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Coloring graphs with forbidden minors

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A R T I C L E I N F O

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A B S T R A C T

Hadwiger's conjecture from 1943 states that for every integer $t \geq 1$, every graph either can be t-colored or has a subgraph that can be contracted to the complete graph on $t+1$ vertices. As pointed out by Paul Seymour in his recent survey on Hadwiger's conjecture, proving that graphs with no K_7 minor are 6-colorable is the first case of Hadwiger's conjecture that is still open. It is not known yet whether graphs with no K_7 minor are 7-colorable. Using a Kempe-chain argument along with the fact that an induced path on three vertices is dominating in a graph with independence number two, we first give a very short and computer-free proof of a recent result of Albar and Gonçalves and generalize it to the next step by showing that every graph with no K_t minor is $(2t-6)$-colorable, where $t \in \{7, 8, 9\}$. We then prove that graphs with no K_8^- minor are 9-colorable, and graphs with no $K_8^=$ minor are 8-colorable. Finally we prove that if Mader's bound for the extremal function for K_t minors is true, then every graph with no K_t minor is $(2t-6)$-colorable for all $t \geq 6$. This implies our first result. We believe that the Kempe-chain method we have developed in this paper is of independent interest.

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1. Introduction

All graphs in this paper are finite and simple. A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. We write $G > H$ if H is a minor of G. In those circumstances we also say that G has an H *minor*.

Our work is motivated by the following Hadwiger's conjecture [6], which is perhaps the most famous conjecture in graph theory, as pointed out by Paul Seymour in his recent survey [18].

Conjecture 1.1. *For every integer $t \geq 1$, every graph with no K_{t+1} minor is t-colorable.*

Hadwiger's conjecture is trivially true for $t \leq 2$, and reasonably easy for $t = 3$, as shown by Dirac [3]. However, for $t \geq 4$, Hadwiger's conjecture implies the Four Color Theorem. Wagner [22] proved that the case $t = 4$ of Hadwiger's conjecture is, in fact, equivalent to the Four Color Theorem, and the same was shown for $t = 5$ by Robertson, Seymour and Thomas [16]. Hadwiger's conjecture remains open for $t \geq 6$. As pointed out by Paul Seymour [18] in his recent survey on Hadwiger's conjecture, proving that graphs with no K_7 minor are 6-colorable is thus the first case of Hadwiger's conjecture that is still open. It is not even known yet whether every graph with no K_7 minor is 7-colorable. Kawarabayashi and Toft [11] proved that every graph with no K_7 or $K_{4,4}$ minor is 6-colorable. Jakobsen [8,9] proved that every graph with no $K_7^{=}$ minor is 6-colorable and every graph with no K_7^- minor is 7-colorable, where for any integer $p > 0$, K_p^- denotes the graph obtained from K_p by removing one edge, and $K_p^{=}$ denotes the family of two non-isomorphic graphs each obtained from K_p by removing two edges. Note that a graph has no $K_p^{=}$ minor if it does not contain any of the two graphs in $K_p^{=}$ as a minor; and a graph G has a $K_p^{=}$ minor or $G > K_p^{=}$ if G contains one of the graphs in $K_p^{=}$ as a minor. For more information on Hadwiger's conjecture, the readers are referred to an earlier survey by Toft [21] and a very recent informative survey due to Seymour [18].

Albar and Gonçalves [1] recently proved the following:

Theorem 1.2. (Albar and Gonçalves [1]) *Every graph with no K_7 minor is 8-colorable, and every graph with no K_8 minor is 10-colorable.*

The proof of Theorem 1.2 is computer-assisted and not simple. In this paper, we apply a Kempe-chain argument (see Lemma 1.7 below), along with the fact that an induced path on three vertices is dominating in a graph with independence number two, to give a much shorter and computer-free proof of Theorem 1.2. In addition, we generalize it to the next step by proving the following.

Theorem 1.3. *Every graph with no K_t minor is $(2t-6)$-colorable, where $t \in \{7, 8, 9\}$.*

We want to point out that our proof of Theorem 1.3 does not rely on Mader's deep result on the connectivity of contraction-critical graphs (see Theorem 1.8 below).

Theorem 1.3 states that (i) every graph with no K_7 minor is 8-colorable; (ii) every graph with no K_8 minor is 10-colorable; and (iii) every graph with no K_9 minor is 12-colorable. We prove Theorem 1.3 in Section 2.

Applying the method we developed in the proof of Theorem 1.3 and Mader's deep result (Theorem 1.8), we then prove two new results Theorem 1.4 and Theorem 1.5.

Theorem 1.4. *Every graph with no K_8^- minor is 9-colorable.*

Theorem 1.5. *Every graph with no $K_8^=$ minor is 8-colorable.*

Our proofs of Theorem 1.4 and Theorem 1.5 are both short and computer-free and will be presented in Section 3 and Section 4, respectively.

To end this paper, we first propose a conjecture in Section 5. We then apply Lemma 1.7 to prove that if Conjecture 5.1 (see Section 5) is true, then every graph with no K_t minor is $(2t-6)$-colorable for all $t \geq 6$. Our proof of the last result does not rely on the connectivity of contraction-critical graphs and the new idea we introduce yields a different/short proof of Theorem 1.3.

To prove our results, we need to investigate the basic properties of contraction-critical graphs. For a positive integer t, a graph G is *t-contraction-critical* if $\chi(G) = t$ and any proper minor of G is $(t-1)$-colorable. Lemma 1.6 below is a folklore result which is an extension of Dirac's initial work [4] on contraction-critical graphs. A proof of Lemma 1.6 can be easily obtained from the definition of k-contraction-critical graphs.

Lemma 1.6. (Dirac [4]) *Every k-contraction-critical graph G satisfies the following:*

(i) for any $v \in V(G)$, $\alpha(G[N(v)]) \leq d(v) - k + 2$, where $\alpha(G[N(v)])$ denotes the independence number of the subgraph of G induced by $N(v)$;
(ii) no separating set of G is a clique.

Lemma 1.7 below on contraction-critical graphs turns out to be very powerful, as the existence of pairwise vertex-disjoint paths is guaranteed without using the connectivity of such graphs. If two vertices u, v in a graph G are not adjacent, we say that uv is a *missing edge* of G. One possible application of Lemma 1.7 is depicted in Fig. 1.

Lemma 1.7. *Let G be any k-contraction-critical graph. Let $x \in V(G)$ be a vertex of degree $k+s$ with $\alpha(G[N(x)]) = s+2$ and let $S \subset N(x)$ with $|S| = s+2$ be any independent set, where $k \geq 4$ and $s \geq 0$ are integers. Let M be a set of missing edges of $G[N(x)\backslash S]$. Then there exists a collection $\{P_{uv} : uv \in M\}$ of paths in G such that for each $uv \in M$, P_{uv} has ends $\{u, v\}$ and all its internal vertices in $G\backslash N[x]$. Moreover, if vertices u, v, w, z with $uv, wz \in M$ are distinct, then the paths P_{uv} and P_{wz} are vertex-disjoint.*

Proof. Let G, x, S and M be given as in the statement. Let H be obtained from G by contracting $S \cup \{x\}$ into a single vertex, say w. Then H is $(k-1)$-colorable. Let

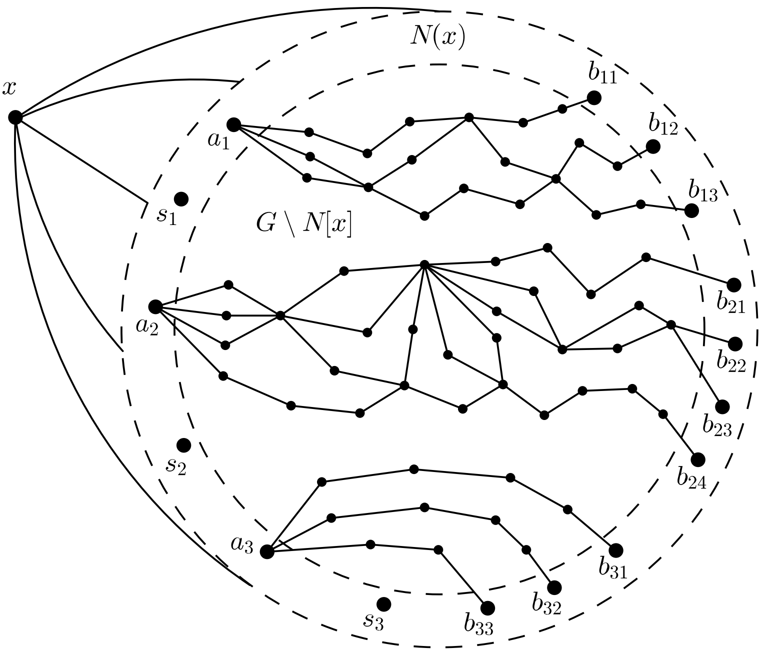

Fig. 1. An application of Lemma 1.7 to x with $S = \{s_1, s_2, s_3\}$ and $M = \{a_1b_{11}, a_1b_{12}, a_1b_{13}, a_2b_{21}, a_2b_{22}, a_2b_{23}, a_2b_{24}, a_3b_{31}, a_3b_{32}, a_3b_{33}\}$.

$c : V(H) \to \{1, 2, \ldots, k-1\}$ be a proper $(k-1)$-coloring of H. We may assume that $c(w) = 1$. Then each of the colors $2, \ldots, k-1$ must appear in $G[N(x)\backslash S]$, else we could assign x the missing color and all vertices in S the color 1 to obtain a proper $(k-1)$-coloring of G, a contradiction. Since $|N(x)\backslash S| = k-2$, we have $c(u) \neq c(v)$ for any two distinct vertices u, v in $G[N(x)\backslash S]$. We next claim that for each $uv \in M$ there must exist a path between u and v with its internal vertices in $G\backslash N[x]$. Suppose not. Let H^* be the subgraph of H induced by the vertices colored $c(u)$ or $c(v)$ under the coloring c. Then $V(H^*) \cap N(x) = \{u, v\}$. Notice that u and v must belong to different components of H^* as there is no path between u and v with its internal vertices in $G\backslash N[x]$. By switching the colors on the component of H^* containing u, we obtain a $(k-1)$-coloring of H with the color $c(u)$ missing on $G[N(x)\backslash S]$, a contradiction. This proves that there must exist a path P_{uv} in H^* with ends u, v and all its internal vertices in $H^*\backslash N[x]$ for each $uv \in M$. Clearly, for any $uv, wz \in M$ with u, v, w, z distinct, the paths P_{uv}, P_{wz} are vertex-disjoint, because no two vertices of u, v, w, z are colored the same under the coloring c. □

Remark. (a) If $M = \{a_1b_{11}, \ldots, a_1b_{1r_1}, a_2b_{21}, \ldots, a_2b_{2r_2}, \ldots, a_mb_{m1}, \ldots, a_mb_{mr_m}\}$, where the vertices $a_1, \ldots, a_m, b_{11}, \ldots, b_{mr_m} \in G[N(x)\backslash S]$ are all distinct, and for any $i \in \{1, 2, \ldots, m\}$, $a_ib_{i1}, \ldots, a_ib_{ir_i}$ are r_i missing edges of $G[N(x)\backslash S]$ with a_i as a common end, then for any $1 \leq i < j \leq m$, the paths $P_{a_ib_{i1}}, \ldots, P_{a_ib_{ir_i}}$ are vertex-disjoint from the paths $P_{a_jb_{j1}}, \ldots, P_{a_jb_{jr_j}}$.

(b) If $M = \{a_1b_1, a_2b_2, \ldots, a_mb_m\}$ is a matching of missing edges of $G[N(x)\backslash S]$, then the paths $P_{a_1b_1}, P_{a_2b_2}, \ldots, P_{a_mb_m}$ are pairwise vertex-disjoint if $m \geq 2$.

Theorem 1.8 below is a deep result of Mader [15] and will be used in the proofs of Theorem 1.4 and Theorem 1.5. It seems very difficult to improve Theorem 1.8 for small values of k. For larger values of k, some better results can be found. Kawarabayashi [12] has shown that any minimal non-complete k-contraction-critical graph with no K_k minor is $\lceil 2k/27 \rceil$-connected, while Kawarabayashi and Yu [13] have shown that any minimal such graph is $\lceil k/9 \rceil$-connected. Chen, Hu and Song [2] recently improved the bound further by showing that any minimal such graph is $\lceil k/6 \rceil$-connected.

Theorem 1.8. (Mader [15]) *For $k \geq 7$, every k-contraction-critical graph is 7-connected.*

We also need the following lemma in the proofs of Theorem 1.4 and Theorem 1.5.

Lemma 1.9. *For any 7-connected graph G, if G contains two different K_6-subgraphs, then $G > K_8^-$.*

Proof. Let H_1, H_2 be two different K_6-subgraphs of G with $V(H_1) = \{v_1, \ldots, v_6\}$ and $V(H_2) = \{w_1, \ldots, w_6\}$. Let $t = |V(H_1) \cap V(H_2)|$. Then $0 \leq t \leq 5$. We may assume that $v_i = w_i$ for all $i \leq t$ if $t \neq 0$. Assume that $t = 5$. Then $H_1 \cup H_2$ has a K_7^--subgraph of G. Since G is 7-connected, it is easy to see that $G > K_8^-$ by contracting a component of $G\backslash(H_1 \cup H_2)$ into a single vertex. So we may assume that $t \leq 4$. Then there exist $6-t$ pairwise disjoint paths $P_{t+1}, \ldots, P_6$ between $H_1\backslash H_2$ and $H_2\backslash H_1$ in $G\backslash(V(H_1) \cap V(H_2))$. We may assume that P_i has ends v_i, w_i for all $i = t+1, \ldots, 6$. Then $G\backslash\{v_1, \ldots, v_5, w_6\}$ is connected, so there must exist a path Q with one end, say u, in $(P_{t+1}\backslash v_{t+1}) \cup \cdots \cup (P_5\backslash v_5)$, the other end, say v, in $P_6\backslash w_6$, and no internal vertices in any of $\{v_1, \cdots, v_t\}, P_{t+1}, \ldots, P_6$. We may assume that u lies on the path $P_5\backslash v_5$. Let P_5^* be the subpath of P_5 with ends u, w_5, and P_6^* be the subpath of P_6 with ends v, v_6. Now contracting P_5^* onto w_5, $P_5\backslash P_5^*$ onto v_5, P_6^* and $Q\backslash u$ onto v_6, $P_6\backslash P_6^*$ onto w_6, and each of $P_{t+1}, \ldots, P_4$ to a single vertex, together with $v_1, \ldots, v_t$ if $t \neq 0$, yields a K_8^- minor in G, as desired. □

We need to introduce more notation. For a graph G we use $|G|$ and $\delta(G)$ to denote the number of vertices and minimum degree of G, respectively. For a subset S of $V(G)$, the subgraph induced by S is denoted by $G[S]$ and $G\backslash S = G[V(G)\backslash S]$. The degree and neighborhood of a vertex v in G are denoted by $d(v)$ and $N(v)$, respectively. We define $N[v] = N(v) \cup \{v\}$. For $S \subseteq V(G)$, if $G[S]$ is connected, then we denote by G/S the graph obtained from G by contracting $G[S]$ into a single vertex and deleting all resulting parallel edges and loops. For $A, B \subseteq V(G)$, we say that A is *complete* to B if each vertex in A is adjacent to all vertices in B, and A is *anti-complete* to B if no vertex in A is adjacent to any vertex in B. If $A = \{a\}$, we simply say a is complete to B or a is anti-complete to B. The *join* $G + H$ (resp. *union* $G \cup H$) of two vertex disjoint graphs G and H is the graph having vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy \,|\, x \in V(G), y \in V(H)\}$ (resp. $E(G) \cup E(H)$).

2. Proof of Theorem 1.3: coloring K_t minor free graphs

Results on the extremal function for K_t minors will be needed to prove Theorem 1.3. Before doing so, we need to define (H_1, H_2, k)-cockade. For graphs H_1, H_2 and an integer k, let us define an (H_1, H_2, k)-*cockade* recursively as follows. Any graph isomorphic to H_1 or H_2 is an (H_1, H_2, k)-cockade. Now let G_1, G_2 be (H_1, H_2, k)-cockades and let G be obtained from the disjoint union of G_1 and G_2 by identifying a clique of size k in G_1 with a clique of the same size in G_2. Then the graph G is also an (H_1, H_2, k)-cockade, and every (H_1, H_2, k)-cockade can be constructed in this way. If $H_1 = H_2 = H$, then G is simply called an (H, k)-cockade. The following Theorem 2.1 was first shown by Dirac [5] for $p \leq 5$ and by Mader [14] for $p = 6, 7$.

Theorem 2.1. (Mader [14]) *For every integer $p = 1, 2, \ldots, 7$, a graph on $n \geq p$ vertices and at least $(p-2)n - \binom{p-1}{2} + 1$ edges has a K_p minor.*

The edge bound in Theorem 2.1 is referred to as Mader's bound for the extremal function for K_p minors. Jørgensen [10] and later the second author and Thomas [20] generalized Theorem 2.1 to $p = 8$ and $p = 9$, respectively, as follows.

Theorem 2.2. (Jørgensen [10]) *Every graph on $n \geq 8$ vertices with at least $6n - 20$ edges either has a K_8 minor or is a $(K_{2,2,2,2,2}, 5)$-cockade.*

Theorem 2.3. (Song and Thomas [20]) *Every graph on $n \geq 9$ vertices with at least $7n-27$ edges either has a K_9 minor, or is a $(K_{1,2,2,2,2,2}, 6)$-cockade, or is isomorphic to $K_{2,2,2,3,3}$.*

It seems hard to generalize Theorem 2.1 for all values of p. In 2003, Seymour and Thomas [20] proposed the following conjecture.

Conjecture 2.4. (Seymour and Thomas [20]) *For every $p \geq 1$ there exists a constant $N = N(p)$ such that every $(p-2)$-connected graph on $n \geq N$ vertices and at least $(p-2)n - \binom{p-1}{2} + 1$ edges has a K_p minor.*

By Theorem 2.3, Conjecture 2.4 is true for $p \leq 9$.

We next prove the following Lemma 2.5, which can be obtained from the (computer-assisted) proof of Lemma 3.7 in [20]. Here we give a computer-free proof of Lemma 2.5 so that the proof of Theorem 1.3 is also computer-free.

Lemma 2.5. *For $7 \leq t \leq 9$, let H be a graph with $2t-5$ vertices and $\alpha(H) = 2$. Then $H > K_{t-2} \cup K_1$.*

Proof. Suppose that H has no $K_{t-2} \cup K_1$ minor. Then $\omega(H) \leq t-3$. We claim that

(1) $\omega(H) \leq t-4$.

Suppose that $\omega(H) = t-3$. Let $K \subseteq H$ be isomorphic to K_{t-3}. Then $|H \backslash K| = t-2 \geq 5$. Clearly, $H \backslash K$ is not a clique. If $H \backslash K$ contains an induced 3-path, say P, with ends y, z, where a 3-path is a path with three vertices, then every vertex of K is adjacent to either y or z because $\alpha(H) = 2$. By contracting the path P into a single vertex, we see that $H[K \cup P] > K_{t-2}$ and so $H > K_{t-2} \cup K_1$, a contradiction. Thus $H \backslash K$ does not contain an induced path on three vertices. Since $\alpha(H) = 2$, it follows that $H \backslash K$ is a disjoint union of two cliques, say A_1 and A_2. For $i = 1, 2$, let $K_i = \{v \in V(K) : v \text{ is not adjacent to some vertex in } A_{3-i}\}$. Since $\alpha(H) = 2$, K_i is complete to A_i for each i. Thus $H \backslash (K_i \cup A_i)$ is a clique for each $i \in \{1, 2\}$ and so either $H \backslash (K_1 \cup A_1)$ or $H \backslash (K_2 \cup A_2)$ is a clique of size at least $t-2$, contrary to the fact that $\omega(H) \leq t-3$. This proves (1).

Let $q = \delta(H)$ and let $y \in V(H)$ be a vertex with $d(y) = q$. Let $J = H \backslash N[y]$. Since $\alpha(H) = 2$, J is a clique of size $2t-q-6$. By (1), $|J| = 2t-q-6 \leq t-4$ and so $q \geq t-2$. We next claim that

(2) for any $A \subseteq N(y)$ with $|A| \geq 6$, either $H[A \cup \{y\}]$ contains two vertex-disjoint induced 3-paths or $H[A]$ is a disjoint union of two cliques.

Suppose $H[A]$ is not a disjoint union of two cliques. Then $H[A]$ is connected because $\alpha(H) = 2$. We next show that $H[A \cup \{y\}]$ contains two vertex-disjoint induced 3-paths. By (1), $H[A]$ is not a clique and thus contains an induced 3-path, say abc, with ends a, c. Let $\{d_1, d_2, \ldots, d_s\} = A \backslash \{a, b, c\}$, where $s = |A| - 3 \geq 3$. Clearly $H[A \cup \{y\}]$ contains two vertex-disjoint induced 3-paths if $H[\{d_1, d_2, \ldots, d_s\}]$ is not a clique, since yd_i is an edge for $1 \leq i \leq s$. So we may assume that $H[\{d_1, d_2, \ldots, d_s\}] = K_s$. First assume that a is complete to $\{d_1, d_2, \ldots, d_s\}$. By (1), b is not complete to $\{d_1, d_2, \ldots, d_s\}$. We may assume that $bd_1 \notin E(H)$. Clearly $H[\{a, y, c\}]$ and $H[\{d_1, b, d_i\}]$ are two vertex-disjoint induced 3-paths if $bd_i \in E(H)$ for some $i \neq 1$. So we may assume that $bd_i \notin E(H)$. Now either $H[\{b, a, d_1\}]$ and $H[\{c, y, d_2\}]$ (if $cd_2 \notin E(H)$) or $H[\{a, d_2, c\}]$ and $H[\{b, y, d_1\}]$ (if $cd_2 \in E(H)$) are two vertex-disjoint induced 3-paths. Next assume that a is not complete to $\{d_1, d_2, \ldots, d_s\}$. We may assume that $ad_1 \notin E(H)$. Then $cd_1 \in E(H)$ because $\alpha(H) = 2$. By symmetry, we may assume that $cd_2 \notin E(H)$. Then $ad_2 \in E(H)$. Now either $H[\{c, d_1, d_2\}]$ and $H[\{a, y, d_3\}]$ (if $ad_3 \notin E(H)$) or $H[\{a, d_3, d_1\}]$ and $H[\{c, y, d_2\}]$ (if $ad_3 \in E(H)$) are two vertex-disjoint induced 3-paths, as desired. This completes the proof of (2).

Finally we claim that

(3) $q = t-2$.

Suppose $q \geq t-1$. By Theorem 2.1, $(t-4)(2t-6) - \binom{t-3}{2} \geq e(H\backslash y) \geq q|H|/2 - q = q(|H|-2)/2 \geq (t-1)(2t-7)/2$, which yields that $t = 9$ and $q = t-1 = 8$. Then H is a graph on thirteen vertices. Clearly, $J = K_4$. Let $z \in N(y)$ be such that $|N(z) \cap V(J)|$ is maximum. Since $e_H(J, N(y)) \geq 20$, we have $|N(z) \cap V(J)| \geq 3$. If $|N(z) \cap V(J)| = 4$, then $H[\{z\} \cup V(J)] = K_5$ and $|N(y)\backslash z| = 7$. Clearly $H > K_7 \cup K_1$ if $G[N[y]]\backslash z$ has two vertex-disjoint induced 3-paths. By (2), $G[N(y)]\backslash z$ is thus a disjoint union of two cliques, say A_1, A_2. By (1), we may assume that $A_1 = K_3$ and $A_2 = K_4$. Let $a \in A_1$. By (1) again, a is not complete to $\{z\} \cup J$ and thus $d_H(a) \leq 7$, contrary to the fact that $q = 8$. Thus $|N(z) \cap V(J)| = 3$. Let $z' \in V(J)$ be the non-neighbor of z. By the choice of z, every vertex in $N(y)$ has at least one non-neighbor in J and so $\delta(N(y)) \geq 4$. Since $d(z) \geq 8$, $|N(z) \cap N(y)| \geq 4$. By (1), $N(z) \cap N(y)$ is not a clique and so z' is adjacent to at least one vertex, say w, in $N(z) \cap N(y)$, because $\alpha(H) = 2$. Now the edge zw is dominating J, i.e., every vertex in J is adjacent to either z or w. Notice that $|N(y)\backslash\{z, w\}| = 6$. If $N[y]\backslash\{z, w\}$ contains two vertex-disjoint induced 3-paths, say P_1 and P_2, then $H > K_7 \cup K_1$ by contracting the edge zw and the two induced 3-paths P_1 and P_2 into three distinct vertices, respectively, a contradiction. Thus $N[y]\backslash\{z, w\}$ does not contain two vertex-disjoint induced 3-paths. By (2), $N(y)\backslash\{z, w\}$ is a disjoint union of two cliques, say B_1 and B_2. Since $\delta(N(y)) \geq 4$, we have $B_1 = B_2 = K_3$. By (1), $H[B_1 \cup \{z, w, y\}]$ is not a clique. Let $w' \in B_1$ be such that either $ww' \notin E(H)$ or $zw' \notin E(H)$. Since w' is adjacent to at most three vertices of J, we see that $d_H(w') \leq 7$, contrary to the fact that $q = 8$. This proves (3).

By (3), $q = t-2$. If $t = 7$, then H is a graph on nine vertices with $\delta(H) = 5$. Thus there exists a vertex $z \in V(H)$ such that $d_H(z) \geq 6$ and so $N[z]$ contains a K_4-subgraph because $\alpha(G[N(z)]) = 2$, contrary to (1). Hence $t \geq 8$. Now J is a clique of size $t-4$ and $|N(y)| = t-2 \geq 6$. Clearly $H > K_{t-2} \cup K_1$ if $N[y]$ contains two vertex-disjoint induced 3-paths, a contradiction. Thus by (2), $N(y)$ is a disjoint union of two cliques, say A_1 and A_2. For $i = 1, 2$, let $K_i = \{v \in H\backslash N[y] : v$ is not adjacent to some vertex in $A_{3-i}\}$. Since $\alpha(H) = 2$, K_i is complete to A_i for each i. Thus $H\backslash(K_i \cup A_i \cup \{y\})$ is a clique for each $i \in \{1, 2\}$ and so at least one of them is of size at least $t-3$, contrary to (1).

This completes the proof of Lemma 2.5. $\square$

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Suppose the assertion is false. Let G be a graph with no K_t minor such that G is not $(2t-6)$-colorable. We may choose such a graph G so that it is $(2t-5)$-contraction-critical. Let $x \in V(G)$ be of minimum degree. Since $K_{2,2,2,3,3}$ and each $(K_{2,2,2,2,2}, 5)$-cockade are 5-colorable, and every $(K_{1,2,2,2,2,2}, 6)$-cockade is 6-colorable, it follows from Theorem 2.1, Theorem 2.2 and Theorem 2.3 that $d(x) \leq 2t-5$. On the other hand, since G is $(2t-5)$-contraction-critical, by Lemma 1.6(i) applied to $G[N(x)]$, we see that $d(x) \geq (2t-5) - 2 + \alpha(G[N(x)])$. Clearly, $\alpha(G[N(x)]) \geq 2$, otherwise $G[N[x]]$ is a clique of size at least $2t-5 > t$, contrary to the fact that G has no

K_t minor. Thus $d(x) \geq 2t-5$ and so $d(x) = 2t-5 \geq t+2$. By Lemma 1.6(i) applied to $G[N(x)]$ again, we have $\alpha(G[N(x)]) = 2$. We claim that

(1) G has no K_{t-1}-subgraph.

Suppose G contains K_{t-1} as a subgraph. Let $H \subseteq G$ be isomorphic to K_{t-1}. Since $\delta(G) = d(x) \geq t+2$, every vertex in H is adjacent to at least one vertex in $G\backslash H$. Then $G\backslash H$ is disconnected, since otherwise $G > K_t$ by contracting $G\backslash H$ into a single vertex, a contradiction. Let G_1 be a component of $G\backslash H$. Then $N(G_1) := \{v \in V(H) :$ v is adjacent to a vertex in $G_1\}$ is a minimal separating set of G. In particular, $G[N(G_1)]$ is a clique, contrary to Lemma 1.6(ii). This proves (1). We next claim that

(2) for any $u \in N(x)$, $|N(x) \cap N(u)| \geq t-3$.

Suppose that there exists a vertex $u \in N(x)$ such that $|N(x) \cap N(u)| \leq t-4$. Since $\alpha(G[N(x)]) = 2$, $G[N(x)]$ contains a clique of size $|N(x)\backslash N[u]| \geq t-2$ and so $G[N[x]]$ has a K_{t-1}-subgraph, contrary to (1). This proves (2).

By Lemma 2.5, $G[N(x)] > K_{t-2} \cup K_1$. Let $y \in N(x)$ be such that $G[N(x)\backslash y] > K_{t-2}$. Clearly, y is not adjacent to every vertex in $N(x)\backslash y$, otherwise $G > G[N[x]] > K_t$, a contradiction. Let $\{y_1, \ldots, y_p\} = N(x)\backslash N[y]$, where $p = 2t-5-|N(x) \cap N[y]|$. Then y is not adjacent to $y_1, y_2, \ldots, y_p$. By (1) and (2), $G[N[y] \cap N(x)]$ is not a clique. Let uw be a missing edge in $G[N(y) \cap N(x)]$. By Lemma 1.7 applied to $G[N(x)]$ with $k = 2t-5$, $S = \{u, w\}$ and $M = \{yy_1, yy_2, \ldots, yy_p\}$, there exists a path P_i with ends y and y_i and all its internal vertices in $G\backslash N[x]$ for each $i \in \{1, 2, \ldots, p\}$. Note that the paths $P_1\backslash y_1, \ldots, P_p\backslash y_p$ have y as a common end. By contracting all $P_i\backslash y_i$ onto y, we see that $G > K_t$, a contradiction.

This completes the proof of Theorem 1.3. □

3. Proof of Theorem 1.4: coloring K_8^- minor free graphs

The primary purpose of this section is to prove Theorem 1.4 which states that every graph with no K_8^- minor is 9-colorable. We need the following results. Theorem 3.1 was proved by Dirac [5] for the cases $p = 5, 6$, and by Jakobsen [9] for the case $p = 7$.

Theorem 3.1. (Dirac [5], Jakobsen [8,9]) *For $p = 5, 6, 7$, if G is a graph with $n \geq p$ vertices and at least $(p-\frac{5}{2})n - \frac{1}{2}(p-3)(p-1)$ edges, then $G > K_p^-$, or G is a $(K_{p-1}, p-3)$-cockade when $p \neq 7$, or $p = 7$ and G is a $(K_{2,2,2,2}, K_6, 4)$-cockade.*

Jakobsen [9] also conjectured that Theorem 3.1 extends to $p = 8$, which was confirmed by the second author [19]:

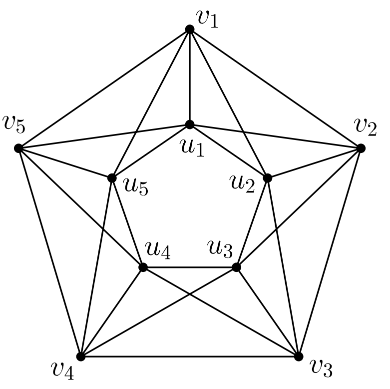

Fig. 2. The graph J.

Theorem 3.2. (Song [19]) *If G is a graph with $n \geq 8$ vertices and at least $\frac{1}{2}(11n - 35)$ edges, then $G > K_8^-$ or G is a $(K_{1,\,2,\,2,\,2,\,2},\, K_7,\, 5)$-cockade.*

The extremal function for K_p^- minors remains open for $p \geq 9$. The following lemma from [19] will also be needed.

Lemma 3.3. (Song [19]) *Let G be a graph with $8 \leq |G| \leq 10$ and $\delta(G) \geq 5$. Then either $G > K_6^- \cup K_1$ or G is isomorphic to one of $\overline{C_8}$, $\overline{C_4} + \overline{C_4}$, $\overline{K_3} + C_5$, $\overline{K_2} + \overline{C_6}$, $K_{2,\,3,\,3}$, or J, where J is the graph depicted in Fig. 2. In particular, all of these graphs are edge maximal (subject to not having a $K_6^- \cup K_1$-minor) with maximum degree $\leq |G| - 2$. Moreover, $\overline{C_8} > K_6$, $\overline{C_4} + \overline{C_4} > K_6$, and $J > K_6$.*

Notice that of the counterexamples listed in Lemma 3.3, only the graph J has ten vertices, and none has exactly nine vertices. We first prove the following lemma.

Lemma 3.4. *Let G be a graph with $n = 10$ vertices and $\alpha(G) = 2$. Then either $G > K_6^- \cup K_1$, or G contains a $K_5 \cup K_5$-subgraph, or G is isomorphic to the graph J depicted in Fig. 2.*

Proof. If $\delta(G) \geq 5$, then by Lemma 3.3, either $G > K_6^- \cup K_1$ or G is isomorphic to J. So we may assume that $\delta(G) \leq 4$. Let $x \in V(G)$ be such that $d(x) = \delta(G)$. Since $\alpha(G) = 2$, one can easily see that $G > K_6 \cup K_1$ if $d(x) \leq 3$. Hence we may further assume that $d(x) = 4$. Then $G \backslash N[x]$ must be a K_5 as $\alpha(G) = 2$. If $G[N[x]]$ is isomorphic to K_5, then G contains a $K_5 \cup K_5$-subgraph. Otherwise, some edge is missing from $G[N(x)]$, say $y, z \in N(x)$ with $yz \notin E(G)$. Then since $\alpha(G) = 2$, each vertex in $G \backslash N[x]$ must be adjacent to either y or z. Thus $G/\{x, y, z\} > K_6 \cup K_1$, as desired. This completes the proof of Lemma 3.4. $\square$

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. Let G be a graph with no K_8^- minor. Suppose for a contradiction that $\chi(G) \geq 10$. We may choose such a graph G so that it is 10-contraction-critical. Let $x \in V(G)$ be of minimum degree. Since G is 10-contraction-critical and has no K_8^- minor, by Lemma 1.6(i) applied to $G[N(x)]$, we see that $\alpha(G[N(x)]) \geq 2$ and $\delta(G) \geq 10$. On the other hand, since every $(K_{1,2,2,2,2}, K_7, 5)$-cockade is 7-colorable, by Theorem 3.2 we see that $\delta(G) \leq 10$. Thus $\delta(G) = 10$. By Lemma 1.6(i) applied to $G[N(x)]$, we have

(1) $\alpha(G[N(x)]) = 2$.

We next prove two claims.

(2) $G[N(x)]$ is not isomorphic to the graph J.

Suppose that $G[N(x)]$ is isomorphic to the graph J. Let the vertices of J be labeled as depicted in Fig. 2. By Lemma 1.7 applied to J with $S = \{v_2, v_5\}$ and $M = \{u_1u_3, u_1u_4, u_1v_3, u_1v_4, u_2u_5\}$ with $m = 2$, $r_1 = 4, r_2 = 1$, there exist paths P_{11}, P_{12}, P_{13}, P_{14}, P_{21} such that the paths P_{11}, P_{12}, P_{13}, P_{14}, P_{21} have ends $\{u_1, u_3\}$, $\{u_1, u_4\}$, $\{u_1, v_3\}$, $\{u_1, v_4\}$, and $\{u_2, u_5\}$, respectively, and all their internal vertices in $G\backslash N[x]$. Moreover, the paths P_{11}, P_{12}, P_{13}, P_{14} are vertex-disjoint from the path P_{21}. By contracting $(P_{11}\backslash u_3) \cup (P_{12}\backslash u_4) \cup (P_{13}\backslash v_3) \cup (P_{14}\backslash v_4)$ onto u_1, P_{21} into a single vertex, and $J[\{v_2, v_1, v_5\}]$ into a single vertex, we see that $G > K_8$, a contradiction. This proves (2).

(3) $G[N(x)]$ contains $K_5 \cup K_5$ as a subgraph.

Suppose that $G[N(x)]$ does not contain $K_5 \cup K_5$ as a subgraph. Then by (1), (2) and Lemma 3.4, we see that $G[N(x)] > K_6^- \cup K_1$. Let $y \in N(x)$ be a vertex such that $G[N(x)] \setminus \{y\} > K_6^-$. Clearly, y is not adjacent to every vertex in $N(x)\backslash y$, otherwise $G > G[N[x]] > K_8^-$, a contradiction. Let $\{y_1, \ldots, y_p\} = N(x)\backslash N[y]$, where $p = 10 - |N(x) \cap N[y]| \geq 1$. Then y is not adjacent to $y_1, y_2, \ldots, y_p$. Clearly, $G[N(x)]\backslash\{y, y_i\}$ is not a clique for all $i \in \{1, 2, \ldots, p\}$. By Lemma 1.7 applied p times to $G[N(x)]$ with $k = 10$, $s = 0$ and $m = 1$ (where $M = \{yy_i\}$ for $i = 1, 2, \ldots, p$), there exists a path P_i between y and y_i with its internal vertices in $G\backslash N[x]$ for each $i \in \{1, 2, \ldots, p\}$. Note that the paths $P_1, \ldots, P_p$ have y as a common end. By contracting each $P_i\backslash y_i$ onto y, we see that $G > K_8^-$, a contradiction. This proves (3).

By (3), x belongs to two different K_6-subgraphs of G. By Theorem 1.8, G is 7-connected. By Lemma 1.9, $G > K_8^-$. This contradiction completes the proof of Theorem 1.4. □

4. Proof of Theorem 1.5: coloring $K_8^=$ minor free graphs

We prove Theorem 1.5 in this section. The following result will be needed. Theorem 4.1 for the cases $p = 5, 6$ is due to Dirac [5], and Theorem 4.1 for the cases $p = 7, 8$ is due to Jakobsen [7,8].

Theorem 4.1. (Dirac [5], Jakobsen [7,8]) *For integer p with $5 \le p \le 8$, every graph with $n \ge p$ vertices and at least $(p-3)n - \frac{1}{2}(p-1)(p-4)$ edges either contains a $K_p^=$ minor or is a $(K_{p-1}, p-4)$-cockade.*

We are ready to prove Theorem 1.5.

Proof of Theorem 1.5. Suppose the assertion is false. Let G be a graph with no $K_8^=$ minor such that $\chi(G) \ge 9$. We may choose such a graph G so that it is 9-contraction-critical. Let $x \in V(G)$ be of minimum degree. Since G is 9-contraction-critical and has no $K_8^=$ minor, by Lemma 1.6(i) applied to $G[N(x)]$, we see that $\alpha(G[N(x)]) \ge 2$ and $d(x) \ge 9$. On the other hand, since each $(K_7, 4)$-cockade is 7-colorable, it follows from Theorem 4.1 for $p = 8$ that $d(x) \le 9$. Thus $d(x) = 9$, and so $\delta(G) = 9$. It follows from Theorem 4.1 for $p = 8$ again that

(1) G contains at least 28 vertices of degree 9.

Since G has no $K_8^=$ minor, by Lemma 1.6(i) applied to $G[N(x)]$,

(2) $\alpha(G[N(x)]) = 2$.

We next claim that

(3) $G[N(x)]$ contains K_5 as a subgraph.

Suppose that $G[N(x)]$ does not contain K_5 as a subgraph. Then $\omega(G[N(x)]) \le 4$ and by (2), $\delta(G[N(x)]) \ge 4$. We claim that $\delta(G[N(x)]) = 4$. Suppose that $\delta(G[N(x)]) \ge 5$. By Lemma 3.3 applied to $G[N(x)]$, we see that $G[N(x)] > K_6^- \cup K_1$. Let $y \in N(x)$ be such that $G[N(x)\backslash y] > K_6^-$. Clearly y has at least two non-neighbors in $G[N(x)\backslash y]$, otherwise $G[N[x]] > K_8^=$, a contradiction. Let $y_1, y_2, \dots, y_j \in N(x)\backslash y$ be all non-neighbors of y, where $j = |N(x)\backslash N[y]| \ge 2$. Since $\omega(G[N(x)]) \le 4$, $G[N(x) \cap N(y)]$ must have a missing edge, say uv. By Lemma 1.7 applied to $G[N(x)]$ with $S = \{u, v\}$ and $M = \{yy_1, \dots, yy_j\}$, there exist j paths $P_1, P_2, \dots, P_j$ such that each path P_i has ends $\{y, y_i\}$ and all its internal vertices in $G\backslash N[x]$. By contracting all the edges of each $P_i\backslash y_i$ onto y for all $i \in \{1, 2, \dots, j\}$, we see that $G > K_8^-$, a contradiction. This proves that $\delta(G[N(x)]) = 4$, as claimed.

Let $y \in N(x)$ be such that y has degree four in $G[N(x)]$ with $e(G[N(y) \cap N(x)])$ maximum. Let $Z = \{z_1, z_2, z_3, z_4\}$ be the set of all neighbors of y in $G[N(x)]$. Since $\omega(G[N(x)]) \le 4$, $G[N[y] \cap N(x)]$ is not complete. We may assume that $z_1z_2 \notin E(G)$. By (2), $G[N(x)\backslash N[y]]$ is isomorphic to K_4. Let $W = \{w_1, w_2, w_3, w_4\} = N(x)\backslash N[y]$. We next show that

$(*)$ each of z_3, z_4 has at most one neighbor in W.

Suppose, say z_4, is adjacent to at least two vertices in W. Then the subgraph induced on $W \cup \{z_4\}$ has a $K_5^=$ minor and thus $G[N[x]] > K_8^=$ if z_3 is adjacent to all vertices in $W \cup \{z_4\}$ (by contracting the path z_1yz_2 into a single vertex), a contradiction. Thus we may assume that z_3 is not adjacent to $u_1, \dots, u_i \in W \cup \{z_4\}$, where $1 \le i \le 5$. By Lemma 1.7 applied to $G[N(x)]$ with $S = \{z_1, z_2\}$ and $M = \{z_3u_1, \dots, z_3u_i\}$, there exist i paths $P_1, P_2, \dots, P_i$ such that for each $j = 1, 2, \dots, i$, the path P_j has ends $\{z_3, u_j\}$ and all its internal vertices in $G\backslash N[x]$. By contracting all $P_j\backslash u_j$ onto z_3, and the path z_1yz_2 into a single vertex, we see that $G > K_8^=$, a contradiction. This proves $(*)$.

We next claim that $G[N[y] \cap N(x)] = K_5^-$. Suppose $z_3z_4 \notin E(G)$. By symmetry, we may apply $(*)$ to the missing edge z_3z_4 in $G[N(x)]$ to obtain that each of z_1, z_2 has at most one neighbor in W. Hence $e_G(Z, W) \le 4$. On the other hand, since $\alpha(G[N(x)]) = 2$, each w_i must be adjacent to at least one of the vertices in $\{z_1, z_2\}$ and $\{z_3, z_4\}$, respectively, for all $i = 1, 2, 3, 4$. Thus $e_G(W, Z) \ge 8$, a contradiction. This proves that $z_3z_4 \in E(G)$ and thus $G[N[y] \cap N(x)]$ does not have two independent missing edges. Next if $z_1z_3 \notin E(G)$, then $z_2z_3 \in E(G)$ because $\alpha(G[N(x)]) = 2$. Since $G[N[y] \cap N(x)]$ does not have two independent missing edges, we see that $z_2z_4 \in E(G)$ and so $G[N[y] \cap N(x)]$ is a graph on five vertices with at least two edges missing. Since $\omega(G[N(x)]) \le 4$, we may assume that $z_1w_1 \notin E(G)$. Then w_1 must be adjacent to both z_2 and z_3 by (2). By applying $(*)$ to the missing edges z_1z_2 and z_1z_3, we see that $\{z_2, z_3\}$ is anti-complete to $\{w_2, w_3, w_4\}$ and z_4 has at most one neighbor in W. By (2), z_1 is complete to $\{w_2, w_3, w_4\}$. Since z_4 has at most one neighbor in W, we may assume that $w_4z_4 \notin E(G)$. Now w_4 has degree four in $G[N(x)]$ with $G[N[w_4] \cap N(x)] = K_5^-$, contrary to the choice of y. Thus $G[N[y] \cap N(x)] = K_5^-$, as claimed.

Clearly, z_1z_2 is the only missing edge in $G[N[y] \cap N(x)]$. Since $\delta(G[N(x)]) = 4$, each of z_1 and z_2 has at least one neighbor in W. By (2), each of $w_1, \dots, w_4$ is adjacent to at least one of z_1, z_2, and so either z_1 or z_2 has at least two neighbors in W. By symmetry, we may assume that $|N(z_1) \cap W| \ge |N(z_2) \cap W|$. On the other hand, each vertex in Z has at least one non-neighbor in W as $\omega(G[N(x)]) \le 4$. Thus, z_1 has either two or three neighbors in W. We consider the following two cases.

First, assume that z_1 has exactly two neighbors in W, say w_1, w_2. Then z_2 must have exactly two neighbors w_3, w_4 in W. By $(*)$, each of z_3, z_4 has at most one neighbor in W. We may assume that z_4 is not adjacent to w_3, w_4, and $z_3w_2 \notin E(G)$. By Lemma 1.7 applied to $G[N(x)]$ with $S = \{z_1, z_2\}$ and $M = \{yw_1, yw_2, z_3w_2, z_4w_3, z_4w_4\}$, there exist three pairwise vertex-disjoint paths P_1, P_2, Q_1 and two vertex-disjoint paths P_3, Q_2 such that the paths P_1, P_2, P_3, Q_1, and Q_2 have ends $\{y, w_1\}$, $\{z_3, w_2\}$, $\{y, w_2\}$, $\{z_4, w_3\}$ and

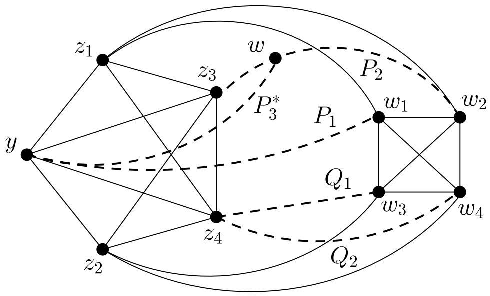

Fig. 3. z_1 has exactly two neighbors in W.

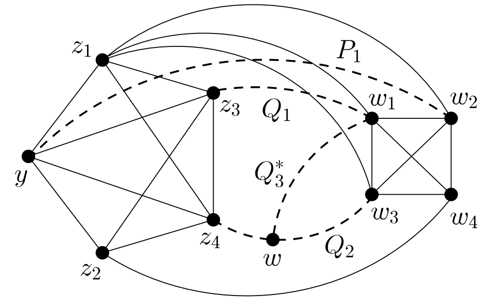

Fig. 4. z_1 has exactly three neighbors in W.

$\{z_4, w_4\}$, respectively, and all their internal vertices in $G\backslash N[x]$, as depicted in Fig. 3. Notice that P_2 and P_3 are not necessarily vertex-disjoint. If P_2 and P_3 have only w_2 in common, then contracting $P_1\backslash w_1, P_3\backslash w_2$ onto y, $P_2\backslash w_2$ onto z_3, and $Q_1\backslash w_3, Q_2\backslash w_4$ onto z_4, and w_1w_3, w_2w_4 into two distinct vertices yields a $K_8^=$ minor in G, a contradiction. Thus P_3 and P_2 must have an internal vertex in common. Let w be the first vertex on P_3 (when P_3 is read from y to w_2) that is also on P_2. Then $w \notin V(P_1)$. Let P_3^* be the subpath of P_3 from y to w, P_2^* be the subpath of P_2 from w to w_2. Notice that $P_3^*\backslash w$ is vertex-disjoint from P_2 but not necessarily internally disjoint from P_1. Now contracting $P_1\backslash w_1$, $P_3^*\backslash w$ onto y, P_2^* onto w_2, $P_2\backslash P_2^*$ onto z_3, $Q_1\backslash w_3, Q_2\backslash w_4$ onto z_4, and w_1w_3, w_2w_4 into two distinct vertices yields another $K_8^=$ minor in G, a contradiction.

It remains to consider the case when z_1 has exactly three neighbors, say w_1, w_2, w_3 in W. Then z_2 is adjacent to w_4. By (∗), we may assume that w_1 is not adjacent to z_3, z_4, and $z_4w_3 \notin E(G)$. By Lemma 1.7 applied to $G[N(x)]$ with $S = \{z_1, z_2\}$ and $M = \{yw_2, z_3w_1, z_4w_3, z_4w_1\}$, there exist three pairwise vertex-disjoint paths P_1, Q_1, Q_2, and another path Q_3 such that the paths P_1, Q_1, Q_2, Q_3 have ends $\{y, w_2\}$, $\{z_3, w_1\}$, $\{z_4, w_3\}$, and $\{z_4, w_1\}$, respectively, and all their internal vertices in $G\backslash N[x]$, as depicted in Fig. 4. Notice that P_1 is vertex-disjoint from Q_j for $j = 1, 2, 3$, but that Q_3 is not necessarily internally vertex-disjoint from either Q_1 or Q_2. If Q_3 and Q_2 have only z_4 in common, then we obtain a $K_8^=$ minor by contracting P_1 and z_2w_4 into two distinct vertices, $Q_1\backslash z_3$, $Q_3\backslash z_4$ onto w_1, and $Q_2\backslash z_4$ onto w_3, a contradiction. Thus Q_3 and Q_2

must have an internal vertex in common. Let w be the first vertex on Q_3 (when Q_3 is read from w_1 to z_4) that is also on Q_2. Then $w \notin V(Q_1)$. Let Q_3^* be the subpath of Q_3 from w_1 to w, Q_2^* be the subpath of Q_2 from w to z_4. Notice that $Q_3^* \backslash w$ is vertex-disjoint from Q_2 but not necessarily internally disjoint from Q_1. Now we obtain another $K_8^=$ minor by contracting P_1 and z_2w_4 into two distinct vertices, $Q_1 \backslash z_3$, $Q_3^* \backslash w$ onto w_1, Q_2^* onto z_4, and $Q_2 \backslash Q_2^*$ onto w_3, a contradiction. This proves (3).

By (3), every vertex of degree 9 belongs to some K_6-subgraph of G. By (1), G contains at least five different K_6-subgraphs. By Theorem 1.8, G is 7-connected and thus $G > K_8^-$ by Lemma 1.9. This contradiction completes the proof of Theorem 1.5. □

5. Concluding remarks

It seems very difficult to prove that every graph with no K_7 minor is 7-colorable. We establish in [17] the properties of 8-contraction-critical graphs with no K_7 minor to shed some light on this open problem. As pointed out by Robin Thomas (personal communication with the second author), one might be able to settle this open problem by using the key ideas in [16]. However, it seems very hard to prove that every 8-contraction-critical graph is 8-connected. Theorem 2.1 is such a nice result. We believe that Mader's bound for the extremal function for K_p minors is true as follows:

Conjecture 5.1. *For every $p \geq 1$, every graph G on n vertices and at least $(p-2)n - \binom{p-1}{2} + 1$ edges either has a K_p minor or is $(p-1)$-colorable.*

By Theorem 2.3, Conjecture 5.1 is true for $p \leq 9$. As mentioned earlier, Lemma 1.7 turns out to be very powerful. We believe that the application of Lemma 1.7 that we have developed in this paper is of independent interest. To end this section, we apply Lemma 1.7 along with a new idea (namely, considering the chromatic number of $G[N(x)]$) to prove that the truth of Conjecture 5.1 implies that every graph with no K_p minor is $(2p-6)$-colorable for all $p \geq 6$. Since Conjecture 5.1 is true for $p \leq 9$, we see that Theorem 5.2 implies Theorem 1.3.

Theorem 5.2. *If Conjecture 5.1 is true, then every graph with no K_p minor is $(2p-6)$-colorable for all $p \geq 6$.*

Proof. Suppose the assertion is false. Among all minimum counterexamples, we choose G so that G has no K_p minor and G is $(2p-5)$-contraction-critical. Let $x \in V(G)$ be such that $d(x) = \delta(G)$. By the assumed truth of Conjecture 5.1, $d(x) \leq 2p-5$. On the other hand, by Lemma 1.6(i) applied to $G[N(x)]$, we see that $\alpha(G[N(x)]) \geq 2$ and $d(x) \geq 2p-5$. Hence $d(x) = 2p-5$. By Lemma 1.6(i) again, we have

(1) $\alpha(G[N(x)]) = 2$.

Our strategy now will be to examine the subgraph $G[N(x)]$ and its chromatic number. We next prove a series of claims.

(2) $\omega(G[N(x)]) \le p-3$, and so $\delta(G[N(x)]) \ge p-3$.

Suppose that $\omega(G[N(x)]) \ge p-2$. Let $H \subseteq G[N[x]]$ be isomorphic to K_{p-1}. Since $\delta(G) = 2p-5$, every vertex in H is adjacent to $p-3$ vertices in $G\backslash H$. Then $G\backslash H$ is disconnected, otherwise $G > K_p$ by contracting $G\backslash H$ into a single vertex, a contradiction. Let G_1 be a component of $G\backslash H$. Then $N(G_1) := \{v \in V(H) : v \text{ is adjacent to a vertex in } G_1\}$ is a minimal separating set of G, in particular, $G[N(G_1)]$ is a clique, contrary to Lemma 1.6(ii). Thus $\omega(G[N(x)]) \le p-3$. By (1), we see that $\delta(G[N(x)]) \ge p-3$. This proves (2).

(3) $\chi(G[N(x)]) = p-2$.

Suppose to the contrary that $\chi(G[N(x)]) \ne p-2$. By (1), it is clear that $\chi(G[N(x)]) \ge p-2$. Thus $\chi(G[N(x)]) = t$ for some $t \ge p-1$. Let $V_1, \dots, V_t$ be the color classes of any proper t-coloring of $G[N(x)]$. By (1), we may assume that the color classes are ordered so that $V_i = \{a_i\}$ for $i = 1, 2, \dots, 2t-2p+5$ and $V_j = \{a_j, b_j\}$ for $j = 2t-2p+6, \dots, t$. Let $r = 2t-2p+6 \ge 4$. Notice that $\{a_1, a_2, \dots, a_{r-1}\}$ induces a clique in $G[N(x)]$, and so $r \le p-2$ by (2). Since $\chi(G[N(x)]) = t$, we see that there is at least one edge between any pair of color classes $V_1, \dots, V_t$ in G. In particular, a_i is adjacent to either a_j or b_j for each $i \in \{1, 2, \dots, r-1\}$ and each $j \in \{r, \dots, t\}$. Notice that if $t = p-1$, then $r = 4$. Suppose that $t \ge p$ or that $t = p-1$ and a_1, a_2, a_3 have a common neighbor in $N(x)\backslash\{a_1, a_2, a_3\}$, say a_4. By Lemma 1.7 applied to $G[N(x)]$ with $S = \{a_r, b_r\}$ and $M = \{a_{r+1}b_{r+1}, \dots, a_tb_t\}$, there exist $t-r$ pairwise vertex-disjoint paths $P_{r+1}, \dots, P_t$ such that each P_j has ends $\{a_j, b_j\}$ and all its internal vertices in $G\backslash N[x]$. By contracting each P_j to a single vertex for all $j \in \{r+1, \dots, t\}$, together with $x, a_1, \dots, a_{r-1}$ if $t \ge p$; and together with x, a_1, a_2, a_3, a_4 if $t = p-1$ where $a_4 = a_r$ is a common neighbor of a_1, a_2, a_3, we obtain a clique minor on $(t-r)+r = t \ge p$ vertices in the former case and $(t-r)+r+1 = t+1 = p$ vertices in the latter case, a contradiction. Thus $t = p-1$ and a_1, a_2, a_3 have no common neighbor in $N(x)\backslash\{a_1, a_2, a_3\}$.

Since each of a_1, a_2, a_3 is adjacent to either a_5 or b_5, by symmetry, we may assume that a_5 is adjacent to a_1 and a_2, but not to a_3. Then b_5 is adjacent to a_3. We may assume that b_5 is not adjacent to a_1 because a_1, a_2, a_3 have no common neighbor in $N(x)\backslash\{a_1, a_2, a_3\}$. For the worst case scenario, we may further assume that b_5 is not adjacent to a_2. By Lemma 1.7 applied to $G[N(x)]$ with $S = \{a_4, b_4\}$ and $M = \{a_6b_6, \dots, a_tb_t, b_5a_1, b_5a_2, b_5a_5, a_5a_3\}$, there exist pairwise vertex-disjoint paths $P_6, \dots, P_t$ such that each P_j has ends $\{a_j, b_j\}$ and all its internal vertices in $G\backslash N[x]$; and paths Q_1, Q_2, Q_5, Q with ends $\{b_5, a_1\}$, $\{b_5, a_2\}$, $\{b_5, a_5\}$ and $\{a_5, a_3\}$, respectively, and all their internal vertices in $G\backslash N[x]$. Notice that each P_j is vertex-disjoint from Q_1, Q_2, Q_5, Q; Q is vertex-disjoint from Q_1, Q_2 but not necessarily from Q_5; and Q_5 is

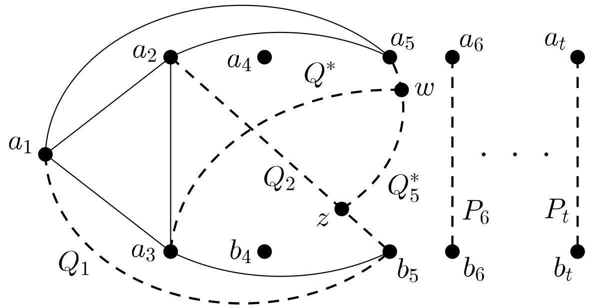

Fig. 5. Finding a K_p minor when $\chi(G[N(x)]) = p - 1$ and $z \in V(Q_2)$.

not necessarily vertex-disjoint from Q_1, Q_2. Let z be the first vertex on Q_5 (when read from a_5 to b_5) that is also on $Q_1 \cup Q_2$. Note that z could be b_5. Let Q_5^* be the subpath of Q_5 between a_5 and z. Then Q is vertex-disjoint from $Q_5 \backslash Q_5^*$. Let w be the first vertex on Q (when read from a_3 to a_5) that is also on Q_5^*. Note that w could be a_5, and that $w \neq z$ because w, z are colored differently when applying Lemma 1.7. Let Q^* be the subpath of Q between w and a_3, as depicted in Fig. 5. By contracting each P_j to a single vertex for all $j \in \{6, \ldots, t\}$, $Q_1 \backslash a_1$ and $Q_2 \backslash a_2$ onto b_5, $Q^* \backslash w$ onto a_3, and $Q_5^* \backslash z$ onto a_5, together with the vertices x, a_1, a_2, a_3, we obtain a K_p minor, a contradiction. This proves (3).

(4) $\delta(G[N(x)]) \geq p - 2$.

Suppose that there exists a vertex $y \in N(x)$ such that $\delta(G[N(x)]) \leq p - 3$. By (2), we have $\delta(G[N(x)]) \geq p - 3$. Thus y has exactly $p - 3$ neighbors in $G[N(x)]$. Then $G[N(x) \backslash N[y]]$ is a clique of $G[N(x)]$ with $p - 3$ vertices. Furthermore, by (2), $G[N(x) \cap N(y)]$ must have some missing edge, say uv. By (1), every vertex of $N(x) \backslash N[y]$ is adjacent to u or v in G. Let $w \in N(x) \cap N(y)$ be such that $w \neq u, v$. This is possible because $|N(x) \cap N(y)| = p - 3 \geq 3$. Now by contracting the path uyv into a single vertex, we see that $G[N(x) \backslash w] > K_{p-2}$. Clearly w is not adjacent to every vertex in $N(x) \backslash w$ in G, or else $G > G[N[x]] > K_p$, a contradiction. Let $\{w_1, \ldots, w_\ell\} = N(x) \backslash N[w]$, where $\ell \geq 1$. Again, by (2), $G[N(w) \cap N(x)]$ must have some missing edge, say ab. By Lemma 1.7 applied to $G[N(x)]$ with $S = \{a, b\}$ and $M = \{ww_1, \ldots, ww_\ell\}$, there exist paths $P_1, \ldots, P_\ell$ such that each P_i has ends $\{w, w_i\}$ and all internal vertices in $G \backslash N[x]$. Now by contracting each $P_i \backslash w_i$ onto w, we see that $G > K_p$, a contradiction. This completes the proof of (4).

By (2) and (4), $G[N(x)]$ does not contain K_{p-2} as a subgraph and $\delta(G[N(x)]) \geq p - 2$. By (3), $\chi(G[N(x)]) = p - 2$. Let $V_1, \ldots, V_{p-2}$ be the color classes of any proper $(p-2)$-coloring of $G[N(x)]$. By (1), we may assume that the color classes are ordered so that $V_1 = \{a_1\}$ and $V_j = \{a_j, b_j\}$ for $j = 2, \ldots, p - 2$. Since $\chi(G[N(x)]) = p - 2$, we see that there is at least one edge between any pair of color classes $V_1, \ldots, V_{p-2}$ in G. By (4), a_1 must be complete to some color class $V_i \in \{V_2, \ldots, V_{p-2}\}$, say V_2. By (4) again, a_2 and b_2 must have one common neighbor in some color class $V_i \in$

$\{V_3, \ldots, V_{p-2}\}$, say V_3. We may further assume that a_3 is adjacent to both a_2, b_2. By symmetry, we may assume that b_3 is adjacent to a_2. By Lemma 1.7 applied to $G[N(x)]$ with $S = \{a_2, b_2\}$ and $M = \{b_3a_1, b_3a_3, a_4b_4, \ldots, a_{p-2}b_{p-2}\}$, there exist paths P_1, P_2 and pairwise vertex-disjoint paths $Q_4, \ldots, Q_{p-2}$ such that P_1, P_2 have ends $\{b_3, a_1\}$ and $\{b_3, a_3\}$, respectively; each Q_j has ends $\{a_j, b_j\}$ and all such paths have their internal vertices in $G\backslash N[x]$. By contracting $P_1\backslash a_1, P_2\backslash a_3$ onto b_3, the edge b_2a_3 onto a_3, and each Q_j into a single vertex for $4 \le j \le p-2$, we see that $G > K_p$, a contradiction.

This completes the proof of Theorem 5.2. □

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