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Double-critical graph conjecture for claw-free graphs

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ABSTRACT

A connected graph G with chromatic number t is *double-critical* if $G \setminus \{x, y\}$ is $(t-2)$-colorable for each edge $xy \in E(G)$. The complete graphs are the only known examples of double-critical graphs. A long-standing conjecture of Erdős and Lovász from 1966, which is referred to as the *Double-Critical Graph Conjecture*, states that there are no other double-critical graphs. That is, if a graph G with chromatic number t is double-critical, then G is the complete graph on t vertices. This has been verified for $t \leq 5$, but remains open for $t \geq 6$. In this paper, we first prove that if G is a non-complete, double-critical graph with chromatic number $t \geq 6$, then no vertex of degree $t+1$ is adjacent to a vertex of degree $t+1$, $t+2$, or $t+3$ in G. We then use this result to show that the Double-Critical Graph Conjecture is true for double-critical graphs G with chromatic number $t \leq 8$ if G is claw-free.

1. Introduction

All graphs considered in this paper are finite and without loops or multiple edges. For a graph G, we will use $V(G)$ to denote the vertex set, $E(G)$ the edge set, $e(G)$ the number of edges, $\alpha(G)$ the independence number, $\omega(G)$ the clique number, $\chi(G)$ the chromatic number, and $\overline{G}$ the complement of G. For a vertex $x \in V(G)$, we will use $N_G(x)$ to denote the set of vertices in G which are adjacent to x. We define $N_G[x] = N_G(x) \cup \{x\}$ and $d_G(x) = |N_G(x)|$. Given vertex sets $A, B \subseteq V(G)$, we say that A is *complete to* (resp. *anti-complete to*) B if for every $a \in A$ and every $b \in B$, $ab \in E(G)$ (resp. $ab \notin E(G)$). The subgraph of G induced by A, denoted $G[A]$, is the graph with vertex set A and edge set $\{xy \in E(G) : x, y \in A\}$. We denote by $B \setminus A$ the set $B - A$, $e_G(A, B)$ the number of edges between A and B in G, and $G \setminus A$ the subgraph of G induced on $V(G) \setminus A$, respectively. If $A = \{a\}$, we simply write $B \setminus a$, $e_G(a, B)$, and $G \setminus a$, respectively. A graph H is an *induced subgraph* of a graph G if $V(H) \subseteq V(G)$ and $H = G[V(H)]$. A graph G is *claw-free* if G does not contain $K_{1,3}$ as an induced subgraph. Given two graphs G and H, the *union* of G and H, denoted $G \cup H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. Given two isomorphic graphs G and H, we may (with a slight but common abuse of notation) write $G = H$. A cycle with $t \geq 3$ vertices is denoted by C_t. Throughout this paper, a proper vertex coloring of a graph G with k colors is called a *k-coloring* of G.

In 1966, the following conjecture of Lovász was published by Erdős [6] and is known as the Erdős–Lovász Tihany Conjecture.

Conjecture 1.1. *For any integers $s, t \geq 2$ and any graph G with $\omega(G) < \chi(G) = s + t - 1$, there exist disjoint subgraphs G_1 and G_2 of G such that $\chi(G_1) \geq s$ and $\chi(G_2) \geq t$.*

To date, Conjecture 1.1 has been shown to be true only for values of $(s, t) \in \{(2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (3, 5)\}$. The case $(2, 2)$ is trivial. The case $(2, 3)$ was shown by Brown and Jung in 1969 [3]. Mozhan [10] and Stiebitz [14] each independently showed the case $(2, 4)$ in 1987. The cases $(3, 3)$, $(3, 4)$, and $(3, 5)$ were also settled by Stiebitz in 1987 [15].

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Recent work on the Erdős–Lovász Tihany Conjecture has focused on proving the conjecture for certain classes of graphs. Kostochka and Stiebitz [9] showed the conjecture holds for line graphs. Balogh, Kostochka, Prince, and Stiebitz [2] then showed that the conjecture holds for all quasi-line graphs and all graphs G with $\alpha(G) = 2$. More recently, Chudnovsky, Fradkin, and Plumettaz [4] proved the following slight weakening of Conjecture 1.1 for claw-free graphs, the proof of which is long and relies heavily on the structure theorem for claw-free graphs developed by Chudnovsky and Seymour [5].

Theorem 1.2. *Let G be a claw-free graph with $\chi(G) > \omega(G)$. Then there exists a clique K with $|V(K)| \leq 5$ such that $\chi(G \setminus V(K)) > \chi(G) - |V(K)|$.*

The most recent result related to the Erdős–Lovász Tihany Conjecture is due to Stiebitz [13], who showed that for integers $s, t \geq 2$, any graph G with $\omega(G) < \chi(G) = s + t - 1$ contains disjoint subgraphs G_1 and G_2 of G with either $\chi(G_1) \geq s$ and $\text{col}(G_2) \geq t$, or $\text{col}(G_1) \geq s$ and $\chi(G_2) \geq t$, where $\text{col}(H)$ denotes the coloring number of a graph H.

If we restrict $s = 2$ in Conjecture 1.1, then the Erdős–Lovász Tihany Conjecture states that for any graph G with $\chi(G) > \omega(G) \geq 2$, there exists an edge $xy \in E(G)$ such that $\chi(G \setminus \{x, y\}) \geq \chi(G) - 1$. To prove this special case of Conjecture 1.1, suppose for a contradiction that no such edge exists. Then $\chi(G \setminus \{x, y\}) = \chi(G) - 2$ for every edge $xy \in E(G)$. This motivates the definition of double-critical graphs. A connected graph G is double-critical if for every edge $xy \in E(G)$, $\chi(G \setminus \{x, y\}) = \chi(G) - 2$. A graph G is *t-chromatic* if $\chi(G) = t$. We are now ready to state the following conjecture, which is referred to as the *Double-Critical Graph Conjecture*, due to Erdős and Lovász [6].

Conjecture 1.3. *Let G be a double-critical, t-chromatic graph. Then $G = K_t$.*

Since Conjecture 1.3 is a special case of Conjecture 1.1, it has been settled in the affirmative for $t \leq 5$ [10,14], for line graphs [9], and for quasi-line graphs and graphs with independence number two [2]. Representing a weakening of Conjecture 1.3, Kawarabayashi, Pedersen, and Toft [8] have shown that any double-critical, t-chromatic graph contains K_t as a minor for $t \in \{6, 7\}$. As a further weakening, Pedersen [11] showed that any double-critical, 8-chromatic graph contains K_8^- as a minor. Albar and Gonçalves [1] later proved that any double-critical, 8-chromatic graph contains K_8 as a minor. Their proof is computer-assisted. The present authors [12] gave a computer-free proof of the same result and further showed that any double-critical, t-chromatic graph contains K_9 as a minor for all $t \geq 9$. We note here that Theorem 1.2 does not completely settle Conjecture 1.3 for all claw-free graphs. Recently, Huang and Yu [7] proved that the only double-critical, 6-chromatic, claw-free graph is K_6. We prove the following main results in this paper. Theorem 1.4 is a generalization of a result obtained in [8] that no two vertices of degree $t + 1$ are adjacent in any non-complete, double-critical, t-chromatic graph.

Theorem 1.4. *If G is a non-complete, double-critical, t-chromatic graph with $t \geq 6$, then for any vertex $x \in V(G)$ with $d_G(x) = t + 1$, the following hold:*

(a) *$e(\overline{G[N_G(x)]}) \geq 8$; and*
(b) *for any vertex $y \in N_G(x)$, $d_G(y) \geq t + 4$. Furthermore, if $d_G(y) = t + 4$, then $|N_G(x) \cap N_G(y)| = t - 2$ and $\overline{G[N_G(x)]}$ contains either only one cycle, which is isomorphic to C_8, or exactly two cycles, each of which is isomorphic to C_5.*

Corollary 1.5 follows immediately from Theorem 1.4.

Corollary 1.5. *If G is a non-complete, double-critical, t-chromatic graph with $t \geq 6$, then no vertex of degree $t + 1$ is adjacent to a vertex of degree $t + 1$, $t + 2$, or $t + 3$ in G.*

We then use Corollary 1.5 to prove the following main result.

Theorem 1.6. *Let G be a double-critical, t-chromatic graph with $t \in \{6, 7, 8\}$. If G is claw-free, then $G = K_t$.*

The rest of this paper is organized as follows. In Section 2, we first list some known properties of non-complete, double-critical graphs obtained in [8] and then establish a few new ones. In particular, Lemma 2.4 turns out to be very useful. Our new lemmas lead to a very short proof of Theorem 1.6 for $t = 6, 7$, which we place at the end of Section 2. We prove the remainder of our main results in Section 3.

2. Preliminaries

The following is a summary of the basic properties of non-complete, double-critical graphs shown by Kawarabayashi, Pedersen, and Toft in [8].

Proposition 2.1. *If G is a non-complete, double-critical, t-chromatic graph, then all of the following are true.*

(a) *G does not contain K_{t-1} as a subgraph.*
(b) *For all edges xy, every $(t - 2)$-coloring $c : V(G) \setminus \{x, y\} \to \{1, 2, \dots, t - 2\}$ of $G \setminus \{x, y\}$, and any non-empty sequence $j_1, j_2, \dots, j_i$ of i different colors from $\{1, 2, \dots, t - 2\}$, there is a path of order $i + 2$ with vertices $x, v_1, v_2, \dots, v_i, y$ in order such that $c(v_k) = j_k$ for all $k \in \{1, 2, \dots, i\}$.*

(c) *For any edge $xy \in E(G)$, x and y have at least one common neighbor in every color class of any $(t-2)$-coloring of $G \setminus \{x, y\}$. In particular, every edge $xy \in E(G)$ belongs to at least $t-2$ triangles.*
(d) *There exists at least one edge $xy \in E(G)$ such that x and y share a common non-neighbor in G.*
(e) *For any edge $xy \in E(G)$, the subgraph of G induced by $N_G(x) \setminus N_G[y]$ contains no isolated vertices. In particular, no vertex of $N_G(x)$ can have degree one in $\overline{G[N_G(x)]}$.*
(f) $\delta(G) \geq t+1$.
(g) *For any vertex $x \in V(G)$, $\alpha(G[N_G(x)]) \leq d_G(x) - t + 1$.*
(h) *For any vertex x with at least one non-neighbor in G, $\chi(G[N_G(x)]) \leq t-3$.*
(i) *For any $x \in V(G)$ with $d_G(x) = t+1$, $\overline{G[N_G(x)]}$ is the union of isolated vertices and cycles of length at least five. Furthermore, there must be at least one such cycle in $\overline{G[N_G(x)]}$.*
(j) *No two vertices of degree $t+1$ are adjacent in G.*

We next establish some new properties of non-complete, double-critical graphs.

Lemma 2.2. *Let G be a double-critical, t-chromatic graph and let $x \in V(G)$. If $d_G(x) = |V(G)|-1$, then $G \setminus x$ is a double-critical, $(t-1)$-chromatic graph.*

Proof. Let uv be any edge of $G \setminus x$. Clearly, $\chi(G \setminus x) = t-1$. Since G is double-critical, $\chi(G \setminus \{u, v\}) = t-2$ and so $\chi(G \setminus \{u, v, x\}) = t-3$ because x is adjacent to all the other vertices in $G \setminus \{u, v\}$. Hence $G \setminus x$ is double-critical and $(t-1)$-chromatic. ■

Lemma 2.3. *If G is a non-complete, double-critical, t-chromatic graph, then for any $x \in V(G)$ with at least one non-neighbor in G, $\chi(G \setminus N_G[x]) \geq 3$. In particular, $G \setminus N_G[x]$ must contain an odd cycle, and so $d_G(x) \leq |V(G)|-4$.*

Proof. Let x be any vertex in G with $d_G(x) < |V(G)|-1$ and let $H = G \setminus N_G[x]$. Suppose that $\chi(H) \leq 2$. Since $d_G(x) < |V(G)|-1$, H contains at least one vertex. Let $y \in V(H)$ be adjacent to a vertex $z \in N_G(x)$. This is possible because G is connected. If H has no edge, then $G \setminus (V(H) \cup \{z\})$ has a $(t-2)$-coloring c, which can be extended to a $(t-1)$-coloring of G by assigning all vertices in $V(H)$ the color $c(x)$ and assigning a new color to the vertex z, a contradiction. Thus H must contain at least one edge, and so $\chi(H) = 2$. Let (A, B) be a bipartition of H. Now $G \setminus H$ has a $(t-2)$-coloring c^*, which again can be extended to a $(t-1)$-coloring of G by assigning all vertices in A the color $c^*(x)$ and all vertices in B the same new color, a contradiction. This proves that $\chi(H) \geq 3$, and so H must contain an odd cycle. Therefore $d_G(x) \leq |V(G)|-4$. ■

Lemma 2.4. *Let G be a double-critical, t-chromatic graph. For any edge $xy \in E(G)$, let c be any $(t-2)$-coloring of $G \setminus \{x, y\}$ with color classes $V_1, V_2, \ldots, V_{t-2}$. Then the following two statements are true.*

(a) *For any $i, j \in \{1, 2, \ldots, t-2\}$ with $i \neq j$, if $N_G(x) \cap N_G(y) \cap V_i$ is anti-complete to $N_G(x) \cap V_j$, then there exists at least one edge between $(N_G(y) \setminus N_G(x)) \cap V_i$ and $N_G(x) \cap V_j$ in G. In particular, $(N_G(y) \setminus N_G(x)) \cap V_i \neq \emptyset$.*
(b) *Assume that $d_G(x) = t+1$ and y belongs to a cycle of length $k \geq 5$ in $\overline{G[N_G(x)]}$.*
 (b1) *If $k \geq 7$, then $d_G(y) \geq t + e(\overline{G[N(x)]}) - 4$;*
 (b2) *If $k = 6$, then $d_G(y) \geq \max\{t+2, t + e(\overline{G[N_G(x)]}) - 5\}$; and*
 (b3) *If $k = 5$, then $d_G(y) \geq \max\{t+2, t + e(\overline{G[N_G(x)]}) - 6\}$.*

Proof. Let G, x, y, c be as given in the statement. For any $i, j \in \{1, 2, \ldots, t-2\}$ with $i \neq j$, if $N_G(x) \cap N_G(y) \cap V_i$ is anti-complete to $N_G(x) \cap V_j$, then G is non-complete. By Proposition 2.1(b), there must exist a path x, u_j, u_i, y in G such that $c(u_j) = j$ and $c(u_i) = i$. Clearly, $u_ju_i \in E(G)$ and $u_j \in N_G(x) \cap V_j$. Since $N_G(x) \cap N_G(y) \cap V_i$ is anti-complete to $N_G(x) \cap V_j$, we see that $u_i \in (N_G(y) \setminus N_G(x)) \cap V_i$. This proves Lemma 2.4(a).

To prove Lemma 2.4(b), let $H = \overline{G[N_G(x)]}$. Assume that $d_G(x) = t+1$ and that y belongs to a cycle, say C_k, of H, where $k \geq 5$. By Proposition 2.1(j), $d_G(y) \geq t+2$, and by Proposition 2.1(i), H is the union of isolated vertices and cycles of length at least five. Clearly, $|N_G(x) \cap N_G(y)| = t-2$. By Proposition 2.1(c), we may assume that $V_i \cap (N_G(x) \cap N_G(y)) = \{v_i\}$ for all $i \in \{1, \ldots, t-2\}$. Then $N_G(x) \cap N_G(y) = \{v_1, \ldots, v_{t-2}\}$. Let $\{a, b\} = N_G(x) \setminus N_G[y]$. Since a and b are neighbors of y in a cycle of length at least 5 in H, $ab \in E(G)$. We may further assume that $a \in V_1$ and $b \in V_2$. Then v_1aybv_2 forms a path on five vertices of C_k, since $v_1, a \in V_1$ and $v_2, b \in V_2$. If $k \geq 6$, then $v_1v_2 \in E(G)$ and both v_1 and v_2 have precisely one non-neighbor in $\{v_3, v_4, \ldots, v_{t-2}\}$. We may assume that $v_1v_3 \notin E(G)$ and $v_2v_\ell \notin E(G)$, where $\ell = 3$ if $k = 6$, and $\ell = 4$ if $k \geq 7$. For any $i, j \in \{3, 4, \ldots, t-2\}$ with $i \neq j$, if $v_iv_j \notin E(G)$, then by Lemma 2.4(a), there exists $v_j' \in V_j \setminus v_j$ such that $v_j'y \in E(G)$. By symmetry, there exists $v_i' \in V_i \setminus v_i$ such that $v_i'y \in E(G)$. Therefore, if C is any cycle in $H \setminus V(C_k)$ and $V_m \cap V(C) \neq \emptyset$ for some $m \in \{3, 4, \ldots, t-2\}$, then y is adjacent to a vertex in $V_m \setminus v_m$.

Assume that $k = 5$. Then $v_1v_2 \notin E(G)$ and so $d_G(y) \geq |N_G(x) \cap N_G(y)| + |\{x\}| + e(H \setminus V(C_k)) = (t-2) + 1 + (e(H) - 5) = t + e(H) - 6$. Next assume that $k = 6$. Then $v_\ell = v_3$. Since both $N_G(x) \cap N_G(y) \cap V_1$ and $N_G(x) \cap N_G(y) \cap V_2$ are anti-complete to $N_G(x) \cap V_3$, by Lemma 2.4(a), $N_G(y) \cap (V_1 \setminus \{a, v_1\}) \neq \emptyset$ and $N_G(y) \cap (V_2 \setminus \{b, v_2\}) \neq \emptyset$. Then $d_G(y) \geq |N_G(x) \cap N_G(y)| + |\{x\}| + |N_G(y) \cap (V_1 \setminus \{a, v_1\})| + |N_G(y) \cap (V_2 \setminus \{b, v_2\})| + e(H \setminus V(C_k)) \geq (t-2) + 1 + 1 + 1 + (e(H) - 6) = t + e(H) - 5$. Finally assume that $k \geq 7$. Then $v_\ell = v_4$. Since $N_G(x) \cap N_G(y) \cap V_1$ is anti-complete to $N_G(x) \cap V_3$ and $N_G(x) \cap N_G(y) \cap V_2$ is anti-complete to $N_G(x) \cap V_4$, by Lemma 2.4(a), $N_G(y) \cap (V_1 \setminus \{a, v_1\}) \neq \emptyset$ and $N_G(y) \cap (V_2 \setminus \{b, v_2\}) \neq \emptyset$. As observed earlier, for any $i, j \in \{3, 4, \ldots, t-2\}$ with $i \neq j$ and $v_iv_j \notin E(G)$, y has at least one neighbor in each of $V_i \setminus v_i$ and $V_j \setminus v_j$ in G. Hence $d_G(y) \geq |N_G(x) \cap N_G(y)| + |\{x\}| + |N_G(y) \cap (V_1 \setminus \{a, v_1\})| + |N_G(y) \cap (V_2 \setminus \{b, v_2\})| + |V(C_k) \setminus \{a, b, v_1, v_2, y\}| + e(H \setminus V(C_k)) \geq (t-2) + 1 + 1 + 1 + (k-5) + (e(H) - k) = t + e(H) - 4$. Note that since $k \geq 7$, we see that $e(H) \geq 7$, and so $d(y) \geq t + e(H) - 4 > t + 2$. This completes the proof of Lemma 2.4(b). ■

Lemma 2.5. *Let G be a double-critical, t-chromatic graph with $t \geq 6$. If G is claw-free, then for any $x \in V(G)$, $d_G(x) \leq 2t - 4$. Furthermore, if $d_G(x) < |V(G)| - 1$, then $d_G(x) \leq 2t - 6$.*

Proof. Let $x \in V(G)$ be a vertex of maximum degree in G, and let uv be any edge of $G \setminus x$. Let c be any $(t-2)$-coloring of $G \setminus \{u, v\}$ with color classes $V_1, V_2, \ldots, V_{t-2}$. We may assume that $x \in V_{t-2}$. Since G is claw-free, x can have at most two neighbors in each of $V_1, \ldots, V_{t-3}$. Additionally, x may be adjacent to u and v in G. Therefore $d_G(x) \leq 2t - 4$. If $d_G(x) < |V(G)| - 1$, then $\chi(G[N_G(x)]) \leq t - 3$ by Proposition 2.1(h). Since G is claw-free, each color class in any $(t-3)$-coloring of $G[N_G(x)]$ can contain at most two vertices, and so $d_G(x) \leq 2t - 6$. ■

It is now an easy consequence of Proposition 2.1 and Lemma 2.5 that Theorem 1.6 is true for $t = 6, 7$.

Proof of Theorem 1.6 for $t = 6, 7$. Let G and $t \in \{6, 7\}$ be as given in the statement. Suppose that $G \neq K_t$. By Proposition 2.1(d), there exists an edge $xy \in E(G)$ such that x and y have a common non-neighbor. By Proposition 2.1(f) and Lemma 2.5, $t + 1 \leq d_G(x) \leq 2t - 6$ and $t + 1 \leq d_G(y) \leq 2t - 6$. Thus $t = 7$ and $d_G(x) = d_G(y) = 8$, which contradicts Proposition 2.1(j). ■

3. Proofs of main results

In this section, we prove our main results, namely, Theorems 1.4 and 1.6 for the case $t = 8$. We first prove Theorem 1.4.

Proof of Theorem 1.4. Let G and x be as given in the statement. Let $H = \overline{G[N_G(x)]}$. Then $|V(H)| = t + 1$. Note that if $d_G(x) = |V(G)| - 1$, then it follows from Proposition 2.1(f) that $G = K_{t+1}$, a contradiction. Thus $d_G(x) < |V(G)| - 1$. Now by Proposition 2.1(g) and Proposition 2.1(h) applied to the vertex x, $\alpha(\overline{H}) \leq 2$ and $\chi(\overline{H}) \leq t - 3$. Let c^* be any $(t-3)$-coloring of $\overline{H}$. Then each color class of c^* contains at most two vertices. Since $|V(H)| = t + 1$, we see that at least four color classes of c^* must each contain two vertices. By Proposition 2.1(e), H has at least eight vertices of degree two and so $e(H) \geq 8$. This proves Theorem 1.4(a).

To prove Theorem 1.4(b), let $y \in N_G(x)$. Since $d_G(x) = t + 1$, by Proposition 2.1(i), either $|N_G(x) \cap N_G(y)| = t$ or $|N_G(x) \cap N_G(y)| = t - 2$. Assume that $|N_G(x) \cap N_G(y)| = t - 2$. Then y belongs to a cycle of length $k \geq 5$ in H because H is a disjoint union of isolated vertices and cycles. By Proposition 2.1(i), y belongs to a cycle of length at least 5 in H. By Theorem 1.4(a), $e(H) \geq 8$. Note that if $5 \leq k \leq 7$, then by Proposition 2.1(i), H has at least two cycles of length at least 5, and so $e(H) \geq k + 5 \geq 10$. Thus by Lemma 2.4(b), $d_G(y) \geq t + 4$. If $d_G(y) = t + 4$, then it follows from Lemma 2.4(b) that either $k = 8$ and H is isomorphic to $C_8 \cup \overline{K}_{t-7}$ or $k = 5$ and H is isomorphic to $C_5 \cup C_5 \cup \overline{K}_{t-9}$. So we may assume that $|N_G(x) \cap N_G(y)| = t$. Let c be any $(t-2)$-coloring of $G \setminus \{x, y\}$ with color classes $V_1, V_2, \ldots, V_{t-2}$. Since $\alpha(\overline{H}) \leq 2$, we may further assume that $N_G(x) \cap V_1 = \{v_1, v_1'\}$, $N_G(x) \cap V_2 = \{v_2, v_2'\}$, and $N_G(x) \cap V_i = \{v_i\}$ for all $i \in \{3, 4, \ldots, t-2\}$. Then $v_1v_1', v_2v_2' \in E(H)$. By Proposition 2.1(i) applied to the vertex x, $e_H(\{v_1, v_1', v_2, v_2'\}, \{v_3, v_4, \ldots, v_{t-2}\}) \leq 4$. By Theorem 1.4(a), $e(H) \geq 8$. Thus by Proposition 2.1(i) and Lemma 2.4(a), there must exist at least four vertices in $\{v_3, v_4, \ldots, v_{t-2}\}$, say v_3, v_4, v_5, v_6, such that $d_H(v_j) = 2$ and y is adjacent to at least one vertex of $V_j \setminus v_j$ in G for all $j \in \{3, 4, 5, 6\}$. Therefore $|N_G(y) \setminus N_G[x]| \geq 4$ and so $d_G(y) = |N_G[x] \cap N_G(y)| + |N_G(y) \setminus N_G[x]| \geq (t+1) + 4 = t + 5$.

This completes the proof of Theorem 1.4. ■

We are now ready to complete the proof of Theorem 1.6.

Proof of Theorem 1.6 for $t = 8$. Let G and $t = 8$ be as given in the statement. Suppose that $G \neq K_8$. We now prove a series of claims.

Claim 1. *G is 10-regular.*

Proof. By Lemma 2.2 and Theorem 1.6 for $t = 7$, $\Delta(G) \leq |V(G)| - 2$. By Proposition 2.1(f) and Lemma 2.5, we see that $9 \leq d_G(x) \leq 10$ for all vertices $x \in V(G)$. By Corollary 1.5, G is 10-regular. ■

Claim 2. *For any $x \in V(G)$, $2 \leq \delta(\overline{G[N_G(x)]}) \leq \Delta(\overline{G[N_G(x)]}) \leq 3$.*

Proof. Let $x \in V(G)$. Then x has at least one non-neighbor in G, otherwise $G = K_{11}$ by Claim 1, contrary to the fact that G is 8-chromatic. By Proposition 2.1(h), $\chi(G[N_G(x)]) \leq 5$. Since G is claw-free, we see that $\alpha(G[N_G(x)]) = 2$, and so $\chi(G[N_G(x)]) = 5$ since every color class can contain at most two vertices. Thus every vertex of $N_G(x)$ has at least one non-neighbor in $G[N_G(x)]$. By Proposition 2.1(e) and Proposition 2.1(c), $2 \leq \delta(\overline{G[N_G(x)]}) \leq \Delta(\overline{G[N_G(x)]}) \leq 3$. ■

Claim 3. *For any $x \in V(G)$, $\Delta(\overline{G[N_G(x)]}) = 3$. That is, $\overline{G[N_G(x)]}$ is not 2-regular.*

Proof. Suppose that there exists a vertex $x \in V(G)$ such that $\overline{G[N_G(x)]}$ is 2-regular. Let $y \in N_G(x)$ and let c be any 6-coloring of $G \setminus \{x, y\}$ with color classes $V_1, V_2, \ldots, V_6$. Let $W = N_G(x) \cap N_G(y)$. Then $|W| = 7$ because $\overline{G[N_G(x)]}$ is 2-regular. By Proposition 2.1(c), we may assume that $|V_1 \cap W| = 2$ and $|V_i \cap W| = 1$ for every $i \in \{2, 3, 4, 5, 6\}$. Let $V_1 \cap W = \{v_1, u_1\}$ and $V_i \cap W = \{v_i\}$ for each $i \in \{2, 3, 4, 5, 6\}$. Since G is claw-free, we may further assume that $N_G(x) \cap V_2 = \{v_2, u_2\}$ and $N_G(x) \cap V_3 = \{v_3, u_3\}$. Clearly, $yu_2, yu_3 \notin E(G)$ and thus $u_2u_3 \in E(G)$ because G is claw-free. Since $\overline{G[N_G(x)]}$ is 2-regular, we see that $G[\{v_4, v_5, v_6\}]$ is not a clique. We may assume that $v_4v_5 \notin E(G)$. By Lemma 2.4(a), $N_G(y) \cap (V_j \setminus \{v_j\}) \neq \emptyset$ for all $j \in \{4, 5\}$. Let $w_4 \in V_4 \setminus v_4$ and $w_5 \in V_5 \setminus v_5$ be two other neighbors of y in G. Then $N_G(y) \setminus N_G[x] = \{w_4, w_5\}$ since G is 10-regular by Claim 1. By Lemma 2.4(a), v_6 must be complete to $\{v_2, v_3, v_4, v_5\}$ in G. Notice that v_6 is complete to $\{u_2, u_3\}$ in G since $\overline{G[N_G(x)]}$ is 2-regular. Thus v_6 must be anti-complete to $\{v_1, u_1\}$ in G and so $G[\{x, v_1, u_1, v_6\}]$ is a claw, a contradiction. ■

From now on, we fix an arbitrary vertex $x \in V(G)$. Let $H = \overline{G[N_G(x)]}$. By Claim 3, let $y \in N_G(x)$ with $|N_G(x) \cap N_G(y)| = 6$. We choose such a vertex $y \in N_G(x)$ so that $N_G(x) \setminus N_G[y]$ contains as many vertices of degree two in H as possible. Let c be any 6-coloring of $G \setminus \{x, y\}$ with color classes $V_1, V_2, \ldots, V_6$. We may assume that $V_i \cap N_G(x) \cap N_G(y) = \{v_i\}$ for all $i \in \{1, 2, 3, 4, 5, 6\}$. Since G is claw-free, we may further assume that $N_G(x) \cap V_j = \{v_j, u_j\}$ for all $j \in \{1, 2, 3\}$. Notice that y is anti-complete to $\{u_1, u_2, u_3\}$ in G, and since G is claw-free, $G[\{u_1, u_2, u_3\}] = K_3$. Let $A = \{u_1, u_2, u_3\}$, $B = \{v_1, v_2, v_3\}$, and $C = \{v_4, v_5, v_6\}$.

Claim 4. *B is not complete to C in G.*

Proof. Suppose that B is complete to C in G. Then $e_H(C, A) = \sum_{v \in C} d_H(v) - 2e(H[C]) \geq 6 - 2e(H[C])$. For each $i \in \{1, 2, 3\}$, $u_iv_i, u_iy \notin E(G)$ and $d_H(u_i) \leq 3$. Thus $e_H(A, C) \leq 3$ and so $e(H[C]) \geq 2$. Since G is claw-free, we have $e(H[C]) = 2$. We may assume that $v_4v_6 \notin E(H)$. Then $v_4v_6 \in E(G)$ and $v_4v_5, v_5v_6 \notin E(G)$. Since $d_H(v_4) \geq 2$, $d_H(v_6) \geq 2$, and B is complete to C in G, we may assume that $u_2v_4, u_3v_6 \notin E(G)$. Note that H is not 3-regular since $e_H(A, C) \leq 3$ and $e_H(B, C) = 0$. By the choice of y, $d_H(u_1) = 2$ and $d_H(v_j) = 2$ for all $j \in \{4, 5, 6\}$. Since $d_H(u_2) = d_H(u_3) = 3$, by the choice of y again, $d_H(v_2) = d_H(v_3) = 3$. Thus $G[B] = \overline{K_3}$ and so $G[\{x\} \cup B]$ is a claw, a contradiction. ■

Claim 5. $G[C] = K_3$.

Proof. Suppose that $G[C]$ contains a missing edge, say $v_4v_5 \notin E(G)$. By Lemma 2.4(a), there exist $w_4 \in V_4 \setminus v_4$ and $w_5 \in V_5 \setminus v_5$ such that $yw_4, yw_5 \in E(G)$. By Claim 4, we may assume that $v_3v_j \notin E(G)$ for some $j \in \{4, 5, 6\}$. By Lemma 2.4(a), y has another neighbor, say w_3, in $V_3 \setminus v_3$. Since G is 10-regular, $\{w_3, w_4, w_5\} = N(y) \setminus N[x]$, so by Lemma 2.4(a), v_4v_5 is the only missing edge in $G[C]$ and $\{v_1, v_2\}$ is complete to C in G. Suppose $e_H(A, C) = 3$, so that $d_H(u_i) = 3$ for all $i \in \{1, 2, 3\}$. By the choice of y, $d_H(v_3) = 3$, or else we could replace y with u_3. Notice that for all $i \in \{4, 5, 6\}$, $e_H(v_i, A \cup \{v_3\}) \geq 1$, and so by the choice of y, $d_H(v_i) = 3$, or else we could replace y with v_3. Thus $e_H(A, C) \geq 5$, which is impossible. Hence $e_H(A, C) \leq 2$. Notice that $e_H(A, C) = (d_H(v_4) - 1) + (d_H(v_5) - 1) + d_H(v_6) - e_H(v_3, C) \geq 2$. It follows that $e_H(A, C) = 2$, $e_H(v_3, C) = 2$ and $d_H(v_i) = 2$ for all $i \in \{4, 5, 6\}$. Then $N_G(x) \setminus N_G[y]$ has at most one vertex of degree two in H, but $N_G(x) \setminus N_G[v_3]$ has two vertices of degree two in H, contradicting the choice of y. ■

Claim 6. *v_1u_1, v_2u_2, and v_3u_3 are the only edges in $H[A \cup B]$.*

Proof. Suppose that $H[A \cup B]$ has at least four edges. By Claims 2 and 5, $e_H(A \cup B, C) \geq 6$. On the other hand, $e_H(A \cup B, C) = \sum_{v \in A \cup B} d_H(v) - 2e(H[A \cup B]) - 3 \leq 15 - 2e(H[A \cup B])$. It follows that $e(H[A \cup B]) = 4$ and $A \cup B$ contains at most one vertex of degree two in H. Thus $e_H(A \cup B, C) \leq 7$ and so at least two vertices of C, say v_4 and v_5, are of degree two in H. Since $e_H(A, C) \leq 3$ and $G[C] = K_3$ by Claim 5, we may assume that $v_4v_3 \notin E(G)$. If $d_H(v_3) = 3$, then since $d_H(v_4) = 2$ and at most one vertex of $A \cup B$ has degree two in H, by the choice of y, exactly one of u_1, u_2, u_3 has degree two in H. Then $e_H(A \cup B, C) = 6$. Thus $d_H(v_j) = 2$ for all $j \in \{4, 5, 6\}$ and by the choice of y, each vertex of B is adjacent to at most one vertex of C in H. Thus $e_H(A \cup B, C) \leq 5$, a contradiction. Hence $d_H(v_3) = 2$. Now $d_H(u_i) = 3$ for all $i \in \{1, 2, 3\}$ because at most one vertex of $A \cup B$ has degree two in H. We see that $N(x) \setminus N[y]$ has no vertex of degree two in H but $N(x) \setminus N[u_3]$ has at least one vertex of degree two in H, contrary to the choice of y. ■

By Claim 6, we see that for any $i \in \{1, 2, 3\}$, $v_iv_j \notin E(G)$ for some $j \in \{4, 5, 6\}$. By Lemma 2.4(a), let $w_i \in V_i \setminus v_i$ be such that $yw_i \in E(G)$ for all $i \in \{1, 2, 3\}$. Let $D = \{w_1, w_2, w_3\}$. Then $N_G(y) \setminus N_G[x] = D$ and $G[D] = K_3$ because G is claw-free. Clearly, D is not complete to C in G, otherwise $G[\{y\} \cup D \cup C] = K_7$, contrary to Proposition 2.1(a). We may assume that $w_3v_4 \notin E(G)$. For each $i \in \{1, 2\}$, $v_iv_3, v_iu_3 \in E(G)$ by Claim 6. Thus $v_1w_3, v_2w_3 \notin E(G)$ because G is claw-free. Notice that $w_3, x, v_1, v_2, v_4 \in N_G(y)$ and w_3 is anti-complete to $\{x, v_1, v_2, v_4\}$ in G. Thus $\Delta(\overline{G[N_G(y)]}) \geq 4$, contrary to Claim 2.

This completes the proof of Theorem 1.6. ■

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