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Clique minors in double-critical graphs

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Abstract

A connected t-chromatic graph G is *double-critical* if $G - \{u, v\}$ is $(t-2)$-colorable for each edge $uv \in E(G)$. A long-standing conjecture of Erdős and Lovász that the complete graphs are the only double-critical t-chromatic graphs remains open for all $t \geq 6$. Given the difficulty in settling Erdős and Lovász's conjecture and motivated by the well-known Hadwiger's conjecture, Kawarabayashi, Pedersen, and Toft proposed a weaker conjecture that every double-critical t-chromatic graph contains a K_t minor and verified their conjecture for $t \leq 7$. Albar and Gonçalves recently proved that every double-critical 8-chromatic graph contains a K_8 minor, and their proof is computer assisted. In this article, we prove that every double-critical t-chromatic graph contains a K_t minor for all $t \leq 9$. Our proof for $t \leq 8$ is shorter and computer free.

KEYWORDS

clique minor, double-critical graph, separating set

1 | INTRODUCTION

All graphs in this article are finite and simple. For a graph G we use $|G|$, $e(G)$, $\delta(G)$ to denote the number of vertices, number of edges, and minimum degree of G, respectively. The degree of a vertex v in a graph is denoted by $d_G(v)$ or simply $d(v)$. For a subset S of $V(G)$, the subgraph induced by S is denoted by $G[S]$ and $G - S = G[V(G) \setminus S]$. If G is a graph and K is a subgraph of G, then by $N(K)$ we denote the set of vertices of $V(G) \setminus V(K)$ that are adjacent to a vertex of K. If $V(K) = \{x\}$, then we use $N(x)$ to denote $N(K)$. By abusing notation we will also denote by $N(x)$ the graph induced by the set $N(x)$. We define $N[x] = N(x) \cup \{x\}$, and similarly will use the same symbol for the graph induced by that set. If u, v are distinct nonadjacent vertices of a graph G, then by $G + uv$ we denote the graph obtained from G by adding an edge with ends u and v. If u, v are adjacent or equal, then we define $G + uv$ to be G.

A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. We write $G \geq H$ if H is a minor of G. In those circumstances we also say that G has an H *minor*. A connected graph G is called *double-critical* if for any edge $uv \in E(G)$, we have $\chi(G - \{u, v\}) = \chi(G) - 2$. The following long-standing *Double-Critical Graph Conjecture* is due to Erdős and Lovász [3].

Conjecture 1.1 (Double-Critical Graph Conjecture (Erdős and Lovász [3])). *For every integer $t \geq 1$, the only double-critical t-chromatic graph is K_t.*

Conjecture 1.1 is a special case of the so-called Erdős-Lovász Tihany Conjecture [3]. It is trivially true for $t \leq 3$ and reasonably easy for $t = 4$. Mozhan [8] and Stiebitz [10] independently proved Conjecture 1.1 for $t = 5$.

Theorem 1.2 (Mozhan [8]; Stiebitz [10]). *The only double-critical 5-chromatic graph is K_5.*

Conjecture 1.1 remains open for all $t \geq 6$. Given the difficulty in settling Conjecture 1.1 and motivated by the well-known Hadwiger's conjecture [4], Kawarabayashi, Pedersen, and Toft proposed a weaker conjecture.

Conjecture 1.3 (Kawarabayashi et al. [6]). *For every integer $t \geq 1$, every double-critical t-chromatic graph contains a K_t minor.*

Conjecture 1.3 is a weaker version of Hadwiger's conjecture [4], which states that for every integer $t \geq 1$, every t-chromatic graph contains a K_t minor. Conjecture 1.3 is true for $t \leq 5$ by Theorem 1.2. In the same article [6], Kawarabayashi, Pedersen, and Toft verified their conjecture for $t \in \{6, 7\}$.

Theorem 1.4 (Kawarabayashi et al. [6]). *For every integer $t \leq 7$, every double-critical t-chromatic graph contains a K_t minor.*

Recently, Albar and Gonçalves [1] announced a proof for the case $t = 8$.

Theorem 1.5 (Albar and Gonçalves [1]). *Every double-critical 8-chromatic graph has a K_8 minor.*

Our main result is the following next step.

Theorem 1.6. *For integers k, t with $1 \leq k \leq 9$ and $t \geq k$, every double-critical t-chromatic graph contains a K_k minor.*

We actually prove a much stronger result, the following.

Theorem 1.7. *For $k \in \{6, 7, 8, 9\}$, let G be a $(k-3)$-connected graph with $k + 1 \leq \delta(G) \leq 2k - 5$. If every edge of G is contained in at least $k - 2$ triangles and for any minimal separating set S of G and any $x \in S$, $G[S \backslash \{x\}]$ is not a clique, then $G \geq K_k$.*

Theorem 1.6 follows directly from Proposition 2.1 (see below) and Theorem 1.7. Our proof of Theorem 1.7 closely follows the proof of the extremal function for K_9 minors by Song and Thomas [9] (see Theorem 1.10 below). Note that the proof of Theorem 1.4 for $k = 7$ is about ten pages long and the proof of Theorem 1.5 is computer assisted. Our proof of Theorem 1.6 is much shorter and computer free for $k \leq 8$. For $k = 9$, our proof is computer-assisted as it applies a computer-assisted lemma from [9] (see Lemma 1.13 below). Note that a computer-assisted proof of Theorem 1.7 for all $k \leq 8$ (and hence computer-assisted proofs of Theorem 1.4 and Theorem 1.5) follows directly from Theorem 1.7 for $k = 9$. (To see that, let G and $k \leq 8$ be as in Theorem 1.7, and let H be obtained from G by adding $9 - k$ vertices, each adjacent to every other vertex of the graph. Then H is 6-connected and satisfies all the other conditions as stated in Theorem 1.7. Thus $H \geq K_9$ and so $G \geq K_k$.) Conjecture 1.3 remains open for all $t \geq 10$. It seems hard to generalize Theorem 1.6.

We need some known results to prove our main results. Before doing so, we need to define (H, k)-cockade. For a graph H and an integer k, let us define an (H, k)-*cockade* recursively as follows. Any graph isomorphic to H is an (H, k)-cockade. Now let G_1, G_2 be (H, k)-cockades and let G be obtained from the disjoint union of G_1 and G_2 by identifying a clique of size k in G_1 with a clique of the same size in G_2. Then the graph G is also an (H, k)-cockade, and every (H, k)-cockade can be constructed this way. We are now ready to state some known results. The following theorem is a result of Dirac [2] for $p \leq 5$ and Mader [7] for $p \in \{6, 7\}$.

Theorem 1.8 (Dirac [2]; Mader [7]). *For every integer $p \in \{1, 2, \dots, 7\}$, a graph on $n \geq p$ vertices and at least $(p-2)n - \binom{p-1}{2} + 1$ edges has a K_p minor.*

Jørgensen [5] and later Song and Thomas [9] generalized Theorem 1.8 to $p = 8$ and $p = 9$, respectively, as follows.

Theorem 1.9 (Jørgensen [5]). *Every graph on $n \geq 8$ vertices with at least $6n - 20$ edges either contains a K_8-minor or is isomorphic to a $(K_{2,2,2,2,2}, 5)$-cockade.*

Theorem 1.10 (Song and Thomas [9]). *Every graph on $n \geq 9$ vertices with at least $7n - 27$ edges either contains a K_9-minor, or is isomorphic to $K_{2,2,2,3,3}$, or is isomorphic to a $(K_{1,2,2,2,2,2}, 6)$-cockade.*

In our proof of Theorem 1.7, we need to examine graphs G such that $k + 1 \leq |G| \leq 2k - 5$, $\delta(G) \geq k - 2$ and $G \ngeq K_k \cup K_1$. We shall use the following results. Lemma 1.11 is a result of Jørgensen [5].

Lemma 1.11 (Jørgensen [5]). *Let G be a graph with $n \leq 11$ vertices and $\delta(G) \geq 6$ such that for every vertex x in G, $G - x$ is not contractible to K_6. Then G is one of the graphs $K_{2,2,2,2}, K_{3,3,3}$ or the complement of the Petersen graph.*

Lemma 1.11 implies Lemma 1.12 below. To see that, let G be a graph satisfying the conditions given in Lemma 1.12. By applying Lemma 1.11 to the graph obtained from G by adding $6 - t$ vertices, each adjacent to every other vertex of the graph, we see that $G \geq K_t \cup K_1$.

Lemma 1.12. *For $t \in \{1, 2, 3, 4, 5\}$, let G be a graph with $n \leq 2t - 1$ vertices and $\delta(G) \geq t$. Then $G \geq K_t \cup K_1$.*

Lemma 1.13 is a result of Song and Thomas [9]. Note that the proof of Lemma 1.13 is computer assisted.

Lemma 1.13 (Song and Thomas [9]). *Let G be a graph with $|G| \in \{9, 10, 11, 12, 13\}$ such that $\delta(G) \geq 7$. Then either $G \geq K_7 \cup K_1$, or G satisfies the following*

- **(A)** *either G is isomorphic to $K_{1,2,2,2,2,2}$, or G has four distinct vertices a_1, b_1, a_2, b_2 such that $a_1a_2, b_1b_2 \notin E(G)$ and for $i = 1, 2$ the vertex a_i is adjacent to b_i, the vertices a_i, b_i have at most four common neighbors, and $G + a_1a_2 + b_1b_2 \geq K_8$,*
- **(B)** *for any two sets $A, B \subseteq V(G)$ of cardinality at least five such that neither is complete and $A \cup B$ includes all vertices of G of degree at most $|G| - 2$, either*
  - **(B1)** *there exist $a \in A$ and $b \in B$ such that $G' \geq K_8$, where G' is obtained from G by adding all edges aa' and bb' for $a' \in A - \{a\}$ and $b' \in B - \{b\}$, or*
  - **(B2)** *there exist $a \in A - B$ and $b \in B - A$ such that $ab \in E(G)$ and the vertices a and b have at most five common neighbors in G, or*
  - **(B3)** *one of A and B contains the other and $G + ab \geq K_7 \cup K_1$ for all distinct nonadjacent vertices $a, b \in A \cap B$.*

2 | BASIC PROPERTIES OF NONCOMPLETE DOUBLE-CRITICAL GRAPHS

We begin with basic properties of noncomplete double-critical k-chromatic graphs established in [6]. We only list those that will be used in our proofs.

Proposition 2.1 (Kawarabayashi et al. [6]). *If G is a noncomplete double-critical k-chromatic graph, then the following hold:*

(a) $\delta(G) \geq k + 1$.

(b) *Every edge $xy \in E(G)$ belongs to at least $k - 2$ triangles.*

(c) *G is 6-connected and no minimal separating set of G can be partitioned into two sets A and B such that $G[A]$ and $G[B]$ are edge-empty and complete, respectively.*

Two proper vertex-colorings c_1 and c_2 of a graph G are *equivalent* if, for all $x, y \in V(G)$, $c_1(x) = c_1(y)$ iff $c_2(x) = c_2(y)$. Two vertex-colorings c_1 and c_2 of a graph G are *equivalent on a set $A \subseteq V(G)$* if the restrictions $c_{1|A}$ and $c_{2|A}$ to A are equivalent on the subgraph $G[A]$. Let S be a separating set of G, and let G_1, G_2 be connected subgraphs of G such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = G[S]$. If c_1 is a k-coloring of G_1 and c_2 is a k-coloring of G_2 such that c_1 and c_2 are equivalent on S, then it is clear that c_1 and c_2 can be combined to a k-coloring of G by a suitable permutation of the color classes of, say c_1. The main technique in the proof of Proposition 2.1(c) involves reassigning and permuting the colors on a separating set S of a noncomplete double-critical k-chromatic graph G so that c_1 and c_2 are equivalent on S to obtain a contradiction, where c_1 is a $(k-1)$-coloring of G_1 and c_2 is a $(k-1)$-coloring of G_2. It seems hard to use this idea to prove that every non-complete double-critical k-chromatic graph is 7-connected, but we can use it to say a bit more about minimal separating sets of size 6 in noncomplete double-critical graphs.

Lemma 2.2. *Suppose G is a noncomplete double-critical k-chromatic graph. If S is a minimal separating set of G with $|S| = 6$, then either $G[S] \subseteq K_{3,3}$ or $G[S] \subseteq K_{2,2,2}$.*

Proof. By Proposition 2.1(c), G is 6-connected. Let $S = \{v_1, \ldots, v_6\} \subset V(G)$ be a minimal separating set of G such that neither $G[S] \subseteq K_{3,3}$ nor $G[S] \subseteq K_{2,2,2}$. Let G_1 and G_2 be subgraphs of G such that $G_1 \cup G_2 = G$, $G_1 \cap G_2 = S$, and there are no edges from $G_1 - S$ to $G_2 - S$. Since $k \geq 6$ by Theorem 1.2, we have $\delta(G) \geq 7$ by Proposition 2.1(a). In particular, since $|S| = 6$, there must exist at least one edge $y_i z_i$ in $G_i - S$ for $i \in \{1, 2\}$. It follows then that G_i is $(k-2)$-colorable since it is a subgraph of $G - \{y_{3-i}, z_{3-i}\}$. Let c_1, c_2 be $(k-2)$-colorings of G_1 and G_2, respectively. For $i = 1, 2$, define $|c_i(A)|$ to be the number of distinct colors assigned to the vertices of A by c_i for any $A \subseteq S$. Clearly c_1 and c_2 are not equivalent on S, otherwise c_1 and c_2, after a suitable permutation of the colors of c_2, can be combined to a $(k-2)$-coloring of G, a contradiction. By Proposition 2.1(c), $\alpha(G[S]) \leq 4$ and so neither c_1 nor c_2 applies the same color to more than four vertices of S. Utilizing a new color, say β, we next redefine the colorings c_1 and c_2 so that c_1 and c_2 are $(k-1)$-colorings of G_1 and G_2, respectively, and are equivalent on S. This yields a contradiction, as c_1 and c_2, after a suitable permutation of the colors of c_2, can be combined to a $(k-1)$-coloring of G.

Suppose that one of the colorings c_1 and c_2, say c_1, assigns the same color to four vertices of S, say $c_1(v_3) = c_1(v_4) = c_1(v_5) = c_1(v_6)$. Then $\{v_3, v_4, v_5, v_6\}$ is an independent set in G. By Proposition 2.1(c), we must have $v_1 v_2 \notin E(G)$. But then $G[S] \subseteq K_{2,2,2}$, a contradiction. Thus neither c_1 nor c_2 assigns the same color to four distinct vertices of S.

Next suppose that one of the colorings c_1 and c_2, say c_1, assigns the same color to three vertices of S, say $c_1(v_4) = c_1(v_5) = c_1(v_6)$. Then $\{v_4, v_5, v_6\}$ is an independent set in G. Since $G[S] \not\subseteq K_{3,3}$, we have $|c_2(\{v_1, v_2, v_3\})| \geq 2$. If $|c_2(\{v_1, v_2, v_3\})| = 2$, we may assume that $c_2(v_2) = c_2(v_3)$. Then $\{v_2, v_3\}$ is an independent set. Then redefining $c_2(v_4) = c_2(v_5) = c_2(v_6) = \beta$ and $c_1(v_2) = c_1(v_3) = \beta$ will make c_1 and c_2 equivalent on S, a contradiction. Thus $|c_2(\{v_1, v_2, v_3\})| = 3$ and so c_2 assigns distinct colors to each of v_1, v_2, v_3. We redefine $c_2(v_4) = c_2(v_5) = c_2(v_6) = \beta$. Clearly c_1 and c_2 are equivalent on S if c_1 assigns distinct colors to each of v_1, v_2, v_3. Thus $|c_1(\{v_1, v_2, v_3\})| \leq 2$. Since $G[S] \not\subseteq K_{3,3}$, we have $|c_1(\{v_1, v_2, v_3\})| = 2$. We may assume that $c_1(v_2) = c_1(v_3)$. Now redefining $c_1(v_3) = \beta$ yields that c_1 and c_2 are equivalent on S. This proves that neither c_1 nor c_2 assigns the same color to three distinct vertices of S. Thus $6 \geq |c_i(S)| \geq 3$ $(i = 1, 2)$. Since $G[S] \not\subseteq K_{2,2,2}$, we have $|c_i(S)| \geq 4$ $(i = 1, 2)$. We may assume that $|c_1(S)| \geq |c_2(S)|$. Then $|c_2(S)| \leq 5$, for otherwise c_1 and c_2 are equivalent on S. Thus $5 \geq |c_2(S)| \geq 4$.

Suppose that $|c_2(S)| = 5$. Then $|c_1(S)| = 5$ or $|c_1(S)| = 6$. We can make c_1 and c_2 equivalent on S by assigning color β to one of the two vertices that are colored the same color by c_1 (if $|c_1(S)| = 5$) and c_2. Thus $|c_2(S)| = 4$. Since neither c_1 nor c_2 assigns the same color to more than two distinct vertices of S, we may assume that $c_2(v_3) = c_2(v_4)$ and $c_2(v_5) = c_2(v_6)$. Then $v_3v_4 \notin E(G)$ and $v_5v_6 \notin E(G)$. Since $G[S] \not\subseteq K_{2,2,2}$, we have $v_1v_2 \in E(G)$. Thus $c_1(v_1) \neq c_1(v_2)$. We may assume that $c_1(v_3) \neq c_1(v_4)$ as c_1 and c_2 are not equivalent on S. If $|c_1(S)| = 6$, then redefining $c_1(v_5) = c_1(v_6) = \beta$ and $c_2(v_3) = \beta$ will make c_1 and c_2 equivalent. If $|c_1(S)| = 5$, then at least one of v_3, v_4, v_5, v_6 shares a color with another vertex of S, say $c_1(v_6) = c_1(v_i)$ for some $i \in \{1, \dots, 5\}$. Then redefining $c_1(v_5) = c_1(v_6) = \beta$ and $c_2(v_3) = \beta$ will again make c_1 and c_2 equivalent. Thus $|c_1(S)| = 4$. Suppose that one of v_1 or v_2 shares a color with another vertex of S. Since $v_1v_2 \in E(G)$, we may assume by symmetry that $c_1(v_1) = c_1(v_3)$. If $c_1(v_5)$ and $c_1(v_6)$ are the two colors each assigned to only a single vertex of S by c_1, then we also have $c_1(v_2) = c_2(v_4)$. Now redefining $c_1(v_3) = c_1(v_4) = \beta$ and $c_2(v_5) = \beta$ will make c_1 and c_2 equivalent. Hence one of the colors $c_1(v_5)$ and $c_1(v_6)$ is assigned to two vertices of S, say $c_1(v_6) = c_1(v_i)$ for some $i \in \{2, 4, 5\}$. If $i = 2$ then redefine $c_1(v_5) = c_1(v_6) = \beta$ and $c_2(v_1) = c_2(v_3) = \beta$, if $i = 4$ then redefine $c_1(v_3) = c_1(v_4) = \beta$ and $c_2(v_6) = \beta$, and if $i = 5$ then redefine $c_1(v_3) = \beta$ and $c_2(v_3) = \beta$, and in each case c_1 is equivalent to c_2. Therefore $c_1(v_1)$ and $c_1(v_2)$ are the two colors assigned to only a single vertex of S by c_1. Since c_1 and c_2 are not equivalent, we must have, say $c_1(v_3) = c_1(v_5)$ and $c_1(v_4) = c_1(v_6)$. Now redefining $c_1(v_5) = c_1(v_6) = \beta$ and $c_2(v_3) = \beta$ will make c_1 and c_2 equivalent. ■

3 | PROOFS OF THEOREM 1.7 AND THEOREM 1.6

In this section, we first prove Theorem 1.7.

Proof. Let G be a graph as in the statement with n vertices. By assumption, we have

(1) $k + 1 \leq \delta(G) \leq 2k - 5$ and $\delta(N(x)) \geq k - 2$ for any x in G; and

(2) G is $(k - 3)$-connected and for any minimal separating set S of G and any $x \in S$, $G[S \backslash \{x\}]$ is not a complete subgraph.

We first show that the statement is true for $k = 6$. Then G is 3-connected with $\delta(G) = 7$. The statement is trivially true if G is complete, so we may assume G is not complete. Let $x \in V(G)$ be a vertex of degree 7. By (1), $\delta(N(x)) \geq 4$, and so $e(N(x)) \geq 14$. If $e(N(x)) \geq 16$, then by Theorem 1.8, $N(x) \geq K_5$ and so $G \geq N[x] \geq K_6$. If $e(N(x)) = 15$, then let K be a component of $G - N[x]$ with $|N(K)|$ minimum. By (2), $|N(K)| \geq 3$ and $N(K)$ is not complete. Let $y, z \in N(K)$ be nonadjacent in

$N(x)$ and let P be a (y,z)-path with interior vertices in K. We see that $G \geq K_6$ by contracting all but one of the edges of P. So we may assume that $e(N(x)) = 14$, and so $N(x)$ is 4-regular and $\overline{N(x)}$ is 2-regular. Thus $\overline{N(x)}$ is then either isomorphic to C_7 or to $C_4 \cup C_3$, and in both cases it is easy to see that $N(x) \geq K_5$ and thus $G \geq K_6$, as desired. Hence we may assume $7 \leq k \leq 9$.

Suppose for a contradiction that $G \ngeq K_k$. We next prove the following.

(3) Let $x \in V(G)$ be such that $k+1 \leq d(x) \leq 2k-5$. Then there is no component K of $G - N[x]$ such that $N(K') \cap M \subseteq N(K)$ for every component K' of $G - N[x]$, where M is the set of vertices of $N(x)$ not adjacent to all other vertices of $N(x)$. ■

Proof. Suppose such a component K exists. Among all vertices x with $k+1 \leq d(x) \leq 2k-5$ for which such a component exists, choose x to be of minimal degree, and among all such components K of $G - N[x]$, choose K such that $|N(K)|$ is minimum. We first prove that $M \subseteq N(K)$. Suppose for a contradiction that $M - N(K) \neq \emptyset$, and let $y \in M \setminus N(K)$ be such that $d(y)$ is minimum. Clearly, $d(y) < d(x)$. Let J be the component of $G - N[y]$ containing K. Since $d(y) < d(x)$ the choice of x implies that $N(x) \setminus N[y] \nsubseteq V(J)$. Let $H = N(x) \setminus (N[y] \cup N(K))$. We have $d_G(z) \geq d_G(y)$ for all $z \in V(H)$ by the choice of y. Let $t = |V(H)|$. Then $t \geq 2$, for otherwise the vertex y and component H contradict the choice of x. On the other hand $t \leq d(x) - d(y) \leq (2k-5) - (k+1) = k-6 \leq 3$ and so $k \geq 8$. Notice that $t = 2$ when $k = 8$. From (1) applied to y we deduce that $N(y) \cap N(x)$ has minimum degree at least $k-3$. Let L be the subgraph of G induced by $(N[y] \cap N(x)) \cup V(H)$. Then the edge-set of L consists of edges of $N(x) \cap N(y)$, edges incident with y, and edges incident with $V(H)$. Clearly, $e(L - V(H), H) = \sum_{z \in V(H)} (d(z) - 1) - 2e(H) \geq t(d(y) - 1) - 2e(H)$. Thus

$$
\begin{aligned}
e(L) &\geq \frac{(k-3)\,(d(y)-1)}{2} + d(y) - 1 + e(L - V(H), H) + e(H) \\
&\geq \frac{(k-3)\,(d(y)-1)}{2} + d(y) - 1 + t(d(y)-1) - e(H) \\
&\geq \frac{(k-3)(d(y)-1)}{2} + d(y) - 1 + t(d(y)-1) - \frac{1}{2}\,t(t-1) \\
&\geq \begin{cases} 5(d(y)+2) + \frac{d(y)}{2} - \frac{33}{2} & \text{if} \quad k = 8 \\ 6(d(y)+t) + (t-2)d(y) - 4 - 7t - \frac{1}{2}\,t(t-1) & \text{if} \quad k = 9 \end{cases} \\
&\geq (k-3)|V(L)| - \binom{k-2}{2} + 1,
\end{aligned}
$$

because $d(y) \geq k+1$ and $2 \leq t \leq k-6$. If $k = 9$, since $12 \leq |V(L)| \leq 13$ the graph L is not a $(K_{2,2,2,2,2}, 5)$-cockade. By Theorem 1.8 and Theorem 1.9, $N(x) \geq L \geq K_{k-1}$. Thus $G \geq N[x] \geq K_k$, a contradiction. This proves that $M \subseteq N(K)$.

If $N(x) \geq K_{k-2} \cup K_1$, then $N(x)$ has a vertex y such that $N(x) - y \geq K_{k-2}$. If $y \notin M$, then $N(x) \geq K_{k-1}$. Otherwise, by contracting the connected set $V(K) \cup \{y\}$ we can contract K_{k-1} onto $N(x)$. Thus in either case $G \geq K_k$, a contradiction. Thus $N(x) \ngeq K_{k-2} \cup K_1$. If $k \leq 8$, by Lemma 1.11 and Lemma 1.12, we have $k = 8$ and $N(x)$ is either $K_{3,3,3}$ or $\overline{P}$, where $\overline{P}$ is the complement of the Petersen graph. If $N(x) = \overline{P}$, it can be easily checked that $\overline{P} + yz \geq K_7$ for any $yz \in E(P)$. By (2), $|N(K)| \geq 5$ and $N(K)$ is not complete. Let $y, z \in N(K)$ be nonadjacent in $N(x)$ and let Q be a (y,z)-path with interior vertices in K. We see that $G \geq K_8$ by contracting all but one of the edges of Q, a contradiction. Thus $N(x) = K_{3,3,3}$, and so $M = N(x)$. Let $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ be the vertex sets of two disjoint triangles of $\overline{N(x)}$. Suppose $G - N[x]$ is 2-connected or has at most two vertices. By the hypothesis, the vertices a_i, b_i (i=1,2) have at least two common neighbors in $G - N[x]$. Let u_1, u_2 (resp. w_1, w_2) be two distinct common neighbors of a_1 and b_1 (resp. a_2 and b_2) in $G - N[x]$. By Menger's Theorem, $G - N[x]$ contains two disjoint paths from $\{u_1, u_2\}$ to $\{w_1, w_2\}$ and so $G \geq N[x] + a_1a_2 + b_1b_2 \geq K_8$, a contradiction. Thus $G - N[x]$ has at least three vertices and is not 2-connected. If $G - N[x]$ is disconnected, let $H_1 = K$ and H_2 be another connected component of $G - N[x]$. If $G - N[x]$ has a cut-vertex, say w, let H_1 be a connected component of $G - N[x] - w$ and let $H_2 = G - N[x] - V(H_1)$. In either case, H_1 and H_2 are disjoint connected subgraphs of $G - N[x]$ such that $M \subseteq N(H_1) \cup N(H_2)$ (because we have shown that $M \subseteq N(K)$). Thus $N(H_1) \cup N(H_2) = N(x)$ because $M = N(x)$. By (2), $N(H_i)$ is not complete and $|N(H_i)| \geq 4$ since $k = 8$. Thus each of $N(H_1)$ and $N(H_2)$ must contain at least one edge of $\overline{N(x)}$. Since $N(x) = K_{3,3,3}$ and $N(H_1) \cup N(H_2) = N(x)$, we may thus assume that $a_1a_2 \in \overline{N(H_1)}$ and $b_1b_2 \in \overline{N(H_2)}$. By contracting H_1 onto a_1 and H_2 onto b_1 we see that $G \geq N[x] + a_1a_2 + b_1b_2 \geq K_8$, a contradiction. This proves that $k = 9$ and so by Lemma 1.13, we may assume that $N(x)$ satisfies properties (A) and (B).

Since $d(x) \geq 10$, $N(x) \neq K_{1,2,2,2,2}$. If $G - N[x]$ is 2-connected or has at most two vertices, then by property (A) and (2), the set $N(x)$ has four distinct vertices a_1, b_1, a_2, b_2 such that $a_1a_2, b_1b_2 \notin E(G)$, $N(x) + a_1a_2 + b_1b_2 \geq K_8$ and for $i = 1, 2$ the vertex a_i is adjacent to b_i, and the vertices a_i, b_i have at least two common neighbors in $G - N[x]$. Let u_1, u_2 (resp. w_1, w_2) be two distinct common neighbors of a_1 and b_1 (resp. a_2 and b_2) in $G - N[x]$. By Menger's Theorem, $G - N[x]$ contains two disjoint paths from $\{u_1, u_2\}$ to $\{w_1, w_2\}$ and so $G \geq N[x] + a_1a_2 + b_1b_2 \geq K_9$, a contradiction. Thus $G - N[x]$ has at least three vertices and is not 2-connected. If $G - N[x]$ is disconnected, let $H_1 = K$ and H_2 be another connected component of $G - N[x]$. If $G - N[x]$ has a cut-vertex, say w, let H_1 be a connected component of $G - N[x] - w$ and let $H_2 = G - N[x] - V(H_1)$. In either case, H_1 and H_2 are disjoint connected subgraphs of $G - N[x]$ such that $M \subseteq N(H_1) \cup N(H_2)$ (because we have shown that $M \subseteq N(K)$). For $i = 1, 2$ let $A_i = N(H_i) \cap N(x)$. By (2), A_i is not complete and $|A_i| \geq 5$ for $i = 1, 2$. By property (B), A_1 and A_2 satisfy properties (B1), (B2), or (B3).

Suppose first that A_1 and A_2 satisfy property (B1). Then there exist $a_i \in A_i$ such that $N(x) + \{a_1a : a \in A_1 \setminus \{a_1\}\} + \{a_2a : a \in A_2 \setminus \{a_2\}\} \geq K_8$. By contracting the connected sets $V(H_1) \cup \{a_1\}$ and $V(H_2) \cup \{a_2\}$ to single vertices, we see that $G \geq K_9$, a contradiction. Suppose next that A_1 and A_2 satisfy property (B2). Then there exist $a_1 \in A_1 \setminus A_2$ and $a_2 \in A_2 \setminus A_1$ such that $a_1a_2 \in E(G)$ and the vertices a_1 and a_2 have at most five common neighbors in $N(x)$. Thus $a_1, a_2 \in M$ by (1), and by another application of (1) there exists a common neighbor $u \in V(G) \setminus N[x]$ of a_1 and a_2. But $a_1 \notin A_2$ and $a_2 \notin A_1$, and hence $u \notin V(H_1) \cup V(H_2)$. Thus $G - N[x]$ is disconnected and $H_1 = K$. But then $a_2 \in M \subseteq N(K) = N(H_1)$, a contradiction. Thus we may assume that A_1 and A_2 satisfy (B3), and hence $A_i \subseteq A_{3-i}$ for some $i \in \{1, 2\}$. As $M \subseteq A_1 \cup A_2$, we have $M \subseteq N(H_{3-i})$. Since A_i is not complete, let $a, b \in A_i$ be distinct and not adjacent. By property (B3), $N(x) + ab \geq K_7 \cup K_1$. Let P be an (a, b)-path with interior in H_i. By contracting all but one of the edges of the path P and by contracting H_{3-i} similarly as above, we see that $G \geq K_9$, a contradiction. ■

(4) $G - N[x]$ is disconnected for every vertex $x \in V(G)$ of degree at most $2k - 5$.

Proof. If $G - N[x]$ is not null, then it is disconnected by (3). Thus we may assume that x is adjacent to every other vertex of G. Let $H = G - x$. Then $|H| = d(x)$ and $\delta(H) \geq k$. Thus $e(H) \geq \frac{k\,d(x)}{2} > (k-3)\,d(x) - \binom{k-2}{2} + 1$ because $d(x) \leq 2k - 5$. By Theorem 1.8 and Theorem 1.9, $G - x$ has a K_{k-1} minor and so the graph G has a K_k minor, a contradiction. ■

(5) Let $x \in V(G)$ be such that $k + 1 \leq d(x) \leq 2k - 5$. Then there is no component K of $G - N[x]$ such that $d_G(y) \geq 2k - 4$ for every vertex $y \in V(K)$.

Proof. Assume that such a component K exists. Let $G_1 = G - V(K)$ and $G_2 = G[V(K) \cup N(K)]$. Let d_1 be the maximum number of edges that can be added to G_2 by contracting edges of G with at least one end in G_1. More precisely, let d_1 be the largest integer so that G_1 contains disjoint sets of vertices $V_1, V_2, \dots, V_p$ so that $G_1[V_j]$ is connected, $|N(K) \cap V_j| = 1$ for $1 \le j \le p = |N(K)|$, and so that the graph obtained from G_1 by contracting $V_1, V_2, \dots, V_p$ and deleting $V(G) \setminus (\bigcup_j V_j)$ has $e(N(K)) + d_1$ edges. Let G_2' be a graph with $V(G_2') = V(G_2)$ and $e(G_2') = e(G_2) + d_1$ edges obtained from G by contracting edges in G_1. By (1), $|G_2'| \ge k + 2$. If $e(G_2') \ge (k-2)\,|G_2'| - \binom{k-1}{2} + 2$, then by Theorem 1.8 and Theorem 1.9, $G \ge G_2' \ge K_k$, a contradiction. Thus

$$e(G_2) = e(G_2') - d_1 \le (k-2)\,|G_2| - \binom{k-1}{2} + 1 - d_1$$
$$= (k-2)|N(K)| + (k-2)|K| - \binom{k-1}{2} + 1 - d_1.$$

By contracting the edge xz, where $z \in N(K)$ has minimum degree d in $N(K)$, we see that $d_1 \ge |N(K)| - d - 1$ and hence

$$e(G_2) \le (k-3)|N(K)| + (k-2)|K| - \binom{k-1}{2} + 2 + d. \tag{a}$$

Let $t = e_G(N(K), K)$. We have $e(G_2) = e(K) + t + e(N(K))$ and

$$2e(K) \ge (2k-4)|K| - t, \tag{b}$$

and hence

$$e(G_2) \ge (k-2)|K| + t/2 + d|N(K)|/2. \tag{c}$$

Since $N(x)$ has minimum degree at least $k-2$, it follows that the subgraph $N(K)$ of $N(x)$ has minimum degree at least $(k-2) - (d(x) - |N(K)|)$. Thus $d \ge (k-2) - (d(x) - |N(K)|) \ge |N(K)| - k + 3$. From (a) and (c) we get

$$-t/2 \ge -(k-3)|N(K)| + d(|N(K)| - 2)/2 + \binom{k-1}{2} - 2 \ge \begin{cases} -8 & \text{if } k = 7 \\ -14 & \text{if } k = 8 \\ -18 & \text{if } k = 9 \end{cases} \tag{d}$$

where the second inequality becomes $\frac{t}{2} \le 11$ when $|N(K)| = 2k - 6$ and $k = 7, 8$, and the second inequality holds with equality only when $|N(K)| = 10$ and $k = 9$. Since G is not contractible to K_k, we deduce from (b) and Theorem 1.8, Theorem 1.9, and Theorem 1.10 that $|K| < 8$. The inequalities $e(K) \ge 5|K| - 8$ when $k = 7$, $e(K) \ge 6|K| - 14$ when $k = 8$, and $e(K) \ge 7|K| - 18$ when $k = 9$ imply $|K| \le 3$. But every vertex of K has degree at least $2k - 4$ and $N(K)$ is a proper subgraph of $N(x)$, and hence $|K| = 3$, $|N(K)| = 2k - 6$ and $\frac{t}{2} = 3(k-3) \ge 12$ when $k = 7, 8$, and (d) holds with equality for $|N(K)| = 12$ when $k = 9$, contrary to our earlier observation of (d) that $\frac{t}{2} \le 11$ when $|N(K)| = 2k - 6$ and $k = 7, 8$, and (d) holds with equality only when $|N(K)| = 10$ and $k = 9$. ■

By (1) there is a vertex x of degree $k+1, k+2, \dots,$ or $2k - 5$ in G. Choose such a vertex x so that $G - N[x]$ has a component K of minimum order. Then choose a vertex $y \in V(K)$ of least degree in G. Thus $k+1 \le d_G(y) \le 2k-5$ by (1) and (5). Let L be the component of $G - N[y]$ containing x. We claim that $N(L)$ contains all vertices of $N(y)$ that are not adjacent to all other vertices of $N(y)$. Indeed, let $z \in N(y)$ be not adjacent to some vertex of $N(y) \setminus \{z\}$. We may assume that $z \notin N(x)$, for otherwise $z \in N(L)$. Thus $z \in V(K)$, and hence $d_G(z) \ge d_G(y)$ by the choice of y. Thus z has a neighbor $z' \in N[x] \cup V(K) \setminus N[y]$. Then $z' \in V(L)$, for otherwise the component of $G - N[y]$ containing z' would be a proper subgraph of K. Thus $z \in N(L)$. This proves our claim that $N(L)$ contains all vertices z as above, contrary to (3). This contradiction completes the proof of Theorem 1.7. ■

We are now ready to prove Theorem 1.6.

Proof. Let G be a double-critical t-chromatic graph with $t \ge k$. The assertion is trivially true if G is complete. By Theorem 1.2, we may assume that $t \ge 6$. By Proposition 2.1(a), $\delta(G) \ge k+1$. By Theorem 1.8, Theorem 1.9, and Theorem 1.10, we have $\delta(G) \le 2k-5$. By Proposition 2.1(b), every edge of G is contained in at least $k-2$ triangles. By Proposition 2.1(c), G is 6-connected and no minimal separating set of G can be partitioned into a clique and an independent set. By Theorem 1.7, $G \ge K_k$, as desired. ■

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REFERENCES

[1] B. Albar and D. Gonçalves, *On triangles in K_r-minor free graphs*, J. Graph Theory https://doi.org/10.1002/jgt.22203 (to appear).

[2] G. A. Dirac, *Homomorphism theorems for graphs*, Math. Ann. **153** (1964), 69–80.

[3] P. Erdős, Problem 2. In *Theory of Graphs (Proc. Colloq., Tihany, 1966)*. Academic Press, New York, 1968, p. 361.

[4] H. Hadwiger, *Über eine Klassifikation der Streckenkomplexe, Vierteljschr*, Naturforsch. Ges Zürich **88** (1943), 133–142.

[5] L. K. Jørgensen, *Contractions to K_8*, J. Graph Theory **18** (1994), 431–448.

[6] K. Kawarabayashi, A. S. Pedersen, and B. Toft, *Double-critical graphs and complete minors*, Electron. J. Combin. **17** (2010), #R87.

[7] W. Mader, *Homomorphiesätze für Graphen*, Math. Ann. **178** (1968), 154–168.

[8] N. N. Mozhan, *On doubly critical graphs with the chromatic number five*, Metody Diskretn. Anal. **46** (1987), 50–59.

[9] Z-X. Song and R. Thomas, *The extremal function for K_9 minors*, J. Combin. Theory Ser. B **96** (2006), 240–252.

[10] M. Stiebitz, *K_5 is the only double-critical 5-chromatic graph*, Discrete Math. **64** (1987), 91–93.

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