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Global-stability problem for coupled systems of differential equations on networks

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ABSTRACT

The global-stability problem of equilibria is investigated for coupled systems of differential equations on networks. Using results from graph theory, we develop a systematic approach that allows one to construct global Lyapunov functions for large-scale coupled systems from building blocks of individual vertex systems. The approach is applied to several classes of coupled systems in engineering, ecology and epidemiology, and is shown to improve existing results.

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1. Introduction

Coupled systems of nonlinear differential equations on networks have been used to model a wide variety of physical, natural, and artificial complex dynamical systems: from biological and artificial neural networks [1,7,10,19], coupled systems of nonlinear oscillators on lattices [2,9], to complex ecosystems [28,33] and the spread of infectious diseases in heterogeneous populations [8,34]. A mathematical description of a network is a directed graph consisting of vertices and directed arcs connecting them. At each vertex, the local dynamics are given by a system of differential equations called vertex system. The directed arcs indicate inter-connections and interactions among vertex systems. In a model system, a vertex can be a single neuron, an oscillator, an ecological community or a patch,

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or a homogeneous group in a heterogeneous host population for an infectious disease; interactions among vertices can be in the form of synaptic connections among neurons, physical connections among oscillators, dispersal among patches, and cross infections among different host groups. The resulting system is a large-scale coupled system of nonlinear differential equations. Many mathematical questions have been investigated for coupled systems, from emergence of patterns, synchronization and clustering, to phase transitions and bifurcations. We refer the reader to [9,19] for introductions to mathematical studies of these problems.

A general approach to the investigation of coupled systems on networks is to determine the collective dynamics based on individual vertex dynamics, often assumed to be simple, and couplings. Of particular interest is to investigate in what degree and fashion the dynamical behaviors are determined by the architecture of the network encoded in the directed graph. From the viewpoint of stability and control of complex dynamical systems, it is of significance to investigate the global-stability problem. Assume that, when isolated, each vertex system has a globally stable equilibrium, the interest is to determine whether the coupled system has a globally stable equilibrium when vertices are connected according to a directed graph. The method of Lyapunov functions is standard for establishing global stability, especially for large systems. One can formulate the global-stability problem as follows: assuming that, when isolated, each vertex system has a globally defined Lyapunov function V_i , under what conditions of the couplings and the network does the coupled system have a globally defined Lyapunov function V , when the vertices are connected? Moreover, is there a systematic approach to the construction of V from individual V_i ?

In this paper, we develop a general approach to the construction of a global Lyapunov function V for the coupled system in the form

$$V(x) = \sum_{i=1}^n c_i V_i,$$

where $c_i \geq 0$ are suitable constants. Using Kirchhoff's Matrix Tree Theorem in graph theory, we present, in Theorem 3.1, a systematic way of deriving constants c_i , based on properties of V_i and structures of the underlying graph. As we will show, the approach is sufficiently general to be applicable to a variety of coupled systems including coupled oscillators, patchy ecological models with dispersal, and multi-group epidemic models. We also show that our approach can be applied to coupled systems of delayed differential equations. Our approach is shown to work with several well known types of Lyapunov functions in the literature, from the traditional energy-type functions to functions of form $x - x^* \ln x$ that are well known in mathematical ecology and epidemiology literature. In all examples considered in this paper, our approach allows significant improvements of existing results in the literature.

Concepts from graph theory related to our development are reviewed in the next section. We also prove two useful combinatorial identities based on Kirchhoff's Matrix Tree Theorem. Our main results are presented in Section 3. In Sections 4–8, our main results are applied to several well-known coupled systems in the literature to demonstrate their applicability and effectiveness.

2. Graph theoretical results

In this section, we prove several combinatorial identities that we will use in later sections. We begin by recalling some definitions from graph theory. We refer the reader to [17,36] for detailed discussions.

A *directed graph* or *digraph* $\mathcal{G} = (V, E)$ contains a set $V = \{1, 2, \dots, n\}$ of vertices and a set E of arcs (i, j) leading from initial vertex i to terminal vertex j . A subgraph \mathcal{H} of \mathcal{G} is said to be *spanning* if \mathcal{H} and \mathcal{G} have the same vertex set. A digraph \mathcal{G} is *weighted* if each arc (j, i) is assigned a positive weight a_{ij} . In our convention, $a_{ij} > 0$ if and only if there exists an arc from vertex j to vertex i in \mathcal{G} . The weight $w(\mathcal{H})$ of a subgraph \mathcal{H} is the product of the weights on all its arcs.

A *directed path* \mathcal{P} in \mathcal{G} is a subgraph with distinct vertices $\{i_1, i_2, \dots, i_m\}$ such that its set of arcs is $\{(i_k, i_{k+1}) : k = 1, 2, \dots, m - 1\}$. If $i_m = i_1$, we call \mathcal{P} a *directed cycle*. A connected subgraph \mathcal{T} is

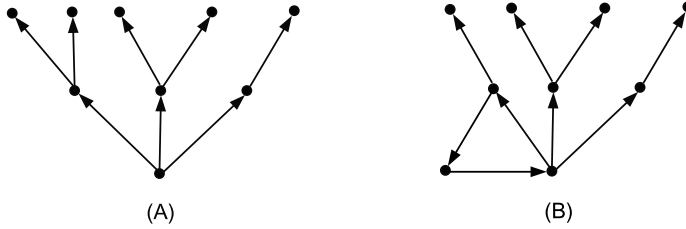


Fig. 1. (a) A rooted tree. (b) A unicyclic graph.

a tree if it contains no cycles, directed or undirected. A tree \mathcal{T} is rooted at vertex i , called the root, if i is not a terminal vertex of any arcs, and each of the remaining vertices is a terminal vertex of exactly one arc. A subgraph \mathcal{Q} is unicyclic if it is a disjoint union of rooted trees whose roots form a directed cycle. Note that every vertex of \mathcal{Q} is the terminal vertex of exactly one arc. A rooted tree and a unicyclic graph are depicted in Fig. 1.

Given a weighted digraph \mathcal{G} with n vertices, define the weight matrix $A = (a_{ij})_{n \times n}$ whose entry a_{ij} equals the weight of arc (j, i) if it exists, and 0 otherwise. For our purpose, we denote a weighted digraph as (\mathcal{G}, A) . A digraph \mathcal{G} is strongly connected if, for any pair of distinct vertices, there exists a directed path from one to the other. A weighted digraph (\mathcal{G}, A) is strongly connected if and only if the weight matrix A is irreducible [5]. The Laplacian matrix of (\mathcal{G}, A) is defined as

$$L = \begin{bmatrix} \sum_{k \neq 1} a_{1k} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \sum_{k \neq 2} a_{2k} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \sum_{k \neq n} a_{nk} \end{bmatrix}. \tag{2.1}$$

Let c_i denote the cofactor of the i -th diagonal element of L . The following result is standard in graph theory, and customarily called Kirchhoff’s Matrix Tree Theorem. We refer the reader to [21,30] for its proof.

Proposition 2.1. Assume $n \geq 2$. Then

$$c_i = \sum_{T \in \mathbb{T}_i} w(T), \quad i = 1, 2, \dots, n, \tag{2.2}$$

where \mathbb{T}_i is the set of all spanning trees \mathcal{T} of (\mathcal{G}, A) that are rooted at vertex i , and $w(T)$ is the weight of \mathcal{T} . In particular, if (\mathcal{G}, A) is strongly connected, then $c_i > 0$ for $1 \leq i \leq n$.

Theorem 2.2. Assume $n \geq 2$. Let c_i be given in Proposition 2.1. Then the following identity holds:

$$\sum_{i, j=1}^n c_i a_{ij} F_{ij}(x_i, x_j) = \sum_{\mathcal{Q} \in \mathcal{Q}} w(\mathcal{Q}) \sum_{(s,r) \in E(C_{\mathcal{Q}})} F_{rs}(x_r, x_s). \tag{2.3}$$

Here $F_{ij}(x_i, x_j)$, $1 \leq i, j \leq n$, are arbitrary functions, \mathcal{Q} is the set of all spanning unicyclic graphs of (\mathcal{G}, A) , $w(\mathcal{Q})$ is the weight of \mathcal{Q} , and $C_{\mathcal{Q}}$ denotes the directed cycle of \mathcal{Q} .

Proof. For a spanning tree \mathcal{T} rooted at vertex i ,

$$w(\mathcal{T})a_{ij} = w(\mathcal{Q}),$$

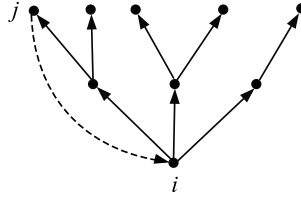


Fig. 2. A unicyclic graph is formed by adding a directed arc (j, i) to a tree rooted at i .

where \mathcal{Q} is the unicyclic graph obtained from \mathcal{T} by adding an arc (j, i) from vertex j to the root vertex i , see Fig. 2. As a consequence,

$$w(\mathcal{T})a_{ij}F_{ij}(x_i, x_j) = w(\mathcal{Q})F_{ij}(x_i, x_j), \quad \text{and } (j, i) \in E(\mathcal{C}_{\mathcal{Q}}).$$

When we perform this operation in all possible ways to all rooted trees in \mathcal{G} , we obtain all unicyclic graphs in \mathcal{G} , and each unicyclic graph \mathcal{Q} is created as many times as the number of arcs in its cycle $\mathcal{C}_{\mathcal{Q}}$. The identity (2.3) follows from (2.2) if we reorganize the double sum on the left-hand side as a sum over all unicyclic graphs in \mathcal{G} . \square

Theorem 2.3. Assume $n \geq 2$. Let c_i be given in Proposition 2.1. Then the following identity holds:

$$\sum_{i,j=1}^n c_i a_{ij} G_i(x_i) = \sum_{i,j=1}^n c_i a_{ij} G_j(x_j), \tag{2.4}$$

where $G_i(x_i), 1 \leq i \leq n$, are arbitrary functions.

Proof. Using Theorem 2.2, we know that both sides of (2.4) are equal to

$$\sum_{\mathcal{Q} \in \mathcal{Q}} w(\mathcal{Q}) \sum_{k \in V(\mathcal{C}_{\mathcal{Q}})} G_k(x_k),$$

where $V(\mathcal{C}_{\mathcal{Q}})$ is the vertex set of $\mathcal{C}_{\mathcal{Q}}$. \square

3. Coupled systems of differential equations on networks

Given a network represented by digraph \mathcal{G} with n vertices, $n \geq 2$, a coupled system can be built on \mathcal{G} by assigning each vertex its own internal dynamics and then coupling these vertex dynamics based on directed arcs in \mathcal{G} . Assume that each vertex dynamics is described by a system of differential equations

$$u'_i = f_i(t, u_i), \tag{3.1}$$

where $u_i \in \mathbb{R}^{m_i}$ and $f_i : \mathbb{R} \times \mathbb{R}^{m_i} \rightarrow \mathbb{R}^{m_i}$. Let $g_{ij} : \mathbb{R} \times \mathbb{R}^{m_i} \times \mathbb{R}^{m_j} \rightarrow \mathbb{R}^{m_i}$ represent the influence of vertex j on vertex i , and $g_{ij} \equiv 0$ if there exists no arc from j to i in \mathcal{G} . Then we obtain the following coupled system on graph \mathcal{G}

$$u'_i = f_i(t, u_i) + \sum_{j=1}^n g_{ij}(t, u_i, u_j), \quad i = 1, 2, \dots, n. \tag{3.2}$$

Here functions f_i, g_{ij} are such that initial-value problems to (3.1) and (3.2) have unique solutions. Many large-scale dynamical systems from science and engineering can be represented as coupled systems on networks in the form of (3.2). Several examples are considered in Sections 4–8.

We assume that each vertex system (3.1) has a globally stable equilibrium and possesses a global Lyapunov function V_i . Our objective is to investigate if the coupled system (3.2) has a globally stable equilibrium, and if a global Lyapunov function V can be systematically constructed for system (3.2) using individual V_i . Such an investigation is significant for the stability and control of large-scale dynamical systems.

Let $D_i \subset \mathbb{R}^{m_i}$ be an open set. For a Lipschitz function $V_i : \mathbb{R} \times D_i \rightarrow \mathbb{R}$, we define the Lyapunov derivative with respect to system (3.2) as

$$\dot{V}_i(t, u_i) := \frac{\partial V_i(t, u_i)}{\partial t} + \frac{\partial V_i(t, u_i)}{\partial u_i} \left(f_i(t, u_i) + \sum_{j=1}^n g_{ij}(t, u_i, u_j) \right). \tag{3.3}$$

Let $D = D_1 \times D_2 \times \dots \times D_n \subset \mathbb{R}^m$, $m = m_1 + m_2 + \dots + m_n$, and $u = (u_1, u_2, \dots, u_n)$. For a Lipschitz function $V : \mathbb{R} \times D \rightarrow \mathbb{R}$, we define

$$\dot{V}(t, u) := \frac{\partial V(t, u)}{\partial t} + \sum_{i=1}^n \frac{\partial V(t, u)}{\partial u_i} \left(f_i(t, u_i) + \sum_{j=1}^n g_{ij}(t, u_i, u_j) \right). \tag{3.4}$$

Let $V_i(t, u_i)$ be a Lyapunov function for each vertex system (3.1). We are particularly interested in constructing Lyapunov functions for coupled system (3.2) of form

$$V(t, u) = \sum_{i=1}^n c_i V_i(t, u_i). \tag{3.5}$$

The following result gives a general and systematic approach for such construction.

Theorem 3.1. *Assume that the following assumptions are satisfied.*

(1) *There exist functions $V_i(t, u_i), F_{ij}(t, u_i, u_j)$, and constants $a_{ij} \geq 0$ such that*

$$\dot{V}_i(t, u_i) \leq \sum_{j=1}^n a_{ij} F_{ij}(t, u_i, u_j), \quad t > 0, u_i \in D_i, i = 1, 2, \dots, n. \tag{3.6}$$

(2) *Along each directed cycle C of the weighted digraph (\mathcal{G}, A) , $A = (a_{ij})$,*

$$\sum_{(s,r) \in E(C)} F_{rs}(t, u_r, u_s) \leq 0, \quad t > 0, u_r \in D_r, u_s \in D_s. \tag{3.7}$$

(3) *Constants c_i are given in (2.2).*

Then the function $V(t, u)$ in (3.5) satisfies $\dot{V}(t, u) \leq 0$ for $t > 0$ and $u \in D$, namely, V is a Lyapunov function for (3.2).

Proof. Using (3.3), (3.4), and assumption (1), we obtain

$$\dot{V}(t, u) = \sum_{i=1}^n c_i \dot{V}_i(t, u_i) \leq \sum_{i,j=1}^n c_i a_{ij} F_{ij}(t, u_i, u_j).$$

Applying Theorem 2.2 with weighted digraph (\mathcal{G}, A) , we obtain

$$\sum_{i,j=1}^n c_i a_{ij} F_{ij}(t, u_i, u_j) = \sum_{\mathcal{Q} \in \mathcal{Q}} w(\mathcal{Q}) \sum_{(s,r) \in E(\mathcal{C}_{\mathcal{Q}})} F_{rs}(t, u_r, u_s). \tag{3.8}$$

Since $w(\mathcal{Q}) > 0$ and along directed cycle $\mathcal{C}_{\mathcal{Q}}$

$$\sum_{(s,r) \in E(\mathcal{C}_{\mathcal{Q}})} F_{rs}(t, u_r, u_s) \leq 0,$$

by assumption (2), we arrive at $\dot{V}(t, u) \leq 0$, completing the proof of Theorem 3.1. \square

A weighted digraph (\mathcal{G}, A) is said to be *balanced* if $w(\mathcal{C}) = w(-\mathcal{C})$ for all directed cycles \mathcal{C} [31]. Here, $-\mathcal{C}$ denotes the reverse of \mathcal{C} and is constructed by reversing the direction of all arcs in \mathcal{C} . For a unicyclic graph \mathcal{Q} with cycle $\mathcal{C}_{\mathcal{Q}}$, let $\tilde{\mathcal{Q}}$ be the unicyclic graph obtained by replacing $\mathcal{C}_{\mathcal{Q}}$ with $-\mathcal{C}_{\mathcal{Q}}$. Suppose that (\mathcal{G}, A) is balanced. Then $w(\mathcal{Q}) = w(\tilde{\mathcal{Q}})$. In the right-hand side of relation (3.8), we can further pair \mathcal{Q} with $\tilde{\mathcal{Q}}$ and obtain

$$\sum_{i,j=1}^n c_i a_{ij} F_{ij}(t, u_i, u_j) = \frac{1}{2} \sum_{\mathcal{Q} \in \mathcal{Q}} w(\mathcal{Q}) \sum_{(s,r) \in E(\mathcal{C}_{\mathcal{Q}})} [F_{rs}(t, u_r, u_s) + F_{sr}(t, u_s, u_r)]. \tag{3.9}$$

The same proof shows that the conclusion of Theorem 3.1 holds if the assumption (2) is replaced by the following:

(2') Along each directed cycle \mathcal{C}

$$\sum_{(s,r) \in E(\mathcal{C})} (F_{rs}(t, u_r, u_s) + F_{sr}(t, u_s, u_r)) \leq 0, \quad t > 0, u_r \in D_r, u_s \in D_s. \tag{3.10}$$

We thus have the following result.

Theorem 3.2. *Suppose that (\mathcal{G}, A) is balanced. Then the conclusion of Theorem 3.1 holds if condition (3.7) is replaced by (3.10).*

Consider a Volterra predator–prey system

$$x'_i = x_i \left(e_i + \sum_{j=1}^n p_{ij} x_j \right), \quad i = 1, 2, \dots, n, \tag{3.11}$$

where $x_i \in \mathbb{R}_+$ represents population density of the i -th species, $e_i \in \mathbb{R}$, $p_{ii} \leq 0$, and $p_{ij} p_{ji} < 0$ if $p_{ij} \neq 0, i \neq j$ [13]. Suppose that (3.11) admits a positive equilibrium $E^* = (x_1^*, x_2^*, \dots, x_n^*)$, $x_i^* > 0, i = 1, 2, \dots, n$. Let

$$V_i(x_i) = x_i - x_i^* - x_i^* \ln \frac{x_i}{x_i^*}.$$

Differentiating V_i along (3.11) gives

$$\begin{aligned} \dot{V}_i &= e_i x_i + \sum_{j=1}^n p_{ij} x_i x_j - e_i x_i^* - \sum_{j=1}^n p_{ij} x_i^* x_j = \sum_{j=1}^n p_{ij} (x_i - x_i^*) (x_j - x_j^*) \\ &= \sum_{j=1}^n a_{ij} F_{ij}(x_i, x_j), \end{aligned}$$

where $a_{ij} = |p_{ij}|$ and $F_{ij}(x_i, x_j) = \text{sgn}(p_{ij})(x_i - x_i^*)(x_j - x_j^*)$. In particular, $F_{ij}(x_i, x_j) = -F_{ji}(x_j, x_i)$, $i \neq j$, and condition (3.10) is satisfied. Let c_i be given in (2.2). Then, by Theorem 3.2, $V(x_1, x_2, \dots, x_n) = \sum_{i=1}^n c_i V_i(x_i)$ is a Lyapunov function for (3.11) provided that (\mathcal{G}, A) is balanced. This extends an earlier result on global Lyapunov functions for (3.11) in [31]. We remark that [31] contains extensive applications of graph theory to global analysis of Volterra systems.

Conditions (3.7) of Theorem 3.1 and (3.10) of Theorem 3.2 can be readily verified if there exist functions $G_i(t, u_i)$, $i = 1, 2, \dots, n$, such that

$$F_{ij}(t, u_i, u_j) \leq G_i(t, u_i) - G_j(t, u_j), \quad 1 \leq i, j \leq n. \tag{3.12}$$

We thus have the following corollary.

Corollary 3.3. *The conclusion of Theorem 3.1 or Theorem 3.2 holds if condition (3.7) or (3.10) is replaced by (3.12).*

If each V_i satisfies a stronger condition

$$\dot{V}_i(t, u_i) \leq -b_i V_i(t, u_i) + \sum_{j=1}^n a_{ij} F_{ij}(t, u_i, u_j), \quad t > 0, u_i \in D_i, 1 \leq i \leq n, \tag{3.13}$$

for constants $b_i \geq 0$, then a stronger conclusion can be drawn for V . The following result can be proved the same way as Theorems 3.1 and 3.2.

Theorem 3.4. *Assume that the following assumptions hold.*

- (1) *There exist $V_i(t, u_i)$, $F_{ij}(t, u_i, u_j)$, $a_{ij} \geq 0$, and $b_i \geq 0$ such that (3.13) holds.*
- (2) *Either (3.7) or (3.12) holds, or if (\mathcal{G}, A) is balanced and (3.10) holds.*
- (3) *Constants c_i are given in (2.2).*

Then the function $V(t, u)$ in (3.5) satisfies

$$\dot{V}(t, u) \leq -bV(t, u) \quad \text{for } t > 0, u \in D,$$

where $b = \min\{b_1, b_2, \dots, b_n\}$.

To demonstrate that mere existence of Lyapunov functions V_i for each vertex system is not sufficient for the existence of V , we consider the following example, which shows that two asymptotically stable linear systems can be coupled through linear diffusion to form an unstable system.

Example. Let $A = \begin{pmatrix} -2 & 3 \\ -1 & 1 \end{pmatrix}$. Two eigenvalues of A are $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$, and the zero solution of the vertex systems

$$\begin{pmatrix} x'_i \\ y'_i \end{pmatrix} = A \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \quad i = 1, 2,$$

is globally asymptotically stable. Consider the following linearly coupled system

$$\begin{aligned} \begin{pmatrix} x'_1 \\ y'_1 \end{pmatrix} &= A \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 - x_1 \\ 0 \end{pmatrix}, \\ \begin{pmatrix} x'_2 \\ y'_2 \end{pmatrix} &= A \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + \begin{pmatrix} x_1 - x_2 \\ 0 \end{pmatrix}, \end{aligned} \tag{3.14}$$

whose coefficient matrix

$$\begin{pmatrix} -3 & 3 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & -3 & 3 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

has a positive eigenvalue $\frac{\sqrt{13}-3}{2}$, and thus the zero solution to the coupled system (3.14) is unstable.

In the rest of the paper, to demonstrate the applicability and effectiveness of the approach described in our main results, we consider the global-stability problem for several classes of coupled systems. We show that Lyapunov functions for these coupled systems can be systematically constructed from well-known Lyapunov functions for individual vertex systems. In all the cases considered, our approach allows significant improvements of the best known results in the literature.

4. Coupled oscillators on a network

Given a weighted digraph (\mathcal{G}, A) with n vertices, $A = (a_{ij})$, $n \geq 2$, a coupled system of nonlinear oscillators on \mathcal{G} can be built as follows: each vertex i is assigned a nonlinear oscillator described by

$$\ddot{x}_i + \alpha_i \dot{x}_i + f_i(x_i) = 0, \tag{4.1}$$

where $\alpha_i \geq 0$ is the damping coefficient, $f_i : \mathbb{R} \rightarrow \mathbb{R}$ is the nonlinear restoring force, and the influence from vertex j to vertex i is provided in the form $a_{ij}(\dot{x}_i - \dot{x}_j)$ [12]. Here weight constants $a_{ij} \geq 0$, and $a_{ij} = 0$ if and only if no arc exists from j to i in \mathcal{G} . We arrive at a coupled system of second order differential equations

$$\ddot{x}_i + \alpha_i \dot{x}_i + f_i(x_i) + \sum_{j=1}^n a_{ij}(\dot{x}_i - \dot{x}_j) = 0, \quad i = 1, 2, \dots, n,$$

or in the form of first order systems

$$\begin{aligned} \dot{x}_i &= y_i, \\ \dot{y}_i &= -\alpha_i y_i - f_i(x_i) - \sum_{j=1}^n a_{ij}(y_i - y_j). \end{aligned} \tag{4.2}$$

Assume that there exists x_i^* such that $f_i(x_i) = 0$ iff $x_i = x_i^*$, and the potential energy $F_i(x_i) = \int^{x_i} f_i(s) ds$ has a global minimum at $x_i = x_i^*$. Then it is standard that, if $\alpha_i > 0$, the total energy

$$V_i(x_i, y_i) = \frac{y_i^2}{2} + F_i(x_i) \tag{4.3}$$

is a global Lyapunov function for the global asymptotic stability of $x_i = x_i^*$ for vertex system (4.1). It can be verified that $E^* = (x_1^*, 0, x_2^*, 0, \dots, x_n^*, 0)$ is an equilibrium of the coupled system (4.2). We investigate conditions under which E^* is globally asymptotically stable.

Theorem 4.1. *Assume (\mathcal{G}, A) is strongly connected. Suppose that there exists k such that $\alpha_k > 0$. Then E^* is globally asymptotically stable in \mathbb{R}^{2n} .*

Proof. We want to verify that $V_i(x_i, y_i)$ in (4.3) satisfies the assumptions of Theorem 3.1. Differentiating V_i along (4.2) gives

$$\begin{aligned} \dot{V}_i &= -\alpha_i y_i^2 - \sum_{j=1}^n a_{ij}(y_i - y_j)y_i \\ &= -\alpha_i y_i^2 + \sum_{j=1}^n a_{ij} \left(-\frac{1}{2}(y_i - y_j)^2 + \frac{1}{2}y_j^2 - \frac{1}{2}y_i^2 \right) \\ &\leq \sum_{j=1}^n a_{ij} \left(\frac{1}{2}y_j^2 - \frac{1}{2}y_i^2 \right). \end{aligned}$$

Let

$$F_{ij}(y_i, y_j) = \frac{1}{2}y_j^2 - \frac{1}{2}y_i^2.$$

We have

$$\dot{V}_i \leq \sum_{j=1}^n a_{ij} F_{ij}(y_i, y_j),$$

and along every directed cycle \mathcal{C} of the weighted digraph (\mathcal{G}, A) ,

$$\sum_{(s,r) \in E(\mathcal{C})} F_{rs}(y_r, y_s) = \sum_{(s,r) \in E(\mathcal{C})} \left(\frac{1}{2}y_s^2 - \frac{1}{2}y_r^2 \right) = 0.$$

Assumptions (1) and (2) of Theorem 3.1 have been verified. Let c_i be the cofactor of the i -th diagonal element in the Laplacian matrix of (\mathcal{G}, A) , as given in (2.2). Then, by Theorem 3.1,

$$V(x_1, y_1, \dots, x_n, y_n) = \sum_{i=1}^n c_i V_i(x_i, y_i)$$

is a Lyapunov function for (4.2), namely, $\dot{V} \leq 0$ for all $(x_1, y_1, \dots, x_n, y_n) \in \mathbb{R}^{2n}$.

To show E^* is globally asymptotically stable, we examine the largest compact invariant set where $\dot{V} = 0$. Since (\mathcal{G}, A) is strongly connected, $c_i > 0$ for $1 \leq i \leq n$. Therefore, $\dot{V} = 0$ implies that $\alpha_i y_i^2 = 0$ and $a_{ij}(y_i - y_j)^2 = 0$ for all $1 \leq i, j \leq n$. As a consequence, $y_i = 0$ if $\alpha_i > 0$; and $y_i = y_j$ if $a_{ij} > 0$, or if there exists an arc from j to i in (\mathcal{G}, A) . By our assumption, there exists k such that $\alpha_k > 0$, thus $y_k = 0$. Let $l \neq k$ denote any vertex of (\mathcal{G}, A) . Then, by the strong connectivity of (\mathcal{G}, A) , there exists a directed path \mathcal{P} from l to k . Applying the relation $y_i = y_j$ to each arc (j, i) of \mathcal{P} , we obtain that $y_l = y_k$. Hence, $\dot{V} = 0$ implies $y_i = 0$ for all i . From the second equation of (4.2), we have $0 = \dot{y}_i = -f_i(x_i)$, and thus $x_i = x_i^*$. This implies that the largest compact invariant subset of $\{(x_1, y_1, \dots, x_n, y_n) \in \mathbb{R}^{2n} \mid \dot{V} = 0\}$ is the singleton $\{E^*\}$. Therefore, by the LaSalle Invariance Principle [24], E^* is globally asymptotically stable in \mathbb{R}^{2n} . \square

Theorem 4.1 shows that in a strongly connected network, the existence of one damped oscillator is sufficient to stop all oscillations in coupled system (4.2).

5. A single-species ecological model with dispersal

Consider the following system that describes the growth and dispersal of a single species among n patches ($n \geq 2$)

$$x'_i = x_i f_i(x_i) + \sum_{j=1}^n d_{ij}(x_j - \alpha_{ij}x_i), \quad i = 1, 2, \dots, n. \tag{5.1}$$

Here $x_i \in \mathbb{R}_+$ represents population density of the species in patch i , $f_i \in C^1(\mathbb{R}_+, \mathbb{R})$ represents the density dependent growth rate in patch i , constant $d_{ij} \geq 0$ is the dispersal rate from patch j to patch i , and constants $\alpha_{ij} \geq 0$ can be selected to represent different boundary conditions in the continuous diffusion case [27]. Hastings [18] studied the local stability of a positive equilibrium of (5.1). Sufficient conditions for uniqueness and global stability of the positive equilibrium were derived in Beretta and Takeuchi [3] and in Lu and Takeuchi [27]. In this section, we interpret (5.1) as a coupled system on a network. Using Theorem 3.1, we prove a global-stability result that is stronger than those in [3] and [27].

A digraph \mathcal{G} with n vertices can be constructed for system (5.1) as follows: each vertex represents a patch, a directed arc (j, i) is assigned if the dispersal rate d_{ij} from patch j to patch i is positive, and no such arc exists if $d_{ij} = 0$. The dynamics at each vertex are defined by the scalar ordinary differential equation $x'_i = x_i f_i(x_i)$. The coupling among vertices are provided by the dispersal among patches. We remark that the dispersal network \mathcal{G} is strongly connected if and only if the dispersal matrix (d_{ij}) is irreducible.

Theorem 5.1. *Assume that the following assumptions hold.*

- (1) *Dispersal matrix (d_{ij}) is irreducible.*
- (2) *$f'_i(x_i) \leq 0, x_i > 0, i = 1, 2, \dots, n$, and there exists k such that $f'_k(x_k) \neq 0$ in any open interval of \mathbb{R}_+ .*
- (3) *System (5.1) is uniformly persistent.*
- (4) *Solutions of (5.1) are uniformly ultimately bounded.*

Then system (5.1) has a globally asymptotically stable positive equilibrium E^ in \mathbb{R}_+^n .*

Proof. Uniform persistence, together with uniform ultimate boundedness of all solutions, implies that (5.1) has at least one positive equilibrium [6,32]. Let $E^* = (x_1^*, x_2^*, \dots, x_n^*), x_i^* > 0, i = 1, 2, \dots, n$, denote a positive equilibrium of (5.1). Then x_i^* satisfies

$$f_i(x_i^*) = - \sum_{j=1}^n d_{ij} \left(\frac{x_j^*}{x_i^*} - \alpha_{ij} \right). \tag{5.2}$$

We show that E^* is globally asymptotically stable in \mathbb{R}_+^n , and thus is unique.

Set $V_i(x_i) = x_i - x_i^* + x_i^* \ln \frac{x_i}{x_i^*}$. It can be verified that $V_i(x_i) > 0$ for all $x_i > 0$ and $V_i(x_i) = 0$ if and only if $x_i = x_i^*$. Direct calculation and (5.2) yield

$$\begin{aligned} \dot{V}_i &= (x_i - x_i^*) \left[f_i(x_i) + \sum_{j=1}^n d_{ij} \left(\frac{x_j}{x_i} - \alpha_{ij} \right) \right] \\ &= (x_i - x_i^*) \left[- \sum_{j=1}^n d_{ij} \left(\frac{x_j^*}{x_i^*} - \alpha_{ij} \right) + (f_i(x_i) - f_i(x_i^*)) + \sum_{j=1}^n d_{ij} \left(\frac{x_j}{x_i} - \alpha_{ij} \right) \right] \\ &= (x_i - x_i^*) (f_i(x_i) - f_i(x_i^*)) + \sum_{j=1}^n d_{ij} x_j^* \left(\frac{x_j}{x_j^*} - \frac{x_i}{x_i^*} + 1 - \frac{x_i^* x_j}{x_i x_j^*} \right). \end{aligned} \tag{5.3}$$

Let $a_{ij} = d_{ij} x_j^*$, $F_{ij}(x_i, x_j) = \frac{x_j}{x_j^*} - \frac{x_i}{x_i^*} + 1 - \frac{x_i^* x_j}{x_i x_j^*}$, and $G_i(x_i) = -\frac{x_i}{x_i^*} + \ln \frac{x_i}{x_i^*}$. Then we have

$$\dot{V}_i \leq \sum_{j=1}^n a_{ij} F_{ij}(x_i, x_j),$$

and

$$F_{ij}(x_i, x_j) = G_i(x_i) - G_j(x_j) + 1 - \frac{x_i^* x_j}{x_i x_j^*} + \ln \frac{x_i^* x_j}{x_i x_j^*} \leq G_i(x_i) - G_j(x_j).$$

Here we use two facts: $(x_i - x_i^*)(f_i(x_i) - f_i(x_i^*)) \leq 0$, and $1 - a + \ln a \leq 0$ for $a > 0$ with equality holding iff $a = 1$. We have shown that V_i, F_{ij}, G_i , and a_{ij} satisfy the assumptions of Theorem 3.1 and Corollary 3.3. Therefore,

$$V(x_1, \dots, x_n) = \sum_{i=1}^n c_i V_i(x_i)$$

as defined in Theorem 3.1 is a Lyapunov function for (5.1), namely, $\dot{V} \leq 0$ for all $(x_1, \dots, x_n) \in \mathbb{R}_+^n$. Using the strong connectivity of (\mathcal{G}, A) and a similar argument as in Section 4, we can show that $\dot{V} = 0$ if and only if $x_i = x_i^*$ for all i . By the classical Lyapunov stability theory, E^* is globally asymptotically stable in \mathbb{R}_+^n . This completes the proof of Theorem 5.1. \square

Theorem 5.1 contains an earlier result in Lu and Takeuchi [27], in which the global stability of E^* was proved under much stricter conditions that $f'_i(x_i) < 0$ in $(0, +\infty)$ for all i .

6. An n -patch predator–prey model

In this section, we consider a predator–prey model in which preys disperse among n patches ($n \geq 2$),

$$\begin{aligned} x'_i &= x_i(r_i - b_i x_i - e_i y_i) + \sum_{j=1}^n d_{ij}(x_j - \alpha_{ij} x_i), \\ y'_i &= y_i(-\gamma_i - \delta_i y_i + \epsilon_i x_i), \quad i = 1, 2, \dots, n. \end{aligned} \tag{6.1}$$

Here, x_i, y_i denote the densities of preys and predators on the patch i , respectively. The parameters in the model are nonnegative constants, and e_i, ϵ_i are positive. The dispersal constants d_{ij}, α_{ij} are similarly defined as in the previous section. We refer the reader to [13,23] for interpretations of predator–prey models and parameters. When $n = 2$, Kuang and Takeuchi [23] proved the uniqueness and global stability of a positive equilibrium by constructing a Lyapunov function. In this section, we regard (6.1) as a coupled system on a network. Using a Lyapunov function for single–patch predator–prey models [20] and our main result Theorem 3.1, we establish that a positive equilibrium of the n -patch model (6.1) is globally asymptotically stable in \mathbb{R}_+^{2n} as long as it exists.

A digraph \mathcal{G} with n vertices for system (6.1) can be constructed similarly as in the previous section. Each vertex represents a patch and $(j, i) \in E(\mathcal{G})$ if and only if $d_{ij} > 0$. At each vertex of \mathcal{G} , the vertex dynamics is described by a predator–prey system. The coupling among these predator–prey systems are provided by dispersal among prey populations.

Theorem 6.1. *Assume (d_{ij}) is irreducible. If there exists k such that $b_k > 0$ or $\delta_k > 0$, then, whenever a positive equilibrium E^* exists, it is unique and globally asymptotically stable in the positive cone \mathbb{R}_+^{2n} .*

Proof. Let $E^* = (x_1^*, y_1^*, \dots, x_n^*, y_n^*)$, $x_i^*, y_i^* > 0$ for $1 \leq i \leq n$, denote the positive equilibrium. Consider a Lyapunov function in [20] for a single–patch predator–prey model

$$V_i(x_i, y_i) = \epsilon_i(x_i - x_i^* \ln x_i) + e_i(y_i - y_i^* \ln y_i).$$

We show that V_i satisfies the assumptions of Theorem 3.1. Following similar steps as in (5.3) we can verify

$$\dot{V}_i = -\epsilon_i b_i (x_i - x_i^*)^2 - e_i \delta_i (y_i - y_i^*)^2 + \sum_{j=1}^n d_{ij} \epsilon_i x_j^* \left(\frac{x_j}{x_j^*} - \frac{x_i}{x_i^*} + 1 - \frac{x_j x_i^*}{x_j^* x_i} \right).$$

Set $a_{ij} = d_{ij} \epsilon_i x_j^*$, $F_{ij}(x_i, x_j) = \frac{x_j}{x_j^*} - \frac{x_i}{x_i^*} + 1 - \frac{x_i x_j^*}{x_i x_j^*}$, and $G_i(x_i) = -\frac{x_i}{x_i^*} + \ln \frac{x_i}{x_i^*}$. Then, as in the previous section, V_i, F_{ij}, G_i , and a_{ij} satisfy the assumptions of Theorem 3.1 and Corollary 3.3. Therefore, the function $V(x_1, y_1, \dots, x_n, y_n) = \sum_{i=1}^n c_i V_i(x_i, y_i)$ as defined in Theorem 3.1 is a Lyapunov function for (6.1), and $\dot{V} \leq 0$ for all $(x_1, y_1, \dots, x_n, y_n) \in \mathbb{R}_+^{2n}$. Using a similar argument as in Section 4, we can show that the only compact invariant set on which $\dot{V} = 0$ is the singleton $\{E^*\}$. The LaSalle Invariance Principle [24] implies that E^* is globally asymptotically stable in \mathbb{R}_+^{2n} . This also implies that E^* is unique in \mathbb{R}_+^{2n} , completing the proof of Theorem 6.1. \square

The existence requirement for E^* in Theorem 6.1 can be satisfied through persistence analysis, which only involves dynamics on the boundary. Theorem 6.1 generalizes a global–stability result in [23] from 2 patches to arbitrary n patches.

7. A multi-group epidemic model with nonlinear incidence

In this section, we consider a coupled n -group epidemic model in which nonlinear couplings are provided by inter-group cross infections. The model

$$\begin{aligned}
 S'_i &= \Lambda_i - d_i^S S_i - \sum_{j=1}^n \beta_{ij} f_{ij}(S_i, I_j), \\
 E'_i &= \sum_{j=1}^n \beta_{ij} f_{ij}(S_i, I_j) - (d_i^E + \epsilon_i) E_i, \quad i = 1, 2, \dots, n, \\
 I'_i &= \epsilon_i E_i - (d_i^I + \gamma_i) I_i,
 \end{aligned}
 \tag{7.1}$$

describes the spread of an infectious disease in a heterogeneous population, which is partitioned into n homogeneous groups. Each group i is further compartmentalized into S_i , E_i , and I_i , which denote the subpopulations that are susceptible to the disease, infected but non-infectious, and infectious, respectively. The nonlinear coupling term $\beta_{ij} f_{ij}(S_i, I_j)$ represents the cross infection from group j to group i . All parameters in (7.1) are nonnegative constants. For detailed discussions of the model and interpretations of parameters, we refer the reader to [15,34]. Let \mathcal{G} be a digraph with n vertices, in which each vertex represents a group. An arc (j, i) exists if and only if $\beta_{ij} > 0$, namely, if the disease can be transmitted from group j to group i . System (7.1) can thus be regarded as a coupled system on \mathcal{G} . We note that \mathcal{G} is strongly connected if and only if transmission matrix (β_{ij}) is irreducible.

Assume that $\epsilon_i > 0$ and $d_i^* > 0$, where $d_i^* = \min\{d_i^S, d_i^E, d_i^I + \gamma_i\}$. Based on biological considerations, we assume that $f_{ij}(0, I_j) = 0$, $f_{ij}(S_i, 0) = 0$, and $f_{ij}(S_i, I_j) > 0$ for $S_i > 0, I_j > 0$. We also assume that $f_{ij}(S_i, I_j)$ are sufficiently smooth. For each i , adding the three equations in (7.1) gives $(S_i + E_i + I_i)' \leq \Lambda_i - d_i^*(S_i + E_i + I_i)$. Hence $\limsup_{t \rightarrow \infty} (S_i + E_i + I_i) \leq \Lambda_i / d_i^*$. Similarly, from the S_i equation we obtain $\limsup_{t \rightarrow \infty} S_i \leq \Lambda_i / d_i^S$. Therefore, omega limit sets of system (7.1) are contained in the following bounded region in the nonnegative cone of \mathbb{R}^{3n}

$$\Gamma = \left\{ (S_1, E_1, I_1, \dots, S_n, E_n, I_n) \in \mathbb{R}_+^{3n} \mid S_i \leq \frac{\Lambda_i}{d_i^S}, S_i + E_i + I_i \leq \frac{\Lambda_i}{d_i^*}, 1 \leq i \leq n \right\}.
 \tag{7.2}$$

It can be verified that region Γ is positively invariant. System (7.1) always has the *disease-free equilibrium* $P_0 = (S_1^0, 0, 0, \dots, S_n^0, 0, 0)$, on the boundary of Γ , where $S_i^0 = \Lambda_i / d_i^S$. An equilibrium $P^* = (S_1^*, E_1^*, I_1^*, \dots, S_n^*, E_n^*, I_n^*)$ in the interior $\overset{\circ}{\Gamma}$ of Γ is called an *endemic equilibrium*, where $S_i^*, E_i^*, I_i^* > 0$ satisfy the equilibrium equations

$$\Lambda_i = d_i^S S_i^* + \sum_{j=1}^n \beta_{ij} f_{ij}(S_i^*, I_j^*),
 \tag{7.3}$$

$$(d_i^E + \epsilon_i) E_i^* = \sum_{j=1}^n \beta_{ij} f_{ij}(S_i^*, I_j^*),
 \tag{7.4}$$

$$\epsilon_i E_i^* = (d_i^I + \gamma_i) I_i^*.
 \tag{7.5}$$

In the rest of this section we consider the following basic assumptions on functions $f_{ij}(S_i, I_j)$:

(H1) $0 < \lim_{I_j \rightarrow 0^+} \frac{f_{ij}(S_i, I_j)}{I_j} = C_{ij}(S_i) \leq +\infty, 0 < S_i \leq S_i^0;$

(H2) $f_{ij}(S_i, I_j) \leq C_{ij}(S_i) I_j$ for sufficiently small I_j ;

(H₃) $f_{ij}(S_i, I_j) \leq C_{ij}(S_i)I_j$ for all $I_j > 0$;

(H₄) $C_{ij}(S_i) < C_{ij}(S_i^0)$, $0 < S_i < S_i^0$.

Classes of $f_{ij}(S_i, I_j)$ satisfying (H₁)–(H₄) include common incidence functions such as $f_{ij}(S_i, I_j) =$

$$I_j S_i, f_{ij}(S_i, I_j) = I_j^{p_j} S_i^{q_i}, \text{ and } f_{ij}(S_i, I_j) = \frac{I_j^{p_j} S_i^{q_i}}{I_j + A_j S_i + B_i}.$$

Assume that $f_{ij}(S_i, I_j)$ satisfies (H₁), and let

$$R_0 = \rho(M_0) \tag{7.6}$$

denote the spectral radius of the matrix

$$M_0 = M(S_1^0, S_2^0, \dots, S_n^0) = \left(\frac{\beta_{ij} \epsilon_i C_{ij}(S_i^0)}{(d_i^E + \epsilon_i)(d_i^I + \gamma_i)} \right)_{1 \leq i, j \leq n}.$$

If $C_{ij}(S_i^0) = +\infty$ for some i and j , we set $R_0 = +\infty$. The parameter R_0 is referred to as the basic reproduction number. Its biological significance is that if $R_0 < 1$ the disease dies out while if $R_0 > 1$ the disease becomes endemic [11,35]. The following results for system (7.1) are standard and can be proved the same way as in [14,34].

Proposition 7.1. Assume that $B = (\beta_{ij})$ is irreducible and $f_{ij}(S_i, I_j)$ satisfies (H₁).

- (1) If $R_0 \leq 1$ and assumptions (H₂) and (H₄) hold, then for system (7.1), P_0 is locally asymptotically stable.
- (2) If $R_0 \leq 1$ and assumptions (H₃) and (H₄) hold, then P_0 is the unique equilibrium and it is globally asymptotically stable in Γ .
- (3) If $R_0 > 1$, then P_0 is unstable and system (7.1) is uniformly persistent. Furthermore, there exists an endemic equilibrium P^* for system (7.1).

A mathematical challenging question for system (7.1) is that whether the endemic equilibrium P^* is unique when $R_0 > 1$, and whether P^* is globally asymptotically stable when it is unique. We have the following result.

Theorem 7.2. Assume that $B = (\beta_{ij})$ is irreducible and $f_{ij}(S_i, I_j)$ satisfies (H₁). If $R_0 > 1$ and $f_{ij}(S_i, I_j)$ satisfies the following conditions

$$(S_i - S_i^*)(f_{ii}(S_i, I_i^*) - f_{ii}(S_i^*, I_i^*)) > 0, \quad S_i \neq S_i^*, \tag{7.7}$$

$$\begin{aligned} & (f_{ij}(S_i, I_j) f_{ii}(S_i^*, I_i^*) - f_{ij}(S_i^*, I_j^*) f_{ii}(S_i, I_i^*)) \\ & \cdot \left(\frac{f_{ij}(S_i, I_j) f_{ii}(S_i^*, I_i^*)}{I_j} - \frac{f_{ij}(S_i^*, I_j^*) f_{ii}(S_i, I_i^*)}{I_j^*} \right) \leq 0, \quad S_i, I_j > 0, \end{aligned} \tag{7.8}$$

then there exists a unique endemic equilibrium P^* for system (7.1), and P^* is globally asymptotically stable in Γ^* .

Proof. The case $n = 1$ is proved in [22]. We only consider $n \geq 2$. Let $P^* = (S_1^*, E_1^*, I_1^*, \dots, S_n^*, E_n^*, I_n^*)$, $S_i^*, E_i^*, I_i^* > 0$ for $1 \leq i \leq n$, denote an endemic equilibrium which exists from Proposition 7.1(3). We prove that P^* is globally asymptotically stable in Γ^* . In particular, this implies that the endemic equilibrium is unique. Let

$$V_i(S_i, E_i, I_i) = \int_{S_i^*}^{S_i} \frac{f_{ii}(\xi, I_i^*) - f_{ii}(S_i^*, I_i^*)}{f_{ii}(\xi, I_i^*)} d\xi + E_i - E_i^* \ln E_i + \frac{d_i^E + \epsilon_i}{\epsilon_i} (I_i - I_i^* \ln I_i),$$

a Lyapunov function for a single-group model considered in [22]. We verify that V_i satisfies the assumptions of Theorem 3.1. Using equilibrium equations (7.3)–(7.5), we obtain

$$\begin{aligned} \dot{V}_i &= \left(1 - \frac{f_{ii}(S_i^*, I_i^*)}{f_{ii}(S_i, I_i^*)}\right) \left(\Lambda_i - d_i^S S_i - \sum_{j=1}^n \beta_{ij} f_{ij}(S_i, I_j)\right) + \left(1 - \frac{E_i^*}{E_i}\right) \left(\sum_{j=1}^n \beta_{ij} f_{ij}(S_i, I_j) \right. \\ &\quad \left. - (d_i^E + \epsilon_i) E_i\right) + \frac{d_i^E + \epsilon_i}{\epsilon_i} \left(1 - \frac{I_i^*}{I_i}\right) (\epsilon_i E_i - (d_i^I + \gamma_i) I_i) \\ &= \left(1 - \frac{f_{ii}(S_i^*, I_i^*)}{f_{ii}(S_i, I_i^*)}\right) \left(d_i^S S_i^* + \sum_{j=1}^n \beta_{ij} f_{ij}(S_i^*, I_j^*) - d_i^S S_i - \sum_{j=1}^n \beta_{ij} f_{ij}(S_i, I_j)\right) \\ &\quad + \left(1 - \frac{E_i^*}{E_i}\right) \left(\sum_{j=1}^n \beta_{ij} f_{ij}(S_i, I_j) - \sum_{j=1}^n \beta_{ij} f_{ij}(S_i^*, I_j^*) \frac{E_i}{E_i^*}\right) \\ &\quad + \sum_{j=1}^n \beta_{ij} \frac{f_{ij}(S_i^*, I_j^*)}{\epsilon_i E_i^*} \left(1 - \frac{I_i^*}{I_i}\right) \left(\epsilon_i E_i - \frac{\epsilon_i E_i^* I_i}{I_i^*}\right) \\ &= -\frac{d_i^S}{f_{ii}(S_i, I_i^*)} (S_i - S_i^*) (f_{ii}(S_i, I_i^*) - f_{ii}(S_i^*, I_i^*)) + \sum_{j=1}^n \beta_{ij} f_{ij}(S_i^*, I_j^*) \left(3 - \frac{f_{ii}(S_i^*, I_i^*)}{f_{ii}(S_i, I_i^*)}\right) \\ &\quad + \frac{f_{ij}(S_i, I_j) f_{ii}(S_i^*, I_i^*)}{f_{ij}(S_i^*, I_j^*) f_{ii}(S_i, I_i^*)} - \frac{f_{ij}(S_i, I_j) E_i^*}{f_{ij}(S_i^*, I_j^*) E_i} - \frac{I_i}{I_i^*} - \frac{E_i I_i^*}{E_i^* I_i}. \end{aligned} \tag{7.9}$$

Let $a_{ij} = \beta_{ij} f_{ij}(S_i^*, I_j^*)$, $G_i(I_i) = -\frac{I_i}{I_i^*} + \ln \frac{I_i}{I_i^*}$, and

$$F_{ij}(S_i, E_i, I_i, I_j) = 3 - \frac{f_{ii}(S_i^*, I_i^*)}{f_{ii}(S_i, I_i^*)} + \frac{f_{ij}(S_i, I_j) f_{ii}(S_i^*, I_i^*)}{f_{ij}(S_i^*, I_j^*) f_{ii}(S_i, I_i^*)} - \frac{f_{ij}(S_i, I_j) E_i^*}{f_{ij}(S_i^*, I_j^*) E_i} - \frac{I_i}{I_i^*} - \frac{E_i I_i^*}{E_i^* I_i}.$$

Then, by condition (7.7),

$$\dot{V}_i \leq \sum_{i,j=1}^n a_{ij} F_{ij}(S_i, E_i, I_i, I_j).$$

Let $\Phi(a) = 1 - a + \ln a$. Then $\Phi(a) \leq 0$ for $a > 0$ and equality holds only at $a = 1$. Furthermore,

$$\begin{aligned} F_{ij} &= G_i(x_i) - G_j(x_j) + \Phi\left(\frac{f_{ii}(S_i^*, I_i^*)}{f_{ii}(S_i, I_i^*)}\right) + \Phi\left(\frac{E_i I_i^*}{E_i^* I_i}\right) + \Phi\left(\frac{I_j f_{ij}(S_i^*, I_j^*) f_{ii}(S_i, I_i^*)}{I_j^* f_{ij}(S_i, I_j) f_{ii}(S_i^*, I_i^*)}\right) \\ &\quad + \Phi\left(\frac{f_{ij}(S_i, I_j) E_i^*}{f_{ij}(S_i^*, I_j^*) E_i}\right) + \left(\frac{f_{ij}(S_i, I_j) f_{ii}(S_i^*, I_i^*)}{f_{ij}(S_i^*, I_j^*) f_{ii}(S_i, I_i^*)} - 1\right) \left(1 - \frac{I_j f_{ij}(S_i^*, I_j^*) f_{ii}(S_i, I_i^*)}{I_j^* f_{ij}(S_i, I_j) f_{ii}(S_i^*, I_i^*)}\right) \\ &\leq G_i(x_i) - G_j(x_j) + \left(\frac{f_{ij}(S_i, I_j) f_{ii}(S_i^*, I_i^*)}{f_{ij}(S_i^*, I_j^*) f_{ii}(S_i, I_i^*)} - 1\right) \left(1 - \frac{I_j f_{ij}(S_i^*, I_j^*) f_{ii}(S_i, I_i^*)}{I_j^* f_{ij}(S_i, I_j) f_{ii}(S_i^*, I_i^*)}\right). \end{aligned}$$

Under condition (7.8), we can show that V_i, F_{ij}, G_i, a_{ij} satisfy the assumptions of Theorem 3.1 and Corollary 3.3. Therefore, the function $V = \sum_{i=1}^n c_i V_i(S_i, E_i, I_i)$ as defined in Theorem 3.1 is a Lyapunov function for (7.1), namely, $\dot{V} \leq 0$ for all $(S_1, E_1, I_1, \dots, S_n, E_n, I_n) \in \dot{\Gamma}$. It can be verified similarly as in Section 4 that the only compact invariant set where $\dot{V} = 0$ is the singleton $\{P^*\}$. By the LaSalle Invariance Principle [24], P^* is globally asymptotically stable in $\dot{\Gamma}$. This completes the proof of Theorem 7.2. \square

Remarks.

1. Condition (7.7) holds if $f_{ii}(S_i, I_i^*)$ is strictly monotonically increasing with respect to S_i .
2. In the special case $f_{ij}(S_i, I_j) = h_i(S_i)g_j(I_j)$, condition (7.8) becomes

$$(g_j(I_j) - g_j(I_j^*)) \left(\frac{g_j(I_j)}{I_j} - \frac{g_j(I_j^*)}{I_j^*} \right) \leq 0. \tag{7.10}$$

If $g_j(I_j)$ is C^1 for $I_j > 0$, then a sufficient condition for (7.10) is

$$0 \leq g'_j(I_j) \leq \frac{g_j(I_j)}{I_j}, \quad I_j > 0. \tag{7.11}$$

Furthermore, if $g_j(I_j)$ is monotonically increasing and concave down, then (7.11) holds, so does (7.10).

3. In the special case $f_{ij}(S_i, I_j) = S_i I_j$, system (7.1) becomes the standard multi-group SEIR model studied in [15]. Theorem 7.2 generalizes Theorem 1.1 in [15].
4. When $n = 1$, Theorem 7.2 contains earlier results on single-group SEIR models, see [22,25,26] and references therein.

8. A multi-group epidemic model with time delays

In this section, we demonstrate that the general approach described in Section 3 can also be applied to establish global stability of coupled systems of delay differential equations. We consider a multi-group epidemic model with time delays that is described by the following system of functional differential equations

$$\begin{aligned} S'_i &= \Lambda_i - d_i^S S_i - \sum_{j=1}^n \beta_{ij} S_i I_j(t - \tau_j), \\ I'_i &= \sum_{j=1}^n \beta_{ij} S_i I_j(t - \tau_j) - (d_i^I + \gamma_i) I_i, \quad i = 1, 2, \dots, n. \end{aligned} \tag{8.1}$$

The parameters in (8.1) are nonnegative and interpreted the same as in (7.1). Here the time delays τ_i are the result of disease latency. Similarly to (7.1), we regard (8.1) as a coupled system of differential equations on a digraph \mathcal{G} with n vertices. In \mathcal{G} , $(j, i) \in E(\mathcal{G})$ if and only if $\beta_{ij} > 0$. The vertex dynamics at each vertex is defined by a system of delay differential equations describing a single-group SIR model with latency [4],

$$\begin{aligned} S'_i &= \Lambda_i - d_i^S S_i - \beta_{ii} S_i I_i(t - \tau_i), \\ I'_i &= \beta_{ii} S_i I_i(t - \tau_i) - (d_i^I + \gamma_i) I_i, \quad i = 1, 2, \dots, n. \end{aligned} \tag{8.2}$$

The coupling between vertices i and j are provided by cross infections $\beta_{ij}S_iI_j(t - \tau_j)$ and $\beta_{ji}S_jI_i(t - \tau_i)$. For each vertex system (8.2), a Lyapunov functional is considered in [29]

$$\begin{aligned}
 V_i(S_i, I_i(\cdot)) &= S_i - S_i^* - S_i^* \ln \frac{S_i}{S_i^*} + I_i - I_i^* - I_i^* \ln \frac{I_i}{I_i^*} \\
 &+ \sum_{j=1}^n \beta_{ij} S_i^* \int_0^{\tau_j} \left(I_j(t-r) - I_j^* - I_j^* \ln \frac{I_j(t-r)}{I_j^*} \right) dr.
 \end{aligned}
 \tag{8.3}$$

We will use V_i and our approach in Section 3 to construct a global Lyapunov functional for system (8.1).

Denote $\tau = \max\{\tau_i: i = 1, 2, \dots, n\}$. Let C be the Banach space of continuous functions on $[-\tau, 0]$ with uniform norm. We consider system (8.1) in the phase space

$$X = \prod_{k=1}^n (\mathbb{R} \times C).
 \tag{8.4}$$

We consider nonnegative initial conditions for system (8.1)

$$S_i(0) = s_{i,0}, \quad I_{i0} = \phi_i, \quad i = 1, 2, \dots, n,
 \tag{8.5}$$

where $s_{i,0} \in \mathbb{R}_+$ and $\phi_i \in C$ satisfies $\phi_i(s) \geq 0$ for $-\tau_i \leq s \leq 0$. It can be verified that solutions with initial condition (8.5) remain nonnegative. Furthermore, from the first equation of (8.1) we obtain $S_i(t)' \leq \Lambda_i - d_i^S S_i(t)$. Hence, $\limsup_{t \rightarrow \infty} S_i(t) \leq \frac{\Lambda_i}{d_i^S}$. For each i , adding the two equations in (8.1) gives $(S_i(t) + I_{it}(0))' \leq \Lambda_i - d_i^*(S_i(t) + I_{it}(0))$, which implies that $\limsup_{t \rightarrow \infty} (S_i(t) + I_{it}(0)) \leq \frac{\Lambda_i}{d_i^*}$, where $d_i^* = \min\{d_i^S, d_i^I + \gamma_i\}$. Therefore, the following set is positively invariant for system (8.1)

$$\begin{aligned}
 \Theta = \left\{ (S_1, I_1(\cdot), \dots, S_n, I_n(\cdot)) \in X \mid 0 \leq S_i \leq \frac{\Lambda_i}{d_i^S}, 0 \leq S_i + I_i(0) \leq \frac{\Lambda_i}{d_i^*}, \right. \\
 \left. I_i(s) \geq 0, s \in (-\tau_i, 0], i = 1, \dots, n \right\}.
 \end{aligned}
 \tag{8.6}$$

Let

$$\begin{aligned}
 \hat{\Theta} = \left\{ (S_1, I_1(\cdot), \dots, S_n, I_n(\cdot)) \in X \mid 0 < S_i < \frac{\Lambda_i}{d_i^S}, 0 < S_i + I_i(0) < \frac{\Lambda_i}{d_i^*}, \right. \\
 \left. I_i(s) > 0, s \in (-\tau_i, 0], i = 1, \dots, n \right\}.
 \end{aligned}
 \tag{8.7}$$

It can be shown that $\hat{\Theta}$ is the interior of Θ . In Θ , system (8.1) has the disease-free equilibrium $P_0 = (S_1^0, 0, \dots, S_n^0, 0)$, where $S_i^0 = \frac{\Lambda_i}{d_i^S}$, and an endemic equilibrium $P^* = (S_1^*, I_1^*, \dots, S_n^*, I_n^*)$ satisfies $S_i^*, I_i^* > 0$ and

$$\Lambda_i = \sum_{j=1}^n \beta_{ij} S_i^* I_j^* + d_i^S S_i^*, \quad \sum_{j=1}^n \beta_{ij} S_i^* I_j^* = (d_i^I + \gamma_i) I_i^*.$$

Using the method in [11], it can be verified that the basic reproduction number $R_0 = \rho(M_0)$ for (8.1) is the spectral radius of the matrix

$$M_0 = \left(\frac{\beta_{ij} S_i^0}{d_i^l + \gamma_i} \right)_{n \times n}.$$

The following result is standard and its proof is omitted.

Proposition 8.1. Assume that $B = (\beta_{ij})$ is irreducible.

- (1) If $R_0 \leq 1$, then P_0 is the unique equilibrium for system (8.1) and it is globally asymptotically stable in Θ .
- (2) If $R_0 > 1$, then P_0 is unstable and there exists a unique endemic equilibrium P^* for system (8.1).

Our main result in this section is given in the following.

Theorem 8.2. Assume that $B = (\beta_{ij})$ is irreducible. If $R_0 > 1$, then (8.1) has a unique endemic equilibrium P^* , and P^* is globally asymptotically stable in $\dot{\Theta}$.

Proof. The case $n = 1$ is proved in [29]. We consider the case $n \geq 2$. Let $P^* = (S_1^*, I_1^*, \dots, S_n^*, I_n^*)$, where $S_i^*, I_i^* > 0$ for $1 \leq i \leq n$, denote the unique endemic equilibrium of system (8.1). Let V_i be defined in (8.3). We show that V_i satisfies the assumptions of Theorem 3.1. Notice that

$$\int_0^{\tau_j} \frac{\partial}{\partial t} \left(I_j(t-r) - I_j^* - I_j^* \ln \frac{I_j(t-r)}{I_j^*} \right) dr = - \int_0^{\tau_j} \frac{\partial}{\partial r} \left(I_j(t-r) - I_j^* - I_j^* \ln \frac{I_j(t-r)}{I_j^*} \right) dr.$$

Similar steps as in (7.9) lead to

$$\dot{V}_i = - \frac{d_i^S}{S_i} (S_i - S_i^*)^2 + \sum_{j=1}^n \beta_{ij} S_i^* I_j^* \left(2 - \frac{S_i^*}{S_i} - \frac{I_i}{I_i^*} - \frac{S_i I_j(t - \tau_j) I_i^*}{S_i^* I_j^* I_i} + \frac{I_j}{I_j^*} + \ln \frac{I_j(t - \tau_j)}{I_j} \right).$$

Let $a_{ij} = \beta_{ij} S_i^* I_j^*$, $G_i(I_i) = -\frac{I_i}{I_i^*} + \ln \frac{I_i}{I_i^*}$, $\Phi(a) = 1 - a + \ln a$, and

$$F_{ij}(S_i, I_i, I_j(\cdot)) = 2 - \frac{S_i^*}{S_i} - \frac{I_i}{I_i^*} - \frac{S_i I_j(t - \tau_j) I_i^*}{S_i^* I_j^* I_i} + \frac{I_j}{I_j^*} + \ln \frac{I_j(t - \tau_j)}{I_j}.$$

Then $\dot{V}_i \leq \sum_{ij} a_{ij} F_{ij}(S_i, I_i, I_j(\cdot))$, and

$$F_{ij} = G_i(I_i) - G_j(I_j) + \Phi\left(\frac{S_i^*}{S_i}\right) + \Phi\left(\frac{S_i I_j(t - \tau_j) I_i^*}{S_i^* I_j^* I_i}\right) \leq G_i(I_i) - G_j(I_j).$$

Therefore, V_i, F_{ij}, G_i, a_{ij} satisfy the assumptions of Corollary 3.3, and the functional $V = \sum_{i=1}^n c_i V_i(S_i, I_i(\cdot))$ as defined in Theorem 3.1 is a Lyapunov functional for (8.1), namely, $\dot{V} \leq 0$ for all $(S_1, I_1(\cdot), \dots, S_n, I_n(\cdot)) \in \dot{\Theta}$. Using a similar argument as in Section 4, we can show that the only compact invariant set where $\dot{V} = 0$ is singleton $\{P^*\}$. By the LaSalle–Lyapunov Theorem for delayed systems (see [24, Theorem 3.4.7] or [16, Theorem 5.3.1]), we conclude that P^* is globally attractive in $\dot{\Theta}$ if $R_0 > 1$. Furthermore, it can be verified that P^* is locally stable using the same proof as that for Corollary 5.3.1 in [16]. This completes the proof of Theorem 8.2. \square

Remarks.

1. When $\tau_i = 0, i = 1, 2, \dots, n$, system (8.1) becomes the standard multi-group SIR model without delays studied in [14]. Theorem 8.2 generalizes Theorem 3.3 in [14].
2. When $n = 1$, Theorem 8.2 gives a global-stability result of McCluskey [29] for a single-group SIR model with delay.

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