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# An extremal problem on the potentially $P_k$-graphic sequences ☆

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**Abstract**

A simple graph $G$ is said to have property $P_k$ if it contains a complete subgraph of order $k+1$ as its subgraph. A nonincreasing sequence $\pi$ of $n$ nonnegative integers is potentially $P_k$-graphic if it is the degree sequence of a graph of order $n$ with property $P_k$. The degree sum of a graphic sequence $\pi$ is denoted by $\sigma(\pi)$. Moreover we denote by $\sigma(k,n)$ the smallest degree sum such that every positive graphic sequence $\pi$ with $\sigma(\pi)\geqslant\sigma(k,n)$ is potentially $P_k$-graphic. Erdős et al. (Graph Theory, Combinatorics & Applications, Wiley, New York, 1991, pp. 439–449) conjectured $\sigma(k,n)=(k-1)(2n-k)+2$. In this paper, we determine the values of $\sigma(k,n)$ for $k+1\leqslant n\leqslant 2k+1$. We also prove that $\sigma(3,n)=4n-4$ for $n\geqslant 8$. In other words, the Erdős–Jacobson–Lehel conjecture is true for $k=3$. © 2000 Elsevier Science B.V. All rights reserved.

## 1. Introduction

For a nonincreasing sequence $\pi=(d_1,d_2,\ldots,d_n)$ of nonnegative integers, $d_i\leqslant n-1$, $i=1,2,\ldots,n$, let $f=\max\{i\colon d_i\geqslant i\}$ and $\sigma(\pi)=\sum_{i=1}^n d_i$. $f$ and $\sigma(\pi)$ are called the trace and the degree sum of $\pi$, respectively. Define an $n\times n$ matrix $\bar{A}=(a_{ij})$ as follows: for $i\geqslant f+1$,

$$a_{ij}=\begin{cases}1 & \text{if } 1\leqslant j\leqslant d_i,\\ 0 & \text{otherwise}\end{cases}$$

and for $1\leqslant i\leqslant f$,

$$a_{ij}=\begin{cases}1 & \text{if } 1\leqslant j\leqslant d_i+1 \text{ and } j\neq i,\\ 0 & \text{otherwise.}\end{cases}$$

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Then the matrix $\bar{A}$ is called the off-diagonal left-most matrix of $\pi$. Clearly, the row and column sum vectors of $\bar{A}$ are $\pi$ and $\bar{\pi}$, the corrected conjugate vector of $\pi$, respectively. A nonincreasing sequence $\pi$ is graphic if it is the degree sequence of a simple graph $G$ and $G$ is a realization of $\pi$. The following is a criteria for a nonincreasing sequence being graphic.

**Theorem 1.1** (Berge [1]). *A nonincreasing sequence $\pi=(d_1,d_2,\ldots,d_n)$ of nonnegative integers with even sum $\sigma(\pi)$ and $d_i \leqslant n-1$, $i=1,\ldots,n$, is graphic if and only if for each $t=1,2,\ldots,f$,*

$$\sum_{i=1}^{t} d_i \leqslant \sum_{i=1}^{t} \bar{d}_i.$$

Let $\pi=(d_1,d_2,\ldots,d_n)$ be a nonincreasing sequence of nonnegative integers. Denote

$$\pi' = \begin{cases} (d_1-1,\ldots,d_{k-1}-1,d_{k+1}-1,\ldots,d_{d_k+1}-1,d_{d_k+2},\ldots,d_n) & \text{if } d_k \geqslant k, \\ (d_1-1,\ldots,d_{d_k}-1,d_{d_k+1},\ldots,d_{k-1},d_{k+1},\ldots,d_n) & \text{if } d_k \leqslant k-1. \end{cases}$$

Then $\pi'$ is the residual sequence obtained by laying off $d_k$ from $\pi$. The following is a famous theorem due to Kleitman and Wang [3].

**Theorem 1.2.** *A sequence $\pi$ is graphic if and only if $\pi'$ is graphic.*

According to Erdős et al. [2], a graph $G$ is said to have property $P_k$ if $G$ contains a complete subgraph of order $k+1$ as its subgraph and a sequence $\pi$ is said to be potentially $P_k$-graphic if $\pi$ has a realization $G$ with property $P_k$. Moreover, Rao [4] introduced the definition as follows: Let $G$ be a graph with vertex set $V(G)=\{v_1,v_2,\ldots,v_n\}$ and let $\pi=(d_1,d_2,\ldots,d_n\}$ be the degree sequence of $G$, where $d_i$ is the degree of $v_i$. Then $G$ is said to have property $A_k$ if the subgraph induced by $\{v_1,v_2,\ldots,v_{k+1}\}$ is complete. A sequence $\pi$ is said to be potentially $A_k$-graphic if it has a realization having property $A_k$. Rao [4] proved the following

**Theorem 1.3.** *A sequence $\pi$ is potentially $P_k$-graphic if and only if it is potentially $A_k$-graphic.*

In [5], Rao gave a criteria for a sequence $\pi$ to be potentially $A_k$-graphic. The following is his result.

**Theorem 1.4.** *Let $\pi=(d_1,d_2,\ldots,d_n)$ be a sequence of nonnegative integers in which $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_{k+1}$ and $d_{k+2} \geqslant d_{k+3} \geqslant \cdots \geqslant d_n$. Then $\pi$ is potentially $A_k$-graphic if and only if the following conditions hold*:

(1) $d_{k+1} \geqslant k$ *and* $d_n \geqslant 0$:

(2) $\sigma(\pi)$ *is even*;

(3) *for any* $s$ *and* $t$, $0 \leqslant s \leqslant k+1$, $0 \leqslant t \leqslant n-k-1$,

$$\sum_{i=1}^{s} d_i + \sum_{j=1}^{t} d_{k+1+j} \leqslant (s+t)(s+t-1) + \sum_{i=s+1}^{k+1} \min\{s+t, d_i - k + s\} + \sum_{j=k+2+t}^{n} \min\{s+t, d_j\}.$$

Let us denote by $\sigma(k,n)$ the smallest integer such that every graphic sequence $\pi = (d_1, d_2, \ldots, d_n)$ with $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n \geqslant 1$ and $\sigma(\pi) \geqslant \sigma(k,n)$ is potentially $P_k$-graphic. In [3], Erdős et al. proposed a problem: Determine the value $\sigma(k,n)$. Using an example, they pointed out that $\sigma(k,n) \geqslant (k-1)(2n-k)+2$ and conjectured that $\sigma(k,n) = (k-1)(2n-k)+2$. They also proved that the conjecture is true for $k=2$. Their result is the following

**Theorem 1.5.** *If* $n \geqslant 6$ *and the graphic sequence* $\pi = (d_1, d_2, \ldots, d_n)$ *satisfies* $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n \geqslant 1$ *and* $\sigma(\pi) \geqslant 2n$, *then* $\pi$ *is potentially* $P_2$-*graphic. In other words,* $\sigma(2,n) = 2n$ *for* $n \geqslant 6$.

The purpose of this paper is to determine the values of $\sigma(k,n)$ for $k+1 \leqslant n \leqslant 2k+1$ and prove that the Erdős–Jacobson–Lehel conjecture is true for $k=3$.

## 2. The value $\sigma(k,n)$ for small $n$

We first determine the values of $\sigma(k,n)$ for $k+1 \leqslant n \leqslant 2k$.

**Theorem 2.1.** *If* $k+1 \leqslant n \leqslant 2k$, *then*

$$\sigma(k,n) = (k-1)(2n-k) + (n-k)(n-k-1) + 2.$$

**Proof.** Suppose that $\pi = (d_1, d_2, \ldots, d_n)$ is a graphic sequence with $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n \geqslant 1$ and $\sigma(\pi) = (k-1)(2n-k) + (n-k)(n-k-1)$. Moreover, suppose that $G$ is a realization of $\pi$ and $e(G)$ is the size of $G$. Then $2e(G) = \sigma(\pi)$ and $2\,e(G^C) = n(n-1) - \sigma(\pi) = 2(n-k)$, where $G^C$ is the complementary graph of $G$. An extremal case is: $G$ is obtained by deleting a matching with $n-k$ edges from the complete graph $K_n$ of order $n$. Hence the largest vertex number of independent sets in $G^C$ is $k$. In other words the largest possible clique number of $G$ is $k$. Clearly, $1 \leqslant n-k \leqslant k$. Hence, there is no complete subgraph of order $k+1$ in $G$. Thus $\sigma(k,n) \geqslant (k-1)(2n-k) + (n-k)(n-k-1) + 2$.

Now suppose that $\pi = (d_1, d_2, \ldots, d_n)$ is graphic, where $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n \geqslant 1$ and $\sigma(\pi) \geqslant (k-1)(2n-k) + (n-k)(n-k-1) + 2$. Then every realization $G$ of $\pi$ is obtained by removing at most $n-k-1$ edges from $K_n$. Hence, the maximal clique of $G$ has order at least $n-(n-k-1) = k+1$. Thus $G$ has the property $P_k$. In other words, $\sigma(k,n) \leqslant (k-1)(2n-k) + (n-k)(n-k-1) + 2$. □

**Theorem 2.2.** *If* $n = 2k + 1$, *then* $\sigma(k,n) = (k-1)(2n-k) + k(k-1) + 2$.

**Proof.** Take $\pi = ((n-3)^n)$, i.e., $\pi$ consists of $n$ integers $n-3$. Clearly, $\pi$ is graphic and $\sigma(\pi) = n(n-3) = (k-1)(2n-k) + k(k-1)$. Let $G$ is a realization of $\pi$. Then the degree sequence of the complementary graph $G^C$ of $G$ is $\pi^C = (2^n)$. Clearly, $G^C$ is the union of disjoint cycles. Since $n$ is odd, $G^C$ has at least an odd cycle. Hence, any independent vertex set of $G^C$ has at most $k$ vertices. In other words, the clique in $G$ has order at most $k$. Thus $\pi$ is not potentially $P_k$-graphic.

Now suppose that $\pi$ is graphic and $\sigma(\pi) \geqslant (k-1)(2n-k) + k(k-1) + 2$. Then any realization $G$ of $\pi$ has size $e(G) = \frac{1}{2}\sigma(\pi) \geqslant \frac{1}{2}\{(k-1)(2n-k) + k(k-1) + 2\}$. Since $n(n-1) - \{(k-1)(2n-k) + k(k-1) + 2\} = 2(n-1)$, $G$ is obtained by deleting at most $n-1$ edges from $K_n$. An extremal case is: these edges form a path of length $n-1$. Thus $G$ contains a complete subgraph of order at least $k+1$. This shows that $\pi$ is potenially $P_k$-graphic. □

## 3. The value $\sigma(3, n)$

Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphic sequence, $n \geqslant 8$. For any $s$ and $t$, $0 \leqslant s \leqslant 4$ and $0 \leqslant t \leqslant n-4$, let us denote

$$L(s,t) = \sum_{i=1}^{s} d_i + \sum_{j=1}^{t} d_{4+j},$$

$$R(s,t) = (s+t)(s+t-1) + \sum_{i=s+1}^{4} \min\{s+t, d_i - 3 + s\} + \sum_{j=5+t}^{n} \min\{s+t, d_j\}.$$

We first prove the following

**Theorem 3.1.** *Let* $\pi = (d_1, d_2, \ldots, d_n)$ *be a graphic sequence*, $n \geqslant 8$. *If* $d_1 \geqslant 4$ *and* $n-2 \geqslant d_1 \geqslant d_2 \geqslant d_3 = d_4 = \cdots = d_{d_1+2} \geqslant d_{d_1+3} \geqslant \cdots \geqslant d_n \geqslant 3$, *then* $\pi$ *is potentially* $P_3$-*graphic*.

**Proof.** By Theorem 1.4, we only need to verify that for any $s$ and $t$, $0 \leqslant s \leqslant 4$ and $0 \leqslant t \leqslant n-4$,

$$L(s,t) \leqslant R(s,t). \tag{1}$$

Clearly, $L(s,t) \leqslant (s+t)d_1$ and $R(s,t) \geqslant (s+t)(s+t-1)$. If $d_1 \leqslant s+t-1$, then (1) holds. Hence, we may assume that $d_1 \geqslant s+t$. Then $t \leqslant d_1 - s$ and $0 \leqslant s \leqslant 4$. We consider the cases as follows.

*Case* 1: $3 \leqslant s \leqslant 4$. In this case, $t \leqslant d_1 - s \leqslant d_1 - 3$. Hence, $3 \leqslant 5 + t \leqslant d_1 + 2 \leqslant n$ and $d_3 = d_{5+t} = d_{d_1+2}$. If $d_3 \geqslant s+t$, then $d_3 - 3 + s \geqslant (s+t) + (s-3) \geqslant s+t$.

Hence,

$$\begin{aligned} R(s,t) &\geqslant (s+t)(s+t-1) + \sum_{i=s+1}^{4} \min\{s+t, d_i - 3 + s\} + \sum_{j=5+t}^{d_1+2} \min\{s+t, d_j\} \\ &= (s+t)(s+t-1) + (4-s)(s+t) + (d_1 + 2 - 4 - t)(s+t) \\ &= (s+t)(d_1+1) > (s+t)d_1 \geqslant L(s,t). \end{aligned}$$

If $d_3 \leqslant s+t-1$, then $3 \leqslant d_3 \leqslant d_3 - 3 + s \leqslant (s+t) - (4-s) \leqslant s+t$. Moreover, $d_n \geqslant 3$. Hence,

$$\begin{aligned} R(s,t) &\geqslant (s+t)(s+t-1) + 3(4-s+n-4-t) \\ &= (s+t)(s+t-1) + 3(n-s-t) \\ &\geqslant 2(n-2) + (s+t-1)(s+t-2) \\ &\geqslant d_1 + d_2 + (s-2)d_3 + td_3 \geqslant \sum_{i=1}^{s} d_i + \sum_{j=1}^{t} d_{4+j} = L(s,t). \end{aligned}$$

Thus (1) holds for $s+t \leqslant d_1$ and $3 \leqslant s \leqslant 4$.

*Case* 2: $0 \leqslant s \leqslant 2$. In this case we have $t \leqslant d_1 - s \leqslant d_1$. When $t = d_1$, we have $s = 0$ and

$$\begin{aligned} R(0,d_1) &\geqslant d_1(d_1-1) + \sum_{i=1}^{4} (d_i - 3) \\ &\geqslant d_1(d_1-3) + 3d_3 + 3(d_1-4) \\ &\geqslant d_3(d_1-3) + 3d_3 = d_1 d_3 \geqslant \sum_{j=1}^{d_1} d_{4+j} = L(0,d_1). \end{aligned}$$

Hence, (1) holds for $t = d_1$. When $t = d_1 - 1$, we have $0 \leqslant s \leqslant 1$, and $d_1 \leqslant n-3$ since $d_1 - 1 = t \leqslant \min\{d_1 - s, n-4\}$. If $s = 0$, then

$$\begin{aligned} R(0,d_1-1) &\geqslant (d_1-1)(d_1-2) + \sum_{i=1}^{4} (d_i - 3) + 3(n - d_1 - 3) \\ &= d_1(d_1-4) + 3d_3 + 2(n-8) + (n-d_1-3) \\ &\geqslant (d_1-4)d_3 + 3d_3 = (d_1-1)d_3 \geqslant L(0,d_1-1). \end{aligned}$$

If $s = 1$, then

$$\begin{aligned} R(1,d_1-1) &\geqslant d_1(d_1-1) + \sum_{i=2}^{4} (d_i - 2) \\ &\geqslant d_1(d_1-1) + 3(d_3-2) \\ &= d_1 + (d_1-1)d_3 + (d_1-d_3)(d_1-2) + 2(d_3-6) \\ &\geqslant d_1 + (d_1-1)d_3 \geqslant L(1,d_1-1). \end{aligned}$$

Hence, (1) holds for $t = d_1 - 1$.

Now suppose $t \leqslant d_1 - 2$. Then $t + 4 \leqslant d_1 + 2$. Hence $d_3 = d_4 = \cdots = d_{t+4} = \cdots = d_{d_1+2} \geqslant d_{d_1+3} \geqslant \cdots \geqslant d_n \geqslant 3$. We consider the following subcases.

*Subcase* 2.1: $d_3 \leqslant s + t - 1$. From $d_i - 3 + s \geqslant s$ for $1 \leqslant i \leqslant 4$, we obtain

$$\begin{aligned} R(s,t) &\geqslant (s+t)(s+t-1) + (4-s)s + 3(n-t-4) \\ &\geqslant (s+t)(s+t-1) + (4-s)s + s(n-t-4) \\ &= s(s+t-1+4-s+n-t-4) + t(s+t-1) \\ &= s(n-1) + t(s+t-1) \geqslant \sum_{i=1}^{s} d_i + \sum_{j=1}^{t} d_{4+j} = L(s,t). \end{aligned}$$

*Subcase* 2.2: $d_3 \geqslant s + t$. If $d_3 \geqslant t + 3$, then $d_i - 3 + s \geqslant s + t$ for $1 \leqslant i \leqslant 4$. Hence,

$$\begin{aligned} R(s,t) &\geqslant (s+t)(s+t-1) + (4-s)(s+t) + (d_1+2-t-4)(s+t) \\ &= (s+t)(d_1+1) \geqslant L(s,t). \end{aligned}$$

Suppose that $s + t \leqslant d_3 \leqslant t + 2$ and $s \neq 0$. Then $d_3 - 3 + s \leqslant s + t - 1$. Denote $d_3 = t + m, 1 \leqslant s \leqslant m \leqslant 2$. Then $s + t = s + d_3 - m \geqslant 3 - (m - s)$. Also $d_n \geqslant 3 \geqslant 3 - (m - s)$. Hence,

$$\begin{aligned} R(s,t) &\geqslant (s+t)(s+t-1) + (4-s)(d_3-3+s) + (3-m+s)(n-t-4) \\ &= s(n-2) + td_3 + (s+t)(s+t-1) + (4-s)(d_3-3+s) \\ &\quad +(2-m)(n-t-4) + s(n-t-4) + (n-t-4) - s(n-2) - td_3 \\ &= s(n-2) + td_3 + (s+t)(s+t-1) + (4-s)(t+m-3+s) \\ &\quad +(2-m)(n-t-4) + s(n-2-t-2) \\ &\quad +(n-t-4) - t(t+m) - s(n-2) \\ &= s(n-2) + td_3 + (3-m)(n-8) + (4-m)s \\ &\geqslant s(n-2) + td_3 \geqslant L(s,t). \end{aligned}$$

Finally suppose $s = 0$ and $t \leqslant d_3 \leqslant t + 2$. Denote $d_3 = t + m$, $0 \leqslant m \leqslant 2$. Since $t = d_3 - m \geqslant 3 - m$, $d_n \geqslant 3 \geqslant 3 - m$, and $d_3 - 3 = t - (3 - m) \leqslant t$, we have

$$\begin{aligned} R(0,t) &\geqslant t(t-1) + 4(t-3+m) + (3-m)(n-t-4) \\ &= t(t+m) + (3-m)(n-8) \geqslant td_3 \geqslant L(0,t). \end{aligned}$$

Thus (1) holds for $t \leqslant d_1 - 2$. □

**Theorem 3.2.** $\sigma(3,8) = 28$.

**Proof.** As Erdős et al. pointed out that $\sigma(3,8) \geqslant (3-1)(16-3)+2 = 28$, we only need to prove that, if $\pi = (d_1, d_2, \ldots, d_n)$ is graphic, $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_8 \geqslant 1$ and $\sigma(\pi) \geqslant 28$, then $\pi$ is potentially $P_3$-graphic.

First we have $d_1 \geqslant 4$ since $8d_1 \geqslant \sigma(\pi) \geqslant 28$. Next, if $d_4 \leqslant 2$, then by the off-diagonal left-most matrix $\bar{A}$ of $\pi$, we have $\bar{d_3} \leqslant 2$. Clearly $\bar{d_1} = 7$, $\bar{d_2} \leqslant 7$, hence $\bar{d_1} + \bar{d_2} + \bar{d_3} \leqslant 16$. On the other hand, $28 \leqslant \sigma(\pi) \leqslant d_1 + d_2 + d_3 + 5 \times 2$. Hence, $d_1 + d_2 + d_3 \geqslant 18 > 16 \geqslant \bar{d_1} + \bar{d_2} + \bar{d_3}$. By Theorem 1.1, $\pi$ is not graphic, a contradiction. Thus $d_4 \geqslant 3$. Moreover, if $d_6 = 1$, then by the off-diagonal left-most matrix $\bar{A}$ of $\pi$, we have $\bar{d_1} = 7$ and $\bar{d_i} \leqslant 4$ for $2 \leqslant i \leqslant 5$. Hence, $\bar{d_1} + \bar{d_2} + \cdots + \bar{d_5} \leqslant 23$. But $28 \leqslant \sigma(\pi) = d_1 + d_2 + \cdots + d_5 + 3$. Hence, $d_1 + d_2 + \cdots + d_5 \geqslant 25 > 23 \geqslant \bar{d_1} + \bar{d_2} + \cdots + \bar{d_5}$. By Theorem 1.1, $\pi$ is not graphic, a contradiction. Therefore, $d_6 \geqslant 2$.

Let $\pi' = (d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_8)$ be the residual sequence obtained by laying off $d_1$ from $\pi$. If $\pi'$ contains zero terms, then $6 \leqslant d_1 \leqslant 7$ and $\pi'$ has at least 5 non-zero terms since $d_6 \geqslant 2$. If $\pi'$ has at least 6 non-zero terms, then from Theorem 1.5, we have $\sigma(2,6) < \sigma(2,7) = 14$. Notice that $\sigma(\pi') = \sigma(\pi) - 2d_1 \geqslant 28 - 14 = 14 = \sigma(2,7) > \sigma(2,6)$. By Theorem 1.5, $\pi'$ is potentially $P_2$-graphic, thereby $\pi$ is potentially $P_3$-graphic. If $\pi'$ contains exactly 5 non-zero terms, then $d_1 = 7$ and $d_6 > d_7 = d_8 = 1$. From the off-diagonal left-most matrix of $\pi$, we get $4 \leqslant d_2 \leqslant 5$. If $d_2 = 5$, then laying off $d_2$ from $\pi$ we know $\pi$ is potentially $P_3$-graphic. Assume $d_2 = 4$. Then $\pi$ must be $(7^1, 4^4, 3^1, 1^2)$. It is easy to check that $\pi = (7^1, 4^4, 3^1, 1^2)$ is potentially $P_3$-graphic.

Now suppose that $\pi'$ is positive. Then $\sigma(\pi') = \sigma(\pi) - 2d_1 \geqslant 14 = \sigma(2,7)$. From Theorems 1.5 and 1.3, $\pi'$ is potentially $A_2$-graphic. If there exists an integer $t$, $4 \leqslant t \leqslant d_1 + 1$ such that $d_t > d_{t+1}$, then $\pi$ is potentially $A_3$-graphic. Hence we may assume that

$$d_1 \geqslant d_2 \geqslant d_3 \geqslant d_4 = \cdots = d_{d_1+2} \geqslant d_{d_1+3} \geqslant \cdots \geqslant d_8.$$

If $d_3 > d_4$, then the residual sequence obtained by laying off $d_{d+2} = l$ from $\pi$ is $\pi'' = (d_1 - 1, d_2 - 1, \ldots, d_l - 1, d_{l+1}, \ldots, d_{d_1+1}, d_{d_1+3}, \ldots, d_8)$ and $\sigma(\pi'') = \sigma(\pi) - 2l \geqslant 14 = \sigma(2,7)$. By Theorems 1.5 and 1.3, $\pi''$ is potentially $A_2$-graphic, thereby $\pi$ is potentially $A_3$-graphic. Hence we may assume that

$$d_1 \geqslant d_2 \geqslant d_3 = d_4 = \cdots = d_{d_1+2} \geqslant d_{d_1+3} \geqslant \cdots \geqslant d_8.$$

If $d_8 \geqslant 3$, then by Theorem 3.1, $\pi$ is potentially $P_3$-graphic. Hence assume $1 \leqslant d_8 \leqslant 2$. If $6 \leqslant d_1 \leqslant 7$, then $d_4 = \cdots = d_8 \geqslant 3$, a contradicition. Therefore $4 \leqslant d_1 \leqslant 5$. If $d_8 = 2$, then $28 \leqslant \sigma(\pi) \leqslant 2d_1 + 5d_3 + d_8 \leqslant 12 + 5d_3$. Hence $d_3 \geqslant 4$. If $d_3 = 5$, then $d_1 = \cdots = d_3 = \cdots = d_7 = 5$. But $\pi = (5^7, 2^1)$ is not graphic. Hence $d_3 = 4$. It is easy to determine that $\pi$ is one of $(5^2, 4^5, 2^1), (4^7, 2^1)$ and $(4^6, 2^2)$. Fig. 1 shows that these sequences are potentially $P_3$-graphic. If $d_8 = 1$, then $28 \leqslant \sigma(\pi) \leqslant 2d_1 + 5d_3 + d_8 \leqslant 11 + 5d_3$. Hence, $d_3 \geqslant 4$. If $d_3 = 5$, then $\pi = (5^7, 1^1)$. If $d_3 = 4$, then $\pi$ is one of $(5^1, 4^6, 1^1)$ and $(4^6, 3^1, 1^1)$. It is easy to see that $\pi$ is potentially $p_3$-graphic (Fig. 2). □

**Theorem 3.3.** *If* $n \geqslant 8$, *then* $\sigma(3,n) = 4n - 4$.

**Proof.** It is enough to prove that, if the graphic sequence $\pi = (d_1, d_2, \ldots, d_n)$ satisfies $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n \geqslant 1$ and $\sigma(\pi) \geqslant 4n - 4$, then $\pi$ is potentially $P_3$-graphic. For this

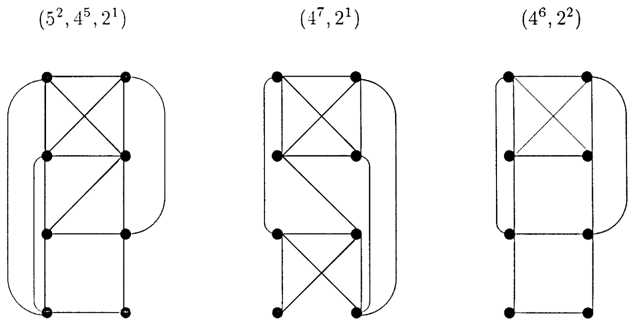


Fig. 1.

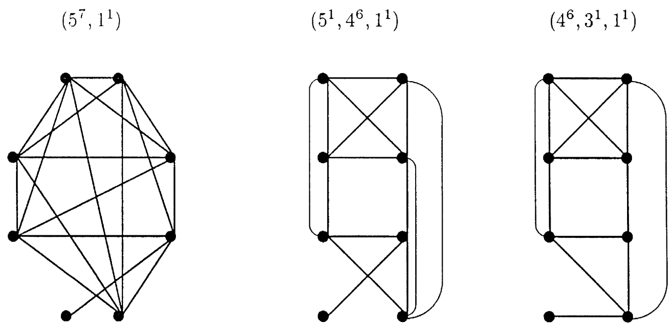


Fig. 2.

reason, we use induction on $n$. By Theorem 3.2, the conclusion holds for $n = 8$. Now suppose $n > 8$. Since $4n - 4 \leqslant \sigma(\pi) \leqslant nd_1$, we have $4 \leqslant d_1 \leqslant n - 1$. If $d_n = 1$, then $\pi' = (d_1 - 1, d_2, \ldots, d_{n-1})$ is graphic and $\sigma(\pi') \geqslant 4(n-1) - 4$. By induction hypothesis, $\pi'$ is potentially $P_3$-graphic. Hence, $\pi$ is potentially $P_3$-graphic. Similarly, $\pi$ is also potentially $P_3$-graphic if $d_n = 2$. Hence, we assume $d_n \geqslant 3$. If $d_1 = n - 1$, then $\pi' = (d_2 - 1, d_3 - 1, \ldots, d_n - 1)$ is graphic and $d_2 - 1 \geqslant d_3 - 1 \geqslant \cdots \geqslant d_n - 1 \geqslant 2$ and $\sigma(\pi') \geqslant 2(n-1)$, $n > 8$. By Theorem 1.5, $\pi'$ is potentially $P_2$-graphic. Hence, $\pi$ is potentially $P_3$-graphic. So we may assume $d_1 \leqslant n-2$. Then the residual sequence after laying off $d_1$ from $\pi$ is $\pi' = (d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$. Clearly $\sigma(\pi') = \sigma(\pi) - 2d_1 \geqslant 2(n-1) = \sigma(2, n-1)$. If there exists an integer $t$, $4 \leqslant t \leqslant d_1 + 1$, such that $d_t > d_{t+1}$, then $\pi$ is potentially $P_3$-graphic. Hence, we assume $n-2 \geqslant d_1 \geqslant d_2 \geqslant d_3 \geqslant d_4 = \cdots = d_{d_1+2} \geqslant d_{d_1+3} \geqslant \cdots \geqslant d_n \geqslant 3$. If $d_3 > d_4$, then the residual sequence after laying off $d_{d_1+2} = l$ from $\pi$ is $\pi' = (d_1 - 1, d_2 - 1, \ldots, d_l - 1, d_{l+1}, \ldots, d_{d_1+1}, d_{d_1+3}, \ldots, d_n)$. Since $\sigma(\pi') = \sigma(\pi) - 2l \geqslant 2(n-1) = \sigma(2, n-1), \pi'$ is potentially $A_2$-graphic. Hence, $\pi$ is potentially $A_3$-graphic. So we may further assume that

$$n - 2 \geqslant d_1 \geqslant d_2 \geqslant d_3 = d_4 = \cdots = d_{d_1+2} \geqslant d_{d_1+3} \geqslant \cdots \geqslant d_n \geqslant 3.$$

By Theorem 3.1, $\pi$ is potentially $P_3$-graphic. □

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