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Chromatic Index Critical Graphs of Even Order with Five Major Vertices

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Abstract. We prove that there does not exist any chromatic index critical graph of even order with exactly five vertices of maximum degree. This extends an earlier result of Chetwynd and Hilton who proved the same with five replaced by four or three.

1. Introduction

Throughout this article, all graphs we deal with are finite, simple, and undirected. We use $V(G)$, $|G|$, $E(G)$, $e(G)$, $\Delta(G)$, and $\delta(G)$ to denote respectively the vertex set, order, edge set, size, maximum degree, and minimum degree of a graph G. We also use K_n, $\overline{G}$ and $G \cup H$ to denote respectively a complete graph of order n, the complement of a graph G, and the union of two vertex-disjoint graphs G and H. For $x, y \in V(G)$, we write $xy \in E(G)$ if x and y are adjacent in G. The *join* $G + H$ of two vertex-disjoint graphs G and H is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy | x \in V(G), y \in V(H)\}$. If $x \in V(G)$, we use $N_G(x)$ (or simply $N(x)$) to denote the neighborhood of x and $d_G(x)$ or simply $d(x)$) the degree of x. Vertices of maximum degree in G are called *major vertices* and others are called *minor vertices*. If a graph G has n_j vertices of degree i_j, where $j = 1, \cdots, \Delta(= \Delta(G))$, then the degree sequence of G is denoted by $i_1^{n_1} i_2^{n_2} \cdots i_\Delta^{n_\Delta}$. If $A \subseteq V(G)$, we use $G - A$ (or simply $G - x$ if $A = \{x\}$) to denote the subgraph obtained from G by deleting the set of vertices A together with their incident edges, and use $G[A]$ (or simply $G[x_1, x_2, \cdots, x_k]$ if $A = \{x_1, x_2, \cdots, x_k\}$) to denote the subgraph of G induced by A. If A and B are disjoint subsets of $V(G)$, we use $e_G(A, B)$ (or simply $e_G(x, B)$ if $A = \{x\}$) to denote the number of edges between A

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and B in G. If $F \subseteq E(G)$, we use $G - F$ to denote the subgraph obtained by deleting F from G.

An *edge coloring* of a graph G is a map $\pi : E(G) \to C$, where C is a set of colors, such that no two adjacent edges receive the same color. The *chromatic index* $\chi'(G)$ of G is the least value of $|C|$ for which an edge coloring $\pi : E(G) \to C$ exists. A well-known theorem of Vizing [12] states that, for any simple graph G, $\Delta(G) \le \chi'(G) \le \Delta(G) + 1$. A graph G is said to be of *class i, i=1,2* if $\chi'(G) = \Delta(G) + i - 1$.

The *core* G_Δ of a graph G is the subgraph of G induced by the set of major vertices of G. For a vertex v of G, we use $d_\Delta(v)$ to denote the number of major vertices of G adjacent to v. If G is a connected class 2 graph with $\Delta(G) = \Delta$ and $\chi'(G - e) < \chi'(G)$ for each edge $e \in E(G)$, then G is said to be Δ-*critical.*

Δ-critical graphs of even order are extremely rare. Beineke and Fiorini [1] proved that there are no Δ-critical graphs of even order at most 10. The nonexistence of Δ-critical graphs of order 12 and 14 were established in [3, 4], where their proofs are computer-assisted. For other partial results regarding the non-existence of Δ-critical graphs of even order, see [2, 9, 13].

A graph G is called *overfull* if $e(G) > \Delta(G)\lfloor\frac{|G|}{2}\rfloor$. By studying the number of major vertices in a graph, Chetwynd and Hilton [6] proved that there does not exist any Δ-critical graph of even order with three or four major vertices, and any Δ-critical graph of odd order with at most four major vertices is overfull (for shorter/alternative proofs, see [14]). In this paper, we prove that there does not exist any Δ-critical graph of even order with exactly five major vertices, which extends the result of Chetwynd and Hilton. Our result is used in [10] to prove that any Δ-critical graph of odd order with five major vertices is overfull. This, together with the result of Chetwynd and Hilton confirms that the fairly long-standing Overfull Conjecture of Chetwynd and Hilton [8] is true for graphs with at most five major vertices.

2. Some Useful Lemmas

In this section we give a list of results, which we shall apply later. Proof of Lemma 2.1 can be found in [13], alternative and shorter proofs of Theorems 2.3, 2.4 and 2.5 can be found in [14].

Lemma 2.1 (Vizing's Adjacency Lemma [12]). *Suppose G is a* Δ*-critical graph and* $vw \in E(G)$, *where* $d(v) = k$. *Then*

(i) $d_\Delta(w) \ge \Delta - k + 1$ *if* $k < \Delta$;
(ii) $d_\Delta(w) \ge 2$ if $k = \Delta$;
(iii) $|G_\Delta| \ge \Delta - \delta(G) + 2$; *and*
(iv) $|G_\Delta| \ge 3$.

Theorem 2.2 (Beineke and Fiorini [1]; Broere and Mynhardt [5]). *If G is a* Δ*-critical graph of even order* $2n$ *with minimum degree* δ, *then*

(i) $\sum_{v \in V(G)}(\Delta(G) - d(v)) \geq 2(\Delta - \delta + 1)$;
(ii) $e(G) \leq (n-1)\Delta + \delta - 1$; *and*
(iii) G has at least three minor vertices.

Theorem 2.3 (Chetwynd and Hilton [7]). *Let G be a connected graph of order n with* $\Delta = \Delta(G) \geq 3$. *Suppose* $|G_\Delta| = 3$. *Then G is of class 2 if and only if the degree sequence of G is* $(n-2)^{n-3}(n-1)^3$ (*and thus n is odd*).

Theorem 2.4 (Chetwynd and Hilton [6]). *There does not exist any* Δ*-critical graph of even order with four major vertices.*

Theorem 2.5 (Chetwynd and Hilton [6]). *Let G be a* Δ*-critical graph such that* $|G| = 2n+1$ *and* $|G_\Delta| = 4$. *Then the degree sequence of G is either* $(2n-2)^{2n-3}(2n-1)^4$ *or* $(2n-2)(2n-1)^{2n-4}(2n)^4$. *In particular,* $e(G) = n\Delta + 1$.

The following theorem of Tutte [11] will be used in the proof of Lemma 2.7.

Theorem 2.6. *A graph G has a perfect matching if and only if* $q(G-S) \leq |S|$ *for all* $S \subset V(G)$, *where* $q(G-S)$ *denotes the number of odd components of* $G-S$.

Let J_s be a graph of order $s \geq 1$. Let G_s denote a connected spanning subgraph of $J_s + \overline{K_{s+2}}$ such that each vertex of $\overline{K_{s+2}}$ is adjacent to at least $s-1$ vertices of J_s. Let G'_s denote a connected spanning subgraph of $J_s + (\overline{K_{s+1}} \cup J_3)$ such that each vertex of $\overline{K_{s+1}}$ is joined to every vertex of J_s and each vertex of J_3 is adjacent to at least $s-2$ vertices of J_s.

Theorem 2.7. *A connected graph G of order 2n has a perfect matching if*

(i) $\delta(G) \geq n-1$, *except when G is isomorphic to* $J_{n-1} + \overline{K_{n+1}}$;
(ii) $\delta(G) = n-2$, *except when G is isomorphic to* G_{n-1}, G'_{n-2}, $J_{n-2} + \overline{K_{n+2}}$, *or* $\overline{K_{3,3,3}} + K_1$.

Proof. Suppose G does not have a perfect matching. Then by Tutte's theorem, there exists $S \subset V(G)$ such that $q(G-S) > |S| \geq 1$. Since $|G|$ is even, $q(G-S)$ and $|S|$ have the same parity. Thus $q(G-S) \geq s+2$, where $s = |S|$, and so

$$s + (s+2) \leq s + q(G-S) \leq 2n. \tag{1}$$

Consequently

$$n \geq s+1. \tag{2}$$

Let G_1 be an odd component of $G-S$ with minimum order among all the odd components of $G-S$. Then $|G_1| \leq \frac{2n-s}{s+2}$. It follows that for any $x \in V(G_1)$, $\delta(G) \leq d(x) \leq |G_1| - 1 + s \leq \frac{2n-s}{s+2} - 1 + s$. If $\delta(G) \geq n-1$, then $n-1 \leq \delta(G) \leq \frac{2n-s}{s+2} - 1 + s$. This, together with (2), implies that $n = s+1$ and $\delta = s$. Thus G is isomorphic to $J_{n-1} + \overline{K_{n+1}}$. So we may assume that $\delta(G) = n - 2$. If $n = s + 1$, then by (1), $q(G - S) = s + 2$. Note that $|G| = 2s + 2$ and $\delta(G) = n - 2 = s - 1$. Thus G is isomorphic to G_{n-1}. If $n = s + 2$, then (1) again implies that $q(G - S) = s + 4$ or $q(G - S) = s + 2$ (because s and $q(G - S)$ have the same parity). Since $|G| = 2s + 4$ and $\delta(G) = n - 2 = s$, it follows that G is isomorphic to $J_{n-2} + \overline{K_{n+2}}$, or G'_{n-2}. Hence, by (2), we may assume that

$$n \geq s + 3. \tag{3}$$

Suppose there exists $x \in V(G_1)$ such that $n - 2 < d(x)$ or $d(x) < \frac{2n-s}{s+2} - 1 + s$. Then

$$n - 1 \leq d(x) \leq \frac{2n - s}{s + 2} - 1 + s \tag{4}$$

or

$$n - 2 \leq d(x) < \frac{2n - s}{s + 2} - 1 + s \tag{5}$$

From (4), it follows that $n \leq s + 1$, which contradicts (3). From (5), it follows that

$$ns < s^2 + 2s + 2. \tag{6}$$

Clearly, (6) does not hold when $n \geq s + 4$ because $s \geq 1$. Hence by (3), $n = s + 3$. This, together with (6), implies that $s < 2$. Since $s \geq 1$, we have $s = 1$. Thus $n = s + 3 = 4$ and $\delta(G) \geq n - 2 = 2$. It follows that $q(G - S) \leq 1$, a contradiction.

So we may assume that for any $x \in V(G_1)$,

$$n - 2 = d(x) = \frac{2n - s}{s + 2} - 1 + s \tag{7}$$

However, from (7) we have $|G_1| = \frac{2n-s}{s+2}$ and

$$ns = s^2 + 2s + 2. \tag{8}$$

Now (3) and (8) yield that $s \leq 2$. If $s = 2$, then from (8) it follows that $n = 5$. Hence $|G_1| = \frac{2n-s}{s+2} = \frac{10-2}{2+2} = 2$, contrary to the fact that $|G_1|$ is odd. If $s = 1$, from (8) again, we have $n = 5$, $\delta(G) = n - 2 = 3$, and $|G_1| = \frac{2n-s}{s+2} = \frac{10-1}{2+1} = 3$. Thus G is isomorphic to $\overline{K_{3,3,3}} + K_1$. □

Lemma 2.8 *Let* $\Delta > 1$ *be an integer. Let* G *be a* Δ-critical graph with $|G| = 2n$ *and* $|G_\Delta| = 5$. *If* $\Delta \geq n$ *and* $\delta(G) \geq n - 2$, *then* G *has a perfect matching.*

Proof. Let $\delta = \delta(G)$. By Lemma 2.1(iii), we have $5 \geq \Delta - \delta + 2$. Thus $n \leq \Delta \leq \delta + 3$. Since $\delta \geq n - 2$, by Lemma 2.7, G has a perfect matching except when G is isomorphic to $J_{n-1} + \overline{K_{n+1}}$, $J_{n-2} + \overline{K_{n+2}}$, G_{n-1}, G'_{n-2}, or

$\overline{K_{3,3,3}} + K_1$. Note that $\overline{K_{3,3,3}} + K_1$ has only one major vertex. Thus G is not isomorphic to $\overline{K_{3,3,3}} + K_1$. Let A be the set of major vertices of G. Suppose G is isomorphic to $J_{n-1} + \overline{K_{n+1}}$, or $J_{n-2} + \overline{K_{n+2}}$. Then $A \subseteq V(J_{n-1})$ or $A \subseteq V(J_{n-2})$. By Lemma 2.1(i) and (ii), $\delta(G_\Delta) \geq 2$ and so $\Delta - \delta \geq 4$, contrary to the fact that $\Delta \leq \delta + 3$.

Suppose G is isomorphic to G_{n-1}. Then $\delta = n - 2$. Since $\Delta \geq n$, we have $A \subseteq V(J_{n-1})$. By Lemma 2.1(i) and (ii), $d_\Delta(v) \geq 2$ for any $v \in V(J_{n-1})$, and so

$$e(J_{n-1}) = \sum_{v \in V(J_{n-1}) - A} d_\Delta(v) + e(J_{n-1}[A]) \geq 2((n-1) - 5) + 5 = 2n - 7.$$

Thus by counting the number of edges between $V(J_{n-1})$ and $V(\overline{K_{n+1}})$ in two different ways, we obtain that

$$\begin{aligned}(n+1)(n-2) &\leq e(V(\overline{K_{n+1}}), V(J_{n-1})) \\ &\leq \sum_{v \in V(J_{n-1})} d(v) - 2e(J_{n-1}[A]) \\ &\leq ((n-1-5)(\Delta - 1) + 5\Delta) - 2(2n-7).\end{aligned}$$

This, together with $\Delta \leq \delta + 3 \leq (n-2) + 3 = n + 1$, implies that $n \leq 5$, which contradicts the fact that $n - 1 = |J_{n-1}| \geq |A| = 5$.

So we may assume that G is isomorphic to G'_{n-2}. Then $\delta = n - 2$. Thus $n \leq \Delta \leq \delta + 3 = n + 1$. It follows that $A \subseteq V(J_{n-2}) \cup V(J_3)$ and each vertex in $\overline{K_{n-1}}$ is adjacent to every vertex in J_{n-2}. If there exists a vertex $w \in V(J_3)$ such that $d_G(w) = \Delta$, then $\Delta = n$ and w is adjacent to every vertex in J_{n-2}. Thus there exists a vertex $w' \in V(J_{n-2}) \cap A$ such that w' is adjacent to at least two vertices of $V(J_3)$. Note that w' is adjacent to every vertex of $\overline{K_{n-1}}$. It follows that $\Delta = d_G(w') \geq (n-1) + 2 = n + 1$, a contradiction. So we may assume that $A \subseteq V(J_{n-2})$. By the same argument in the previous paragraph, we have $e(J_{n-2}) \geq 2((n-2) - 5) + 5 = 2n - 9$. Thus

$$\begin{aligned}(n-1+3)(n-2) - 6 &\leq e(V(\overline{K_{n-1}} \cup J_3), V(J_{n-2})) \\ &\leq \sum_{v \in V(J_{n-2})} d(v) - 2e(J_s[A]) \\ &\leq ((n-2-5)(\Delta - 1) + 5\Delta) - 2(2n-9).\end{aligned}$$

This, together with $\Delta \leq n + 1$, implies that $n \leq 5$, contrary to the fact that $n - 2 = |J_{n-2}| \geq |A| = 5$. □

Lemma 2.9 *Let $\Delta > 1$ be an integer. Let G be a Δ-critical graph with $|G| = 2n$ and $|G_\Delta| = 5$. Then G contains a perfect matching.*

Proof. Since G is a Δ-critical graph of even order, by Theorem 2.2(iii), G has at least three minor vertices. Thus $2n = |G| \geq 5 + 3 = 8$ and so $n \geq 4$. Let $\delta = \delta(G)$. By Lemma 2.1(iii), $5 \geq \Delta - \delta + 2$. Hence

$$\delta \geq \Delta - 3 \tag{9}$$

By Theorem 2.2(iii), $\delta \neq \Delta$. If $\delta = \Delta - 1$, then G has either five vertices of odd degree Δ or $2n - 5$ vertices of odd degree $\Delta - 1$, which contradicts the fact that the number of vertices of odd degree in any graph is always even. By (9), $\Delta - 3 \leq \delta \leq \Delta - 2$. We next show that $\Delta \geq n$ and $\delta \geq n - 2$.

Note that $2e(G_\Delta) = \sum_{v \in V(G_\Delta)} d_\Delta(v)$. Now by counting the number of edges joining $V(G_\Delta)$ and $V(G) - V(G_\Delta)$ in two different ways, we have

$$5\Delta - 2e(G_\Delta) = e_G(V(G_\Delta), V(G) - V(G_\Delta)) = \sum_{v \in V(G) - V(G_\Delta)} d_\Delta(v).$$

It follows that

$$5\Delta = \sum_{v \in V(G)} d_\Delta(v). \tag{10}$$

By Lemma 2.1(i) and (ii), we have $d_\Delta(v) \geq 2$ for any $v \in V(G)$. Let $x \in V(G)$ be such that $d(x) = \delta$. Suppose that $\delta = \Delta - 2$. By Lemma 2.1(i), $d_\Delta(w) \geq 3$ for any $w \in N(x)$. By (10),

$$3(\Delta - 2) + 2(2n - (\Delta - 2)) \leq \sum_{w \in N(x)} d_\Delta(w) + \sum_{v \in V(G) - N(x)} d_\Delta(v) = 5\Delta,$$

which yields that $\Delta \geq n$. Hence $\delta = \Delta - 2 \geq n - 2$.

Suppose that $\delta = \Delta - 3$. Since $|G_\Delta| = 5$ and the number of vertices of odd degree in G is even, G has at least one vertex (say y) of degree $\Delta - 2$. By (10),

$$\sum_{w \in N(x)} d_\Delta(w) + \sum_{u \in N(y) - N(x)} d_\Delta(u) + \sum_{v \in V(G) - N(x) \cup N(y)} d_\Delta(v) = 5\Delta. \tag{11}$$

By Lemma 2.1(i), $d_\Delta(w) \geq 4$ for any $w \in N(x)$, and $d_\Delta(u) \geq 3$ for any $u \in N(y)$. It follows from (11) that $4(\Delta - 3) + 3((\Delta - 2) - (\Delta - 3)) + 2(2n - (\Delta - 2)) \leq 5\Delta$, which yields that

$$3\Delta \geq 4n - 5 \tag{12}$$

Since $n \geq 4$, (12) implies that $\Delta \geq n$. Suppose $\Delta = n$. Then (12) again implies that $\Delta \leq 5$. Clearly, $\delta \geq 2$ and so $\Delta = 3 + \delta \geq 5$. Hence $\Delta = n = 5$, $d(x) = \delta = 2$, and $d(y) = 3$. By Lemma 2.1(i), $xy \notin E(G)$ for otherwise $d(x) > 3$, a contradiction. Note that the two neighbors of x, say a, b, are of degree $\Delta = 5$. By Lemma 2.1(i), $d_\Delta(a) = d_\Delta(b) = 4$. Hence $ya, yb \notin E(G)$, for otherwise $d(a) > 5$ or $d(b) > 5$, a contradiction. Thus $N(x) \cap N(y) = \emptyset$. Now (11) becomes $2 \times 4 + 3 \times 3 + 2 \times (10 - 5) \leq 5\Delta = 5 \times 5$, which is false. Therefore $\Delta \geq n + 1$ and $\delta = \Delta - 3 \geq n - 2$. This proves that in either case, we have $\Delta \geq n$ and $\delta \geq n - 2$. By Lemma 2.8, G has a perfect matching. □

3. Main Result

We are now ready to prove our main result.

Theorem 3.1. *Let $\Delta > 1$ be an integer. There does not exist any Δ-critical graph of even order with exactly five major vertices.*

Proof. Suppose such a Δ-critical graph G exists. Clearly, $\Delta \geq 3$, for otherwise the theorem holds. Assume that $2n = |G|$ is minimum among all graphs G which are Δ-critical with $|G_\Delta| = 5$ for some Δ, and let Δ be minimum among all such graphs of order $2n$. By Lemma 2.9, G has a perfect matching, say F.

Let $G^* = G - F$. Then G^* is of class 2. Clearly, G^* has a $(\Delta - 1)$-critical subgraph H, and H obviously has at most five major vertices (of degree $\Delta - 1$). Suppose H has exactly five major vertices. Note that $N_{G^*}(V(G_\Delta)) = V(G^*)$ because in G every vertex is adjacent to at least two vertices of $V(G_\Delta)$. This implies that $|H| = |G^*|$, which contradicts the choice of Δ. Hence, by Lemma 2.1(iv), H has either three or four major vertices.

We now show that $V(G_\Delta) \subseteq V(H)$. Suppose for a contradiction that there exists $a \in V(G_\Delta) - V(H)$. Then a is adjacent to exactly two major vertices in G, say b, c, and either $ab \in F$ or $ac \in F$, say the latter, for otherwise $G^* - a$ (and thus H) would have at most two major vertices of degree $\Delta - 1$. If $xa \in E(G)$, then by Lemma 2.1(i), a is adjacent to at least three major vertices in G, a contradiction. Hence $xa \notin E(G)$. Clearly, $d_{G^*-a}(b) = \Delta - 2 < \Delta(G^*)$. Thus $G^* - a$ has exactly three major vertices and so H has exactly three major vertices. By Theorem 2.3, $\Delta(H) - \delta(H) = 1$. Therefore $d_{G^*-a}(x) = \delta - 1 \leq \Delta - 3 = \Delta(H) - 2$ implies that $x \notin V(H)$. Suppose $xb \notin E(G)$. Since $xa \notin E(G)$ and x is adjacent to at least one major vertex in G^*, it follows that $G^* - a - x$ (and thus H) has at most two major vertices, a contradiction. Hence $xb \in E(G)$. By Lemma 2.1(i), $d_\Delta(b) \geq 3$, and by the same argument in the previous sentence, $b \in V(H)$. However, $d_{G^*-a-x}(b) = (\Delta - 1) - 2 = \Delta(H) - 2$, contrary to Theorem 2.3 applied to H. This proves that $V(G_\Delta) \subseteq V(H)$.

Since H has at most four major vertices, by Theorem 2.3 and Theorem 2.4, $|H|$ is odd. Thus $|G^*| - |H| = 2n - |H|$ is odd. We next show that $|G^*| - |H| = 1$. Suppose that $|G^*| - |H| \geq 3$. Let $v_1, v_2, v_3 \in V(G^*) - V(H)$ and $t = |N_{G^*}(\{v_1, v_2, v_3\}) \cap V(G_\Delta)|$. Clearly, $t \geq 1$ because each v_i, $i = 1, 2, 3$, is adjacent to at least one vertex of $V(G_\Delta)$ in G^*. On the other hand, $G^* - \{v_1, v_2, v_3\}$ has exactly $5 - t$ major vertices. Thus $t \leq 2$, for otherwise H would have at most two major vertices. Hence $1 \leq t \leq 2$. Let $a \in V(G_\Delta)$ have the most neighbors in $\{v_1, v_2, v_3\}$ in G^*. Then a is adjacent to at least $4 - t$ vertices of v_1, v_2, v_3 in G^*. Thus $d_{G^*-\{v_1,v_2,v_3\}}(a) \leq (\Delta - 1) - (4 - t) = \Delta(H) - (4 - t)$. However, $H \subset G^* - \{v_1, v_2, v_3\}$ has at most $5 - t \leq 4$ major vertices. By Theorem 2.3 and Theorem 2.5, $\Delta(H) - \delta(H) \leq 3 - t$. Hence $a \notin V(H)$, contrary to the fact that $V(G_\Delta) \subseteq V(H)$. This proves that $|H| = |G^*| - 1 = 2n - 1 = 2(n - 1) + 1$. By Theorem 2.3 and Theorem 2.5 again, $e(H) = (n - 1)(\Delta - 1) + 1$. Let $\{z\} = V(G^*) - V(H)$. Then $d_{G^*}(z) \geq \delta - 1$. Note that $e(G) = e(G^*) + |F| \geq e(H) + d_{G^*}(z) + |F| \geq$

$((n-1)(\Delta-1)+1)+(\delta-1)+n = (n-1)\Delta+\delta+1$, contrary to Theorem 2.2(ii) applied to G. This completes the proof of Theorem 3.1. □

The following corollary follows immediately from Theorems 2.3, 2.4 and Theorem 3.1.

Corollary 3.2 *If G is a Δ-critical graph of even order, then G has at least six major vertices.*

Remark 3.3 Theorem 3.1 can be used to give an alternative proof of a result of Beineke and Fiorini [1] on the nonexistence of chromatic index critical graphs of even order $2n \leq 10$.

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