, j \leq n$,

$$I_i = 0 \quad \text{and} \quad b_{ij} > 0 \quad \Longrightarrow \quad I_j = 0. \tag{2.5}$$

Since B is irreducible, there exists a sequence of ordered pairs $\{(i, r_1), (r_1, r_2), \cdots, (r_m, j)\}$ such that $b_{ir_1} > 0$, $b_{r_1 r_2} > 0, \cdots, b_{r_m j} > 0$, $1 \leq r_k \leq n$, $k = 1, 2, \ldots, m$, and $m \geq 0$ [**4,** p. 30]. Applying (2.5) to each pair in such a sequence and using $I_i = 0$, we can see that

$$I_{r_1} = 0, \quad I_{r_2} = 0, \quad \cdots, \quad I_{r_m} = 0, \quad I_j = 0.$$

Hence, we have $I_j = 0$ for all j. Therefore, by Proposition 2.1, we know that the only equilibrium that lies on the boundary $\partial\Gamma$ is P_0.

When travel matrix $B = (b_{ij})$ is reducible, system (1.2) can have multiple boundary equilibria and the dynamics of (1.2) can be complicated. We refer the readers to [**1, 2, 6, 19**] for discussions on this issue.

3 Global stability of disease-free equilibrium P_0 In this section, we prove that the disease-free equilibrium P_0 is globally asymptotically stable if $R_0 \leq 1$. In particular, our result generalizes Theorem 2.1 in [**19**] from two patches to arbitrary n patches.

Theorem 3.1. *Assume $R_0 \leq 1$. Suppose that $B = (b_{ij})$ is either irreducible or equal to 0. Then the disease-free equilibrium P_0 is globally asymptotically stable in Γ.*

Proof. We prove the result when travel matrix $B = (b_{ij})$ is irreducible. The case when $B = 0$ can be proved similarly. Let F, V be as given in (2.3) and (2.4), respectively. All off-diagonal entries of V are nonpositive and the sum of the entries in each column of V is positive, and thus V is a non-singular M-matrix. Suppose that B is irreducible, then $V^{-1} > 0$ is also irreducible. By Perron-Frobenious Theorem [**4,** p. 27], nonnegative irreducible matrix $V^{-1}F$ has a positive left eigenvector $(w_1, w_2, \cdots, w_n)$ corresponding to eigenvalue $\rho(V^{-1}F)$. Since F is a diagonal matrix, $\rho(V^{-1}F) = \rho(FV^{-1}) = R_0$. As a consequence, we have

$$(w_1, w_2, \cdots, w_n)V^{-1}F = R_0(w_1, w_2, \cdots, w_n),$$

and thus

$$\frac{1}{R_0}(w_1, w_2, \cdots, w_n) = (w_1, w_2, \cdots, w_n)F^{-1}V. \tag{3.1}$$

Let $c_i = w_i/(\beta_i S_i^0) > 0,\ i = 1, 2, \ldots, n$, and $I = (I_1, I_2, \cdots, I_n)^T$. Set $L = \sum_{i=1}^n c_i I_i$. Differentiating L along system (1.2) and using identity (3.1), we obtain

$$\begin{aligned} L' &= \sum_{i=1}^n c_i\Big(\beta_i S_i I_i - (d_i^I + \gamma_i)I_i + \sum_{j=1}^n b_{ij} I_j - \sum_{j=1}^n b_{ji} I_i\Big) \\ &\le \sum_{i=1}^n c_i\Big(\beta_i S_i^0 I_i - \Big(d_i^I + \gamma_i + \sum_{j\neq i} b_{ji}\Big) I_i + \sum_{j\neq i} b_{ij} I_j\Big) \\ &= \Big(\frac{w_1}{\beta_1 S_1^0}, \frac{w_2}{\beta_2 S_2^0}, \cdots, \frac{w_n}{\beta_n S_n^0}\Big)(F - V)I \\ &= (w_1, w_2, \cdots, w_n)(1 - F^{-1}V)I \\ &= (w_1, w_2, \cdots, w_n)\Big(1 - \frac{1}{R_0}\Big)I \le 0, \qquad \text{if } R_0 \le 1. \end{aligned} \tag{3.2}$$

Therefore, L is a Lyapunov function for system (1.2). Since $c_i > 0$ for $i = 1, 2, \ldots, n$, $L' = 0$ implies that either $S_i = S_i^0$ or $I_i = 0$ for any $1 \le i \le n$. When $S_i = S_i^0$, from the first equation of (1.2) we obtain

$$0 = (S_i^0)' = \Lambda_i - \beta_i S_i^0 I_i - d_i^S S_i^0 + \sum_{j=1}^n a_{ij} S_j^0 - \sum_{j=1}^n a_{ji} S_i^0.$$

Comparing this relation with (2.1), we know $I_i = 0$. Thus, we have shown that $L' = 0$ implies that $I_i = 0$ for all i. It can be verified that the only invariant subset of the set $\{(S_1, I_1, \cdots, S_n, I_n) \in \Gamma \mid I_i = 0, i = 1, 2, \ldots, n\}$ is the singleton $\{P_0\}$. Therefore, by LaSalle Invariance Principle [**11**], P_0 is globally asymptotically stable in Γ.

Suppose that travel matrix $B = (b_{ij})$ is irreducible and $R_0 > 1$. It follows from (3.2) that $L' > 0$ in a neighborhood of P_0 in $\overset{\circ}{\Gamma}$. Therefore, P_0 is unstable and solutions in $\overset{\circ}{\Gamma}$ sufficiently close to P_0 move away from P_0. Using a uniform persistence result from [**7**] and a similar argument as in the proof of Proposition 3.3 of [**12**], we can show that, when B is irreducible, the instability of P_0 implies the uniform persistence of (1.2). Uniform persistence of (1.2), together with uniform boundedness of solutions in $\overset{\circ}{\Gamma}$, implies the existence of an equilibrium of (1.2) in $\overset{\circ}{\Gamma}$ (see Theorem D.3 in [**17**] or Theorem 2.8.6 in [**5**]). Therefore, the following result holds.

Proposition 3.2. *Suppose that $B = (b_{ij})$ is irreducible. If $R_0 > 1$, then system* (1.2) *is uniformly persistent and there exists an endemic equilibrium P^* in $\overset{\circ}{\Gamma}$.*

In Proposition 3.2, the assumption that travel matrix $B = (b_{ij})$ is irreducible is necessary. If $B = 0$, system (1.2) can have an asymptotically stable boundary equilibrium when $R_0 > 1$ and thus is not persistent [**6, 19**]. It is also possible that, if $B = 0$, no endemic equilibrium exists when $R_0 > 1$ [**6**].

4 Uniqueness and global stability of endemic equilibria In this section, under the assumption $R_0 > 1$, we derive sufficient conditions under which, the endemic equilibrium is unique and globally asymptotically stable. Our proof utilizes the graph-theoretical approach developed in [**8, 9, 13**].

Theorem 4.1. *Assume that $R_0 > 1$ and an endemic equilibrium $P^* = (S_1^*, I_1^*, \cdots, S_n^*, I_n^*)$ exists. Suppose that one of the following assumptions is satisfied.*

(1) *$A = 0$ and B is irreducible;*
(2) *$B = 0$ and A is irreducible;*
(3) *A and B are irreducible, and there exists $\lambda > 0$ such that $a_{ij} S_j^* = \lambda b_{ij} I_j^*$ for all $1 \le i, j \le n$.*

Then P^* is unique and globally asymptotically stable in $\overset{\circ}{\Gamma}$.

By Proposition 3.2, the existence of an endemic equilibrium P^* is ensured if the assumption (1) or assumption (3) is satisfied.

Proof. We prove the result when assumption (3) is satisfied. The other two cases can be proved similarly. We prove that P^* is globally asymptotically stable in $\overset{\circ}{\Gamma}$. In particular, this implies that P^* is necessarily unique. Set

$$V_i(S_i, I_i) = S_i - S_i^* - S_i^* \ln \frac{S_i}{S_i^*} + I_i - I_i^* - I_i^* \ln \frac{I_i}{I_i^*}.$$

From equilibrium equations of (1.2), we obtain

$$d_i^S S_i^* = \Lambda_i - \beta_i S_i^* I_i^* + \sum_{j=1}^n a_{ij} S_j^* - \sum_{j=1}^n a_{ji} S_i^*,$$

and

$$(d_i^I + \gamma_i) I_i^* = \beta_i S_i^* I_i^* + \sum_{j=1}^n b_{ij} I_j^* - \sum_{j=1}^n b_{ji} I_i^*.$$

Note that $1 - x + \ln x \leq 0$ for $x > 0$ and equality holds if and only if $x = 1$. Differentiating V_i along the solution of system (1.2), we obtain

$$\begin{aligned}
(4.1) \qquad V_i' &= \Lambda_i - d_i^S S_i + \sum_{j=1}^n a_{ij} S_j - \sum_{j=1}^n a_{ji} S_i - \Lambda_i \frac{S_i^*}{S_i} \\
&\quad + \beta_i S_i^* I_i + d_i^S S_i^* - \sum_{j=1}^n a_{ij} S_j \frac{S_i^*}{S_i} + \sum_{j=1}^n a_{ji} S_i^* \\
&\quad - (d_i^I + \gamma_i) I_i + \sum_{j=1}^n b_{ij} I_j - \sum_{j=1}^n b_{ji} I_i - \beta_i S_i I_i^* \\
&\quad + (d_i^I + \gamma_i) I_i^* - \sum_{j=1}^n b_{ij} I_j \frac{I_i^*}{I_i} + \sum_{j=1}^n b_{ji} I_i^* \\
&= \Lambda_i \left(1 - \frac{S_i}{S_i^*} + \ln \frac{S_i}{S_i^*} + 1 - \frac{S_i^*}{S_i} + \ln \frac{S_i^*}{S_i} \right) \\
&\quad + \sum_{j=1}^n a_{ij} S_j^* \left(1 - \frac{S_i^* S_j}{S_i S_j^*} + \ln \frac{S_i^* S_j}{S_i S_j^*} \right)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{j=1}^{n} a_{ij} S_j^* \left(\frac{S_j}{S_j^*} + \ln \frac{S_j^*}{S_j} - \frac{S_i}{S_i^*} - \ln \frac{S_i^*}{S_i} \right) \\
&\quad + \sum_{j=1}^{n} b_{ij} I_j^* \left(1 - \frac{I_i^* I_j}{I_i I_j^*} + \ln \frac{I_i^* I_j}{I_i I_j^*} \right) \\
&\quad + \sum_{j=1}^{n} b_{ij} I_j^* \left(\frac{I_j}{I_j^*} + \ln \frac{I_j^*}{I_j} - \frac{I_i}{I_i^*} - \ln \frac{I_i^*}{I_i} \right) \\
&\leq \sum_{j=1}^{n} a_{ij} S_j^* \left(\frac{S_j}{S_j^*} + \ln \frac{S_j^*}{S_j} - \frac{S_i}{S_i^*} - \ln \frac{S_i^*}{S_i} \right) \\
&\quad + \sum_{j=1}^{n} b_{ij} I_j^* \left(\frac{I_j}{I_j^*} + \ln \frac{I_j^*}{I_j} - \frac{I_i}{I_i^*} - \ln \frac{I_i^*}{I_i} \right) \\
&= \sum_{j=1}^{n} b_{ij} I_j^* \left[\left(\lambda \frac{S_j}{S_j^*} + \lambda \ln \frac{S_j^*}{S_j} + \frac{I_j}{I_j^*} + \ln \frac{I_j^*}{I_j} \right) \right. \\
&\quad \left. - \left(\lambda \frac{S_i}{S_i^*} + \lambda \ln \frac{S_i^*}{S_i} + \frac{I_i}{I_i^*} + \ln \frac{I_i^*}{I_i} \right) \right] \\
&= \sum_{j=1}^{n} b_{ij} I_j^* [G_j(S_j, I_j) - G_i(S_i, I_i)],
\end{aligned}
$$

where

$$G_i(S_i, I_i) = \lambda \frac{S_i}{S_i^*} + \lambda \ln \frac{S_i^*}{S_i} + \frac{I_i}{I_i^*} + \ln \frac{I_i^*}{I_i}.$$

Consider a weight matrix $W = (w_{ij})$ with entry $w_{ij} = b_{ij} I_j^*$ and denote the corresponding weighted digraph as $(\mathcal{G}, W)$. Let $c_i = \sum_{\mathcal{T} \in \mathbb{T}_i} w(\mathcal{T}) \geq 0$ be as given in (A.1) in the Appendix with $(\mathcal{G}, W)$. Then, by (A.2), the following identity holds

$$\sum_{i=1}^{n} c_i \sum_{j=1}^{n} b_{ij} I_j^* (G_j(S_j, I_j) - G_i(S_i, I_i)) = 0. \tag{4.2}$$

Set

$$V(S_1, I_1, S_2, I_2, \cdots, S_n, I_n) = \sum_{i=1}^{n} c_i V_i(S_i, I_i).$$

Using (4.1) and (4.2) we obtain

$$V' = \sum_{i=1}^{n} c_i V_i' \leq \sum_{i=1}^{n} c_i \sum_{j=1}^{n} b_{ij} I_j^* (G_j(S_j, I_j) - G_i(S_i, I_i)) = 0$$

for all $(S_1, I_1, \cdots, S_n, I_n) \in \overset{\circ}{\Gamma}$. Therefore, V is a Lyapunov function for system (1.2). Since B is irreducible, we know that $c_i > 0$ for all i (see the Appendix), and thus $V' = 0$ implies that $S_i = S_i^*$ for all i. From the first equation of (1.2), we obtain

$$0 = (S_i^*)' = \Lambda_i - \beta_i S_i^* I_i - d_i^S S_i^* + \sum_{j=1}^{n} a_{ij} S_j^* - \sum_{j=1}^{n} a_{ji} S_i^*, \qquad i = 1, 2, \ldots, n,$$

which implies that $I_i = I_i^*$ for all i. The only invariant set on which $V' = 0$ is the singleton $\{P^*\}$. Therefore, by LaSalle Invariance Principle [**11**], P^* is globally asymptotically stable in $\overset{\circ}{\Gamma}$.

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Appendix: A combinatorial identity Let $(\mathcal{G}, W)$ be a weighted digraph with $n \geq 2$ vertices, where $W = (w_{ij})$ is the weight matrix. A weight $w_{ij} > 0$ if the directed arc (j, i) from vertex j to vertex i exists, otherwise $w_{ij} = 0$. Let $\mathbb{T}_i$ be the set of all spanning trees of $(\mathcal{G}, W)$ rooted at vertex i. For $\mathcal{T} \in \mathbb{T}_i$, the weight of $\mathcal{T}$, denoted by $w(\mathcal{T})$, is the product of weights on all arcs of $\mathcal{T}$. Let

$$c_i = \sum_{\mathcal{T} \in \mathbb{T}_i} w(\mathcal{T}), \qquad i = 1, 2, \ldots, n. \tag{A.1}$$

Then $c_i \geq 0$, and for any family of functions $\{G_i(x_i)\}_{i=1}^n$, the following identity holds

$$\sum_{i,j=1}^{n} c_i \, w_{ij} \, G_i(x_i) = \sum_{i,j=1}^{n} c_i \, w_{ij} \, G_j(x_j). \tag{A.2}$$

If $W = (w_{ij})$ is irreducible, then $c_i > 0$ for $i = 1, 2, \ldots, n$. We refer the reader to [**13**] for the proof of (A.2).

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