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HADWIGER'S CONJECTURE FOR GRAPHS WITH FORBIDDEN HOLES*

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Abstract. Given a graph G, the *Hadwiger number* of G, denoted by $h(G)$, is the largest integer k such that G contains the complete graph K_k as a minor. A *hole* in G is an induced cycle of length at least four. Hadwiger's conjecture from 1943 states that for every graph G, $h(G) \geq \chi(G)$, where $\chi(G)$ denotes the chromatic number of G. In this paper we establish more evidence for Hadwiger's conjecture by showing that if a graph G with independence number $\alpha(G) \geq 3$ has no hole of length between 4 and $2\alpha(G) - 1$, then $h(G) \geq \chi(G)$. We also prove that if a graph G with independence number $\alpha(G) \geq 2$ has no hole of length between 4 and $2\alpha(G)$, then G contains an odd clique minor of size $\chi(G)$, that is, such a graph G satisfies the odd Hadwiger's conjecture.

Key words. Hadwiger's conjecture, graph minor, quasi-line graph

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1. Introduction. All graphs considered in this paper are finite and have no loops or parallel edges. We begin with some definitions. Let G be a graph. The *complement* $\overline{G}$ of G is the graph with vertex set $V(G)$, such that two vertices are adjacent in G if and only if they are nonadjacent in $\overline{G}$. A *clique* in G is a set of vertices all pairwise adjacent. A *stable set* in G is a set of vertices all pairwise nonadjacent. We use $\chi(G)$, $\omega(G)$, and $\alpha(G)$ to denote the chromatic number, the clique number, and the independence number of G, respectively. Given a graph H, we say that G is H*-free* if G has no induced subgraph isomorphic to H. For a family $\mathcal{F}$ of graphs, we say that G is $\mathcal{F}$-free if G is F-free for every $F \in \mathcal{F}$. A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. In those circumstances we also say that G has an H *minor*.

In 1943, Hadwiger [16] conjectured that for every integer $t \geq 0$, every graph either can be t-colored or has a K_{t+1} minor. Hadwiger's conjecture is perhaps the most famous conjecture in graph theory, as pointed out by Seymour in his recent survey [26]. It suggests a far reaching generalization of the four color theorem [2, 3, 22]. Hadwiger's conjecture is trivially true for $t \leq 2$ and reasonably easy for $t = 3$, as shown by Dirac [11]. However, for $t \geq 4$, Hadwiger's conjecture implies the four color theorem. Wagner [28] proved that the case $t = 4$ of Hadwiger's conjecture is, in fact, equivalent to the four color theorem, and the same was shown for $t = 5$ by Robertson, Seymour, and Thomas [23] in 1993. Hadwiger's conjecture remains open for $t \geq 6$. As pointed out by Seymour [26] in his recent survey on Hadwiger's conjecture, proving that graphs with no K_7 minor are 6-colorable is thus the first case of Hadwiger's conjecture that is still open. It is not even known yet whether every graph with no K_7 minor is 7-colorable. Kawarabayashi and Toft [18] proved that every graph

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with no K_7 or $K_{4,4}$ minor is 6-colorable. Jakobsen [14, 15] proved that every graph with no $K_7^{=}$ minor is 6-colorable and every graph with no K_7^- minor is 7-colorable, where for any integer $p > 0$, K_p^- (resp., $K_p^{=}$) denotes the graph obtained from K_p by removing one edge (resp., two edges). Recently Albar and Gonçalves [1] proved that every graph with no K_7 minor is 8-colorable and every graph with no K_8 minor is 10-colorable. Their proof is computer-assisted. In [25], Rolek and Song obtained a short and computer-free proof of Albar and Gonçalves' results and extended it to the next step by showing that every graph with no K_9 minor is 12-colorable. Rolek and Song [25] also proved that every graph with no $K_8^{=}$ minor is 8-colorable and every graph with no K_8^- minor is 9-colorable. For more information on Hadwiger's conjecture, readers are referred to an earlier survey by Toft [27] and a very recent informative survey due to Seymour [26].

Hadwiger's conjecture has also been verified to be true for some special classes of graphs. In 2004, Reed and Seymour [21] proved that Hadwiger's conjecture holds for line graphs (possibly with parallel edges). A graph G is a *quasi-line graph* if for every vertex v, the set of neighbors of v can be covered by two cliques, namely, the vertex set of the neighborhood of v can be partitioned into two cliques. A graph G is *claw-free* if G does not contain $K_{1,3}$ as an induced subgraph. It is easy to verify that the class of line graphs is a proper subset of the class of quasi-line graphs, and the class of quasi-line graphs is a proper subset of the class of claw-free graphs. Recently quasi-line graphs attracted more attention; see [9, 6, 7, 4]. In particular, Chudnovsky and Seymour [9] gave a constructive characterization of quasi-line graphs; Chudnovsky and Fradkin [7, 8] proved in 2008 that Hadwiger's conjecture holds for quasi-line graphs and proved in 2010 that if G is claw-free, then G contains a clique minor of size at least $\lceil \frac{2}{3}\chi(G) \rceil$.

One particular interesting case of Hadwiger's conjecture is when graphs have independence number two. It has attracted more attention recently (see section 4 in Seymour's survery [26] for more information). As stated in his survey, Seymour believes that if Hadwiger's conjecture is true for graphs G with $\alpha(G) = 2$, then it is probably true in general. Plummer, Stiebitz, and Toft [20] proved that Hadwiger's conjecture holds for every H-free graph G with $\alpha(G) = 2$, where H is any graph on four vertices and $\alpha(H) = 2$. Later, Kriesell [19] extended their result and proved that Hadwiger's conjecture holds for every H-free graph G with $\alpha(G) = 2$, where H is any graph on five vertices and $\alpha(H) = 2$.

One strengthening of Hadwiger's conjecture is to consider the odd-minor variant of Hadwiger's conjecture. We say that a graph G has an *odd clique minor* of size at least k if there are k vertex-disjoint trees in G such that every two of them are joined by an edge, and in addition, all the vertices of the trees are two-colored in such a way that the edges within the trees are bichromatic, but the edges between trees are monochromatic (and hence the vertices of all trivial trees must receive the same color, where a tree is *trivial* if it has one vertex only). We say that G has an *odd K_k minor* if G has an odd clique minor of size at least k. It is easy to see that any graph that has an odd K_k minor certainly contains K_k as a minor.

Gerards and Seymour (see [13, p. 115]) proposed a well-known strengthening of Hadwiger's conjecture: for every integer $t \geq 0$, every graph either can be t-colored or has an odd K_{t+1} minor. This conjecture is referred to as "the odd Hadwiger's conjecture." The odd Hadwiger's conjecture is substantially stronger than Hadwiger's conjecture. It is trivially true for $t \leq 2$. The case $t = 3$ was proved by Catlin [5] in 1978. Guenin [12] announced at a meeting in Oberwolfach in 2005 a solution of the case $t = 4$. It remains open for $t \geq 5$. Kawarabayashi and Song [17] proved that every graph G on n vertices with $\alpha(G) \geq 2$ has an odd clique minor of size at least $\lceil n/(2\alpha(G)-1)\rceil$. For more information on the odd Hadwiger's conjecture, readers are referred to the very recent survey of Seymour [26].

In this paper, we establish more evidence for Hadwiger's conjecture and the odd Hadwiger's conjecture. We first prove that if a graph G with independence number $\alpha(G)$ has no induced cycles of length between 4 and $2\alpha(G)$, then G satisfies the odd Hadwiger's conjecture. We then prove that if a graph G with independence number $\alpha(G) \geq 3$ has no induced cycles of length between 4 and $2\alpha(G)-1$, then G satisfies Hadwiger's conjecture. We prove these two results in section 2.

We need to introduce more notation. Given a graph G, the *Hadwiger number* (resp., *odd Hadwiger number*) of G, denoted by $h(G)$ (resp., $oh(G)$), is the largest integer k such that G contains the complete graph K_k as a minor (resp., an odd minor). We use $|G|$, $\delta(G)$, and $\Delta(G)$ to denote the number of vertices, the minimum degree, and the maximum degree of G, respectively. Given vertex sets $A, B \subseteq V(G)$, we say that A is *complete to* (resp., *anticomplete to*) B if for every $a \in A$ and every $b \in B$, $ab \in E(G)$ (resp., $ab \notin E(G)$). The subgraph of G induced by A, denoted $G[A]$, is the graph with vertex set A and edge set $\{xy \in E(G) : x, y \in A\}$. We denote by $G\backslash A$ the subgraph of G induced on $V(G)\backslash A$. If $A = \{a\}$, we simply write $G\backslash a$. An (A, B)*-path* of G is a path with one end in A, the other end in B, and no internal vertices in $A\cup B$. A graph G is t*-critical* if $\chi(G) = t$ and $\chi(G\backslash v) < t$ for any $v \in V(G)$. One can easily see that for any t-critical graph G, $\delta(G) \geq t-1$. A *hole* in a graph G is an induced cycle of length at least four. An *antihole* in G is an induced subgraph isomorphic to the complement of a hole. A graph G is *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G. We use C_k to denote a cycle on k vertices. We shall need the following two results. Theorem 1.1 is the well-known strong perfect graph theorem [10] and Theorem 1.2 is a result of Chudnovsky and Fradkin [7].

THEOREM 1.1. *A graph G is perfect if and only if it has no odd hole and no odd antihole.*

THEOREM 1.2. *If G is a quasi-line graph, then $h(G) \geq \chi(G)$.*

We shall need the following corollary.

COROLLARY 1.3. *If G is $\{C_4, C_5, C_7, \ldots, C_{2\alpha(G)+1}\}$-free, then G is perfect.*

Proof. One can easily see that G is C_k-free for all $k \geq 2\alpha(G)+2$. Since G is C_4-free, we see that G is $\overline{C}_k$-free for all $k \geq 7$. Note that $C_5 = \overline{C}_5$. Hence G is C_k-free and $\overline{C}_k$-free for all $k \geq 5$ odd and so G is perfect by Theorem 1.1, as desired. ∎

2. Main results. A graph G is an *inflation of a graph H* if G can be obtained from H by replacing each vertex of H by a clique of order at least one and two such cliques are complete to each other if their corresponding vertices in H are adjacent. Under these circumstances, we define $s(G)$ to be the size of the smallest clique used to replace a vertex of H. We first prove a lemma.

LEMMA 2.1. *Let G be an inflation of an odd cycle C. Then $\chi(G) \leq \omega(G) + s(G)$ and $oh(G) \geq \chi(G)$.*

Proof. Let G and C be given as in the statement and let $x_0, x_1, x_2, \ldots, x_{2t}$ be the vertices of C in order, where $t \geq 1$ is an integer. The statement is trivially true when $t = 1$. So we may assume that $t \geq 2$. Let $A_0, A_1, A_2, \ldots, A_{2t}$ be vertex-disjoint cliques such that A_i is used to replace x_i for all $i \in \{0, 1, \ldots, 2t\}$. We may assume that $s(G) = |A_0|$. Then $|A_0| = \min\{|A_i| : 0 \leq i \leq 2t\}$. Since $t \geq 2$, we have

$\omega(G) = \omega(G\backslash A_0)$. One can easily see that $G\backslash A_0$ is an inflation of a path on $2t$ vertices and so $\chi(G\backslash A_0) = \omega(G\backslash A_0)$. Therefore $\chi(G) \le \chi(G\backslash A_0) + |A_0| \le \omega(G\backslash A_0) + |A_0| = \omega(G) + s(G)$.

We next show that G contains an odd clique minor of size $\omega(G) + |A_0|$. Without loss of generality, we may assume that $\omega(G\backslash A_0) = |A_1| + |A_2|$. By the choice of $|A_0|$, there are $|A_0|$ pairwise vertex-disjoint (A_0, A_3)-paths (each on $2t-1$ vertices) in $G\backslash(A_1 \cup A_2)$. Let $T_1, \ldots, T_{|A_0|}$ be $|A_0|$ such paths and let $T_{|A_0|+1}, \ldots, T_{|A_0|+\omega(G)}$ be $\omega(G)$ pairwise disjoint trivial trees (i.e., trees with one vertex only) in $A_1 \cup A_2$. Now coloring all the vertices in $A_4 \cup A_6 \cup \cdots \cup A_{2t}$ by color 1 and all the other vertices in G by color 2, we see that $T_1, \ldots, T_{|A_0|}, T_{|A_0|+1}, \ldots, T_{|A_0|+\omega(G)}$ yield an odd clique minor of size $|A_0| + \omega(G) \ge \chi(G)$, as desired. ∎

We next use Lemma 2.1 to prove that every graph G with $\alpha(G) \ge 2$ and no hole of length between 4 and $2\alpha(G)$ satisfies the odd Hadwiger's conjecture, the following.

THEOREM 2.2. *Let G be a graph with $\alpha(G) \ge 2$. If G is $\{C_4, C_5, C_6, \ldots, C_{2\alpha(G)}\}$-free, then $oh(G) \ge \chi(G)$.*

Proof. Let $\alpha = \alpha(G)$. We first assume that G contains no odd hole of length $2\alpha + 1$. By Corollary 1.1, G is perfect and so G contains a clique (and thus an odd clique minor) of size $\chi(G)$. So we may assume that G contains an odd hole of length $2\alpha + 1$, say, C, with vertices $v_0, v_1, \ldots, v_{2\alpha}$ in order. We next prove that for every $w \in V(G\backslash C)$, w is either complete to C or adjacent to exactly three consecutive vertices on C. Since $\alpha(G) = \alpha$, we see that w is adjacent to at least one vertex on C. If w is complete to C, then we are done. So we may assume that $wv_0 \notin E(G)$ but $wv_1 \in E(G)$. Then w is not adjacent to $v_{2\alpha}, v_{2\alpha-1}, \ldots, v_4$ as G is $\{C_4, C_5, \ldots, C_{2\alpha}\}$-free. Hence w must be adjacent to v_2, v_3 because $\alpha(G) = \alpha$. This proves that for any $w \in V(G\backslash C)$ not complete to C, w is adjacent to precisely three consecutive vertices on C.

Let J (possibly empty) denote the set of vertices in G that are complete to C, and let A_i (possibly empty) denote the set of vertices in G adjacent to v_i, v_{i+1}, v_{i+2}, where $i = 0, 1, \ldots, 2\alpha$ and all arithmetic on indices here and henceforth is done modulo $2\alpha + 1$. Since G is C_4-free, we see that $G[J]$ is a clique and $G[A_i]$ is a clique for all $i \in \{0, 1, \ldots, 2\alpha\}$. Note that $\{J, V(C), A_0, A_1, \ldots, A_{2\alpha}\}$ partitions $V(G)$. Since $\alpha(G) = \alpha$ and G is $\{C_4, C_5, \ldots, C_{2\alpha}\}$-free, one can easily check that A_i is complete to $A_{i+1} \cup A_{i-1}$ and anticomplete to all A_j, where $j \ne i+1, i, i-1$. We now show that J is complete to all A_i. Suppose there exist $a \in J$ and $b \in A_i$ for some $i \in \{0, 1, \ldots, 2\alpha\}$, say, $i = 0$, such that $ab \notin E(G)$. Then $G[\{a, v_0, b, v_2\}]$ is an induced C_4, a contradiction. Thus J is complete to all A_i and so J is complete to $V(G)\backslash J$. Let $A_i^* = A_i \cup \{v_{i+1}\}$ for $i = 0, 1, \ldots, 2\alpha$. Then $A_i^* \ne \emptyset$ for all $i \in \{0, 1, \ldots, 2\alpha\}$ and $G\backslash J$ is an inflation of an odd cycle of length $2\alpha + 1$. By Lemma 2.1, $oh(G\backslash J) \ge \chi(G\backslash J)$ and so $oh(G) \ge oh(G\backslash J) + |J| \ge \chi(G\backslash J) + |J| = \chi(G)$, as desired.

This completes the proof of Theorem 2.2. ∎

Finally we prove that every graph G with $\alpha(G) \ge 3$ and no hole of length between 4 and $2\alpha(G) - 1$ satisfies Hadwiger's conjecture.

THEOREM 2.3. *Let G be a graph with $\alpha(G) \ge 3$. If G is $\{C_4, C_5, C_6, \ldots, C_{2\alpha(G)-1}\}$-free, then $h(G) \ge \chi(G)$.*

Proof. Suppose for a contradiction that $h(G) < \chi(G)$. Let G be a counterexample with $|V(G)|$ as small as possible. Let $n := |V(G)|$, $t := \chi(G)$, and $\alpha := \alpha(G)$. By Theorem 1.1 and the fact that $h(G) \ge \omega(G)$, G is not perfect. Since G is

$\{C_4, C_5, C_7, \ldots, C_{2\alpha-1}\}$-free, by Corollary 1.3, we see that G must contain an odd hole, say, C, with $2\alpha+1$ vertices. ∎

(1) G is t-critical and so $\delta(G) \geq t-1$.

Proof. Suppose that there exists $x \in V(G)$ such that $\chi(G\backslash x) = t$. If $\alpha(G\backslash x) = \alpha$, then $G\backslash x$ is $\{C_4, C_5, C_6, \ldots, C_{2\alpha(G\backslash x)-1}\}$-free. By the minimality of G, we have $h(G\backslash x) \geq \chi(G\backslash x) = t$ and so $h(G) \geq h(G\backslash x) \geq t$, a contradiction. Thus $\alpha(G\backslash x) = \alpha-1$. Then $G\backslash x$ is $\{C_4, C_5, C_7, \ldots, C_{2\alpha(G\backslash x)+1}\}$-free. By Corollary 1.3, $G\backslash x$ is perfect and so $h(G) \geq h(G\backslash x) \geq \omega(G\backslash x) = \chi(G\backslash x) = t$, a contradiction. Thus G is t-critical and so $\delta(G) \geq t-1$. ∎

(2) $\Delta(G) \leq n-2$.

Proof. Suppose there exists a vertex x in G with $d(x) = n-1$. Then $\chi(G\backslash x) = t-1$ and by the minimality of G, $h(G\backslash x) \geq \chi(G\backslash x) = t-1$ and so $h(G) \geq h(G\backslash x) + 1 \geq (t-1)+1 = t$, a contradiction. ∎

(3) $\omega(G) \leq t-2$.

Proof. Suppose that $\omega(G) \geq t-1$. Since $h(G) < t$, we see that $\omega(G) = t-1$. Let $H \subseteq G$ be isomorphic to K_{t-1}. Then C and H have at most two vertices in common, and if $|C \cap H| = 2$, then the two vertices in $C \cap H$ must be adjacent on C. Let P be a subpath of $C\backslash H$ on $2\alpha-1$ vertices. Then P is an induced path in $G\backslash H$. Since $\alpha(G) = \alpha$, we see that every vertex in H must have a neighbor on P. By contracting the path P into a single vertex, we see that $h(G) \geq t$, a contradiction. ∎

Let $v_0, v_1, \ldots, v_{2\alpha}$ be the vertices of C in order. We next show that

(4) For every $w \in V(G\backslash C)$, either w is complete to C, or w is adjacent to exactly three consecutive vertices on C, or w is adjacent to exactly four consecutive vertices on C.

Proof. Since $\alpha(G) = \alpha$, we see that w is adjacent to at least one vertex on C. Suppose that w is not complete to C. We may assume that $wv_0 \notin E(G)$ but $wv_1 \in E(G)$. Then w is not adjacent to $v_{2\alpha}, v_{2\alpha-1}, \ldots, v_5$ because G is $\{C_4, C_5, \ldots, C_{2\alpha-1}\}$-free. If $wv_4 \in E(G)$, then w must be adjacent to v_2, v_3 because G is $\{C_4, C_5\}$-free. If $wv_4 \notin E(G)$, then again w must be adjacent to v_2, v_3 because $\alpha(G) = \alpha$. Thus w is adjacent to either v_1, v_2, v_3 or v_1, v_2, v_3, v_4 on C, as desired. ∎

Let J denote the set of vertices in G that are complete to C. For each $i \in I := \{0, 1, \ldots, 2\alpha\}$, let $A_i \subseteq V(G\backslash C)$ (possibly empty) denote the set of vertices in G adjacent to precisely v_i, v_{i+1}, v_{i+2} on C, and let $B_i \subseteq V(G\backslash C)$ (possibly empty) denote the set of vertices in G adjacent to precisely $v_i, v_{i+1}, v_{i+2}, v_{i+3}$ on C, where all arithmetic on indices here and henceforth is done modulo $2\alpha+1$. By (4), $\{J, V(C), A_0, A_1, \ldots, A_{2\alpha}, B_0, B_1, \ldots, B_{2\alpha}\}$ partitions $V(G)$.

(5) $J = \emptyset$.

Proof. Suppose that $J \neq \emptyset$. Let $a \in J$. By (2), there exists $b \in V(G)\backslash(V(C) \cup J)$ such that $ab \notin E(G)$. By (4), we may assume that b is adjacent to v_0, v_1, v_2. Then $G[\{a, v_0, b, v_2\}]$ is an induced C_4 in G, a contradiction. ∎

The fact that $\alpha(G) = \alpha$ implies that

(6) For each $i \in I$, both $G[A_i]$ and $G[B_i]$ are cliques; and A_i is complete to $A_{i-1} \cup A_{i+1}$.

Since G is $\{C_4, C_5, \ldots, C_{2\alpha-1}\}$-free, one can easily check that

(7) For each $i \in I$, A_i is anticomplete to each A_j, where $j \in I\backslash\{i-2, i-1, i, i+1, i+2\}$; and

(8) For each $i \in I$, B_i is complete to $B_{i-1} \cup A_i \cup A_{i+1} \cup B_{i+1}$ and anticomplete to each B_j, where $j \in I \backslash \{i-1, i, i+1\}$.

We shall also need the following:

(9) For each $i \in I$, if $B_i \neq \emptyset$, then $B_j = \emptyset$ for any $j \in I \backslash \{i-2, i-1, i, i+1, i+2\}$.

Proof. Suppose B_j is not empty for some $j \in I \backslash \{i+2, i+1, i, i-1, i-2\}$. We may assume that $j > i$. Let $a \in B_i$ and $b \in B_j$. Then $G[\{v_i, a, v_{i+3}, \ldots, v_j, b, v_{j+3}, \ldots, v_{i-1}\}]$ is an odd hole with $2\alpha - 1$ vertices if $ab \notin E(G)$, and $G[\{v_i, a, b, v_{j+3}, \ldots, v_{i-1}\}]$ is a hole with $2\alpha + 1 - j + i \geq 4$ vertices if $ab \in E(G)$. In both cases, we obtain a contradiction. ∎

With an argument similar to that of (9), we see that

(10) For each $i \in I$, if $B_i \neq \emptyset$, then B_i is anticomplete to A_j for any $j \in I \backslash \{i-1, i, i+1, i+2\}$.

We next show that

(11) For any $i \in I$, if $A_i \neq \emptyset$, then each vertex in A_i is either anticomplete to A_{i+2} or anticomplete to A_{i-2}.

Proof. Suppose there exists a vertex $x \in A_i$ such that x is adjacent to a vertex $y \in A_{i-2}$ and a vertex $z \in A_{i+2}$. Then $G[\{x, y, z\} \cup (V(C) \backslash \{v_{i-1}, v_i, v_{i+1}, v_{i+2}, v_{i+3}\})]$ is an odd hole with $2\alpha - 1$ vertices, a contradiction. ∎

(12) For any $i \in I$, if $B_i \neq \emptyset$, then every vertex in B_i is either complete to A_{i-1} or complete to A_{i+2}.

Proof. Suppose for a contradiction, say, $B_2 \neq \emptyset$, and there exists a vertex $b \in B_2$ such that b is not adjacent to a vertex $a_1 \in A_1$ and a vertex $a_4 \in A_4$. By (7), A_1 is anticomplete to A_4. Thus G contains a stable set $\{b, a_1, a_4, v_0\}$ of size four if $\alpha = 3$ or stable set $\{b, a_1, a_4, v_0, v_7, v_9, \ldots, v_{2\alpha-1}\}$ of size $\alpha + 1$ if $\alpha \geq 4$, a contradiction. ∎

(13) There exists an $i \in I$ such that $B_j = \emptyset$ for any $j \in I \backslash \{i, i+1, i+2\}$.

Proof. This is obvious if $B_k = \emptyset$ for any $k \in I$. So we may assume that $B_k \neq \emptyset$ for some $k \in I$, say, $B_2 \neq \emptyset$. Then by (9), $B_j = \emptyset$ for all $j = 5, 6, \ldots, 2\alpha$. By (9) again, either $B_0 \neq \emptyset$ or $B_4 \neq \emptyset$ but not both. By symmetry, we may assume that $B_4 = \emptyset$. Similarly, either $B_0 \neq \emptyset$ or $B_3 \neq \emptyset$ but not both. Thus either $B_j = \emptyset$ for all $j \in I \backslash \{0, 1, 2\}$ or $B_j = \emptyset$ for all $j \in I \backslash \{1, 2, 3\}$. ∎

By (13), we may assume that $B_j = \emptyset$ for all $j \in I \backslash \{1, 2, 3\}$. For any $A_i \neq \emptyset$, where $i \in I$, let $A_i^1 = \{a \in A_i : a \text{ has a neighbor in } A_{i-2}\}$, $A_i^3 = \{a \in A_i : a \text{ has a neighbor in } A_{i+2}\}$, and $A_i^2 = A_i \backslash (A_i^1 \cup A_i^3)$. Then A_i^2 is anticomplete to $A_{i-2} \cup A_{i+2}$. By (11), A_i^1 is anticomplete to A_{i+2} and A_i^3 is anticomplete to A_{i-2}. Clearly, $\{A_i^1, A_i^2, A_i^3\}$ partitions A_i. Next, for any $B_j \neq \emptyset$, where $j \in \{1, 2, 3\}$, by (12), let $B_j^1 = \{b \in B_j : b \text{ is complete to } A_{j-1}\}$ and $B_j^2 = \{b \in B_j : b \text{ is complete to } A_{j+2}\}$. Clearly, B_j^1 and B_j^2 are not necessarily disjoint. Note that B_j^1 and B_j^2 are not symmetrical because B_j^1 is complete to A_{j-1} and B_j^2 is complete to A_{j+2}.

(14) For any $j \in \{1, 2, 3\}$, B_j is anti-complete to $A_{j-1}^1 \cup A_{j+2}^3$.

Proof. Suppose there exist a vertex $b \in B_j$ and a vertex $a \in A_{j-1}^1 \cup A_{j+2}^3$ such that $ba \in E(G)$. By the definitions of A_{j-1}^1 and A_{j+2}^3, we see that a has a neighbor, say, c, in A_{j-3} if $a \in A_{j-1}^1$, or in A_{j+4} if $a \in A_{j+2}^3$. Now $G[\{b, a, c\} \cup (V(C) \backslash \{v_{j-2}, v_{j-1}, v_j, v_{j+1}, v_{j+2}\})]$ is an odd hole of length $2\alpha - 1$ if $a \in A_{j-1}^1$, or $G[\{b, a, c\} \cup (V(C) \backslash \{v_{j+1}, v_{j+2}, v_{j+3}, v_{j+4}, v_{j+5}\})]$ is an odd hole of length $2\alpha - 1$ if $a \in A_{j+2}^3$. In either case, we obtain a contradiction. ∎

(15) G is a quasi-line graph.

Proof. It suffices to show that for any $x \in V(G)$, $N(x)$ is covered by two cliques. By (5), $J = \emptyset$. Since B_j^1 and B_j^2 are not symmetrical for all $j \in \{1,2,3\}$, we consider the following four cases.

Case 1. $x \in A_i$ for some $i \in I \backslash \{0,1,2,3,4,5\}$.

In this case, $x \in A_i^k$ for some $k \in \{1,2,3\}$. We first assume that $k = 1$. Then $x \in A_i^1$. By (11) and the definition of A_i^1, x is anticomplete to A_{i+2} and so $N[x] \subseteq A_{i-2} \cup A_{i-1} \cup A_i \cup A_{i+1} \cup \{v_i, v_{i+1}, v_{i+2}\}$. We see that $N(x)$ is covered by two cliques $G[(A_{i-2} \cap N(x)) \cup A_{i-1} \cup \{v_i\}]$ and $G[(A_i \backslash x) \cup A_{i+1} \cup \{v_{i+1}, v_{i+2}\}]$. By symmetry, the same holds if $k = 3$. So we may assume that $k = 2$. By the definition of A_i^2, x is anticomplete to $A_{i-2} \cup A_{i+2}$. Thus $N[x] = A_{i-1} \cup A_i \cup A_{i+1} \cup \{v_i, v_{i+1}, v_{i+2}\}$ and so $N(x)$ is covered by two cliques $G[A_{i-1} \cup (A_i \backslash x) \cup \{v_i\}]$ and $G[A_{i+1} \cup \{v_{i+1}, v_{i+2}\}]$.

Case 2. $x \in A_i$ for some $i \in \{0,1,2,3,4,5\}$.

In this case, we first assume that $i = 0$. Then $x \in A_0^k$ for some $k \in \{1,2,3\}$. Assume that $x \in A_0^1$. Then x is anticomplete to B_1 by (14) and anticomplete to A_2 by (11). Thus $N[x] \subseteq A_{2\alpha-1} \cup A_{2\alpha} \cup A_0 \cup A_1 \cup \{v_0, v_1, v_2\}$. One can see that $N(x)$ is covered by two cliques $G[(A_{2\alpha-1} \cap N(x)) \cup A_{2\alpha} \cup \{v_0\}]$ and $G[(A_0 \backslash x) \cup A_1 \cup \{v_1, v_2\}]$. It can be easily checked that $N(x)$ is covered by two cliques $G[A_{2\alpha} \cup (A_0 \backslash x) \cup \{v_0\}]$ and $G[A_1 \cup (B_1 \cap N(x)) \cup \{v_1, v_2\}]$ if $k = 2$ and by two cliques $G[A_{2\alpha} \cup (A_0 \backslash x) \cup \{v_0\}]$ and $G[A_1 \cup (B_1 \cap N(x)) \cup (A_2 \cap N(x)) \cup \{v_1, v_2\}]$ if $k = 3$.

Next assume that $i = 1$. Then $x \in A_1^k$ for some $k \in \{1,2,3\}$ and $N[x] \subseteq A_{2\alpha} \cup A_0 \cup A_1 \cup B_1 \cup A_2 \cup B_2 \cup A_3 \cup \{v_1, v_2, v_3\}$. One can see that $N(x)$ is covered by two cliques $G[(A_{2\alpha} \cap N(x)) \cup A_0 \cup \{v_1\}]$ and $G[(A_1 \backslash x) \cup B_1 \cup A_2 \cup \{v_2, v_3\}]$ if $k = 1$; by two cliques $G[A_0 \cup (A_1 \backslash x) \cup \{v_1\}]$ and $G[B_1 \cup A_2 \cup (B_2 \cap N(x)) \cup \{v_2, v_3\}]$ if $k = 2$; and by two cliques $G[A_0 \cup (A_1 \backslash x) \cup B_1^1 \cup \{v_1, v_2\}]$ and $G[B_1^2 \cup A_2 \cup (B_2 \cap N(x)) \cup (A_3 \cap N(x)) \cup \{v_3\}]$ if $k = 3$.

Assume that $i = 2$. Then $x \in A_2^k$ for some $k \in \{1,2,3\}$ and $N[x] \subseteq A_0 \cup A_1 \cup B_1 \cup A_2 \cup B_2 \cup A_3 \cup B_3 \cup A_4 \cup \{v_2, v_3, v_4\}$. One can check that $N(x)$ is covered by two cliques $G[(A_0 \cap N(x)) \cup A_1 \cup B_1^1 \cup \{v_2\}]$ and $G[(A_2 \backslash x) \cup B_1^2 \cup B_2 \cup A_3 \cup \{v_3, v_4\}]$ if $k = 1$; by two cliques $G[A_1 \cup B_1 \cup (A_2 \backslash x) \cup \{v_2\}]$ and $G[B_2 \cup A_3 \cup (B_3 \cap N(x)) \cup \{v_3, v_4\}]$ if $k = 2$; and by two cliques $G[A_1 \cup B_1 \cup (A_2 \backslash x) \cup B_2^1 \cup \{v_2, v_3\}]$ and $G[B_2^2 \cup A_3 \cup (B_3 \cap N(x)) \cup (A_4 \cap N(x)) \cup \{v_4\}]$ if $k = 3$.

Assume that $i = 3$. Then $x \in A_3^k$ for some $k \in \{1,2,3\}$ and $N[x] \subseteq A_1 \cup B_1 \cup A_2 \cup B_2 \cup A_3 \cup B_3 \cup A_4 \cup A_5 \cup \{v_3, v_4, v_5\}$. One can check that $N(x)$ is covered by two cliques $G[(A_1 \cap N(x)) \cup (B_1 \cap N(x)) \cup A_2 \cup B_2^1 \cup \{v_3\}]$ and $G[B_2^2 \cup (A_3 \backslash x) \cup B_3 \cup A_4 \cup \{v_4, v_5\}]$ if $k = 1$; and by two cliques $G[(B_1 \cap N(x)) \cup A_2 \cup B_2 \cup \{v_3\}]$ and $G[(A_3 \backslash x) \cup B_3 \cup A_4 \cup \{v_4, v_5\}]$ if $k = 2$. So we may assume that $x \in A_3^3$. By (14), we have A_3^3 is anticomplete to B_1. Thus $N(x)$ is covered by two cliques $G[A_2 \cup B_2 \cup (A_3 \backslash x) \cup B_3^1 \cup \{v_3, v_4\}]$ and $G[B_3^2 \cup A_4 \cup (A_5 \cap N(x)) \cup \{v_5\}]$ if $k = 3$.

Assume that $i = 4$. Then $x \in A_4^k$ for some $k \in \{1,2,3\}$ and $N[x] \subseteq A_2 \cup B_2 \cup A_3 \cup B_3 \cup A_4 \cup A_5 \cup A_6 \cup \{v_4, v_5, v_6\}$. One can see that $N(x)$ is covered by two cliques $G[(A_2 \cap N(x)) \cup (B_2 \cap N(x)) \cup A_3 \cup B_3^1 \cup \{v_4\}]$ and $G[B_3^2 \cup (A_4 \backslash x) \cup A_5 \cup \{v_5, v_6\}]$ if $k = 1$ and by two cliques $G[(B_2 \cap N(x)) \cup A_3 \cup B_3 \cup \{v_4\}]$ and $G[(A_4 \backslash x) \cup A_5 \cup \{v_5, v_6\}]$ if $k = 2$. So we may assume that $x \in A_4^3$. By (14), we have A_4^3 is anticomplete to B_2. Thus $N(x)$ is covered by two cliques $G[A_3 \cup B_3 \cup (A_4 \backslash x) \cup \{v_4, v_5\}]$ and $G[A_5 \cup (A_6 \cap N(x)) \cup \{v_6\}]$ if $k = 3$.

Finally assume that $i = 5$. Then $x \in A_5^k$ for some $k \in \{1,2,3\}$ and $N[x] \subseteq A_3 \cup B_3 \cup A_4 \cup A_5 \cup A_6 \cup A_7 \cup \{v_5, v_6, v_7\}$. One can check that $N(x)$ is covered by two cliques $G[(A_3 \cap N(x)) \cup (B_3 \cap N(x)) \cup A_4 \cup \{v_5\}]$ and $G[(A_5 \backslash x) \cup A_6 \cup \{v_6, v_7\}]$ if $k = 1$

and by two cliques $G[(B_3 \cap N(x)) \cup A_4 \cup \{v_5\}]$ and $G[(A_5 \backslash x) \cup A_6 \cup \{v_6, v_7\}]$ if $k = 2$. So we may assume that $x \in A_5^3$. By (14), we have A_5^3 is anticomplete to B_3. Thus $N(x)$ is covered by two cliques $G[A_4 \cup (A_5 \backslash x) \cup \{v_5, v_6\}]$ and $G[A_6 \cup (A_7 \cap N(x)) \cup \{v_7\}]$ if $k = 3$.

This completes the proof of Case 2.

Case 3. $x \in B_j$ for some $j \in \{1, 2, 3\}$.

In this case, first assume that $j = 1$. Then $x \in B_1^k$ for some $k \in \{1, 2\}$, and $N[x] \subseteq A_0 \cup A_1 \cup B_1 \cup A_2 \cup B_2 \cup A_3 \cup \{v_1, v_2, v_3, v_4\}$. We see that $N(x)$ is covered by two cliques $G[A_0 \cup A_1 \cup (B_1^1 \backslash x) \cup \{v_1, v_2\}]$ and $G[B_1^2 \cup A_2 \cup B_2 \cup (A_3 \cap N(x)) \cup \{v_3, v_4\}]$ if $k = 1$ and by two cliques $G[(A_0 \cap N(x)) \cup A_1 \cup B_1^1 \cup \{v_1, v_2\}]$ and $G[(B_1^2 \backslash x) \cup A_2 \cup B_2 \cup A_3 \cup \{v_3, v_4\}]$ if $k = 2$.

Next assume that $j = 2$. Then $x \in B_2^k$ for some $k \in \{1, 2\}$, and $N[x] \subseteq A_1 \cup B_1 \cup A_2 \cup B_2 \cup A_3 \cup B_3 \cup A_4 \cup \{v_2, v_3, v_4, v_5\}$. One can see that $N(x)$ is covered by two cliques $G[A_1 \cup B_1 \cup A_2 \cup (B_2^1 \backslash x) \cup \{v_2, v_3\}]$ and $G[B_2^2 \cup A_3 \cup B_3 \cup (A_4 \cap N(x)) \cup \{v_4, v_5\}]$ if $k = 1$ and by two cliques $G[(A_1 \cap N(x)) \cup B_1 \cup A_2 \cup B_2^1 \cup \{v_2, v_3\}]$ and $G[(B_2^2 \backslash x) \cup A_3 \cup B_3 \cup A_4 \cup \{v_4, v_5\}]$ if $k = 2$.

Finally assume $j = 3$. Then $x \in B_3^k$ for some $k \in \{1, 2\}$, and $N[x] \subseteq A_2 \cup B_2 \cup A_3 \cup B_3 \cup A_4 \cup A_5 \cup \{v_3, v_4, v_5, v_6\}$. We see that $N(x)$ is covered by two cliques $G[A_2 \cup B_2 \cup A_3 \cup (B_3^1 \backslash x) \cup \{v_3, v_4\}]$ and $G[B_3^2 \cup A_4 \cup (A_5 \cap N(x)) \cup \{v_5, v_6\}]$ if $k = 1$ and by two cliques $G[(A_2 \cap N(x)) \cup B_2 \cup A_3 \cup B_3^1 \cup \{v_3, v_4\}]$ and $G[(B_3^2 \backslash x) \cup A_4 \cup A_5 \cup \{v_5, v_6\}]$ if $k = 2$.

Case 4. $x \in V(C)$.

In this case, let $x = v_i$ for some $i \in I$. First assume that $i \neq 1, 2, 3, 4, 5, 6$. Then $N(v_i) = A_{i-2} \cup A_{i-1} \cup A_i \cup \{v_{i-1}, v_{i+1}\}$ and so $N(v_i)$ is covered by two cliques $G[A_{i-2} \cup A_{i-1} \cup \{v_{i-1}\}]$ and $G[A_i \cup \{v_{i+1}\}]$. Next assume that $i \in \{1, 2, 3, 4, 5, 6\}$. One can easily check that $N(v_1)$ is covered by two cliques $G[A_{2\alpha} \cup A_0 \cup \{v_0\}]$ and $G[A_1 \cup B_1 \cup \{v_2\}]$; $N(v_2)$ by two cliques $G[A_0 \cup A_1 \cup \{v_1\}]$ and $G[B_1 \cup A_2 \cup B_2 \cup \{v_3\}]$; $N(v_3)$ by two cliques $G[A_1 \cup B_1 \cup A_2 \cup \{v_2\}]$ and $G[B_2 \cup A_3 \cup B_3 \cup \{v_4\}]$; $N(v_4)$ by two cliques $G[B_1 \cup A_2 \cup B_2 \cup \{v_3\}]$ and $G[A_3 \cup B_3 \cup A_4 \cup \{v_5\}]$; $N(v_5)$ by two cliques $G[B_2 \cup A_3 \cup B_3 \cup \{v_4\}]$ and $G[A_4 \cup A_5 \cup \{v_6\}]$; and $N(v_6)$ by two cliques $G[B_3 \cup A_4 \cup \{v_5\}]$ and $G[A_5 \cup A_6 \cup \{v_7\}]$, respectively.

This proves that G is a quasi-line graph.

By (15), G is a quasi-line graph. By Theorem 1.2, $h(G) \geq \chi(G) = t$, a contradiction. This completes the proof of Theorem 2.3. □

Remark 2.4. We made no use of the fact $\omega(G) \leq t-2$ in the proof of Theorem 2.3. We kept it in the proof in the hope that one might be able to find a short proof of Theorem 2.3 without using the fact that Hadwiger's conjecture is true for quasi-line graphs (namely, Theorem 1.2).

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