Independence Number and Clique Minors

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Received September 20, 2005; Revised January 19, 2007

Published online 6 September 2007 in Wiley InterScience(www.interscience.wiley.com).
DOI 10.1002/jgt.20268

Abstract: The Hadwiger number $h(G)$ of a graph G is the maximum integer t such that K_t is a minor of G. Since $\chi(G) \cdot \alpha(G) \geq |G|$, Hadwiger's conjecture implies that $h(G) \cdot \alpha(G) \geq |G|$, where $\alpha(G)$ and $|G|$ denote the independence number and the number of vertices of G, respectively. Motivated by this fact, it is shown that $(2\alpha(G) - 2) \cdot h(G) \geq |G|$ for every graph G with $\alpha(G) \geq 3$. This improves a theorem of Duchet and Meyniel and a recent improvement due to Kawarabayashi et al. J Graph Theory 56: 219–226, 2007

Keywords: *Hadwiger's conjecture; independence number; graph minor*

Contract grant sponsor: Japan Society for the Promotion of Science for Young Scientists; Contract grant sponsor: Grant-in-Aid for Scientific Research; Contract grant sponsor: Sumitomo Foundation; Contract grant sponsor: Inoue Research Award for Young Scientists (to K. K.).

1. INTRODUCTION

All graphs considered in this article are finite and simple. Hadwiger's conjecture [7] from 1943 suggests a far reaching generalization of the Four Color Theorem [1,2,12], and is considered by many as one of the deepest open problems in graph theory. Hadwiger's conjecture states the following.

Conjecture 1.1. *For all $k \geq 1$, every k-chromatic graph contains the complete graph K_k as a minor.*

Hadwiger's conjecture is trivially true for $k \leq 3$, and reasonably easy for $k = 4$, as shown by Dirac [5]. However, for $k \geq 5$, Hadwiger's conjecture implies the Four Color Theorem. (To see that, let H be a planar graph, and let G be obtained from H by adding $k - 4$ vertices, each joined to every other vertex of the graph. Then G has no K_{k+1} minor, and hence is k-colorable by Hadwiger's conjecture, and hence H is 4-colorable.) Wagner [14] proved that the case $k = 5$ of Hadwiger's conjecture is, in fact, equivalent to the Four Color Theorem, and the same was shown for $k = 6$ by Robertson et al. [13]. Hadwiger's conjecture remains open for $k \geq 7$.

For a graph G, the *Hadwiger number* $h(G)$ of G is the maximum integer k such that K_k is a minor of G. Let $\chi(G)$ denote the chromatic number of G. Hadwiger's conjecture states that $\chi(G) \leq h(G)$ for every graph G. In a $\chi(G)$-coloring of G, each color-class has size at most $\alpha(G)$, the size of a maximum independent set of G. Hence, $\chi(G) \cdot \alpha(G) \geq |G|$. Hadwiger's conjecture implies the following.

Conjecture 1.2. *$h(G) \cdot \alpha(G) \geq |G|$ for every graph G.*

This conjecture seems weaker than Hadwiger's conjecture, however for $\alpha(G) = 2$ the two conjectures are equivalent [10]. Conjecture 1.2 was explicitly stated and put into context by Woodall [15]. In 1982, Duchet and Meyniel [6] proved the following result.

Theorem 1.3. *$h(G) \cdot (2\alpha(G) - 1) \geq |G|$ for every graph G.*

No improvement of Theorem 1.3 has been obtained in general, although Maffray and Meyniel [9] proved interesting results. Reed and Seymour [11] extended Theorem 1.3 to the fractional version of Hadwiger's conjecture by proving that the fractional chromatic number of a graph is at most twice its Hadwiger number. Recently, Kawarabayashi et al. [8] proved the following.

Theorem 1.4. *$h(G) \cdot (4\alpha(G) - 3) \geq 2|G|$ for every graph G with $\alpha(G) \geq 3$.*

For $\alpha(G) = 2$, Theorem 1.3 states that $h(G) \geq |G|/3$, and it seems quite hard to improve the constant 1/3. Seymour and Mader independently asked if it is possible to obtain a larger constant than 1/3 for this case. As far as we know no such improvement has been achieved so far.

The purpose of our article is to improve Theorem 1.4 further. We first prove the following Theorem 1.5, which improves Theorems 1.3 and 1.4 for the case $\alpha(G) = 3$.

Theorem 1.5. *$h(G) \geq |G|/4$ for every graph G with $\alpha(G) = 3$.*

We then use Theorem 1.5 to prove the following general result.

Theorem 1.6. *$h(G) \cdot (2\alpha(G) - 2) \geq |G|$ for every graph G with $\alpha(G) \geq 3$.*

When $\alpha(G) = 4$, Theorem 1.6 gives $h(G) \geq |G|/6$. It seems difficult to improve 1/6 since if G consists of two disjoint copies of graphs each with independence number two, then it is not hard to see that $h(G) \geq |G|/6$. To improve the constant 1/6, one needs to improve the constant 1/3 in Theorem 1.3 for the case $\alpha(G) = 2$, which as mentioned earlier, seems to be difficult.

We need to introduce more notation. For a graph G, we use $|G|$ to denote the number of vertices of G. For a vertex x of a graph G, the neighborhood of x in G is denoted by $N_G(x)$ or simply $N(x)$, and $d_G(x) = |N_G(x)|$ is the degree of x in G. We define $N[x] = N(x) \cup \{x\}$. For a subset S of $V(G)$, the subgraph induced by S is denoted by $G[S]$ and $G - S = G[V(G) - S]$. For a subgraph H of G, $G - H = G[V(G) - V(H)]$. If x, y are adjacent vertices of a graph G, then we denote by G/xy the graph obtained from G by contracting the edge xy and deleting all resulting parallel edges. We write $H \preceq G$ or $G \succeq H$ if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges. In those circumstances, we say that G has an *H minor*. A graph G is called *claw-free* if G does not contain $K_{1,3}$ as an induced subgraph. For graph-theoretic terminology not explained in this article, we refer the reader to [4].

2. PROOF OF THEOREM 1.5

The proof is by induction on $n = |G|$. The result is trivially true for $n \leq 8$. Suppose there exists a graph G with $\alpha(G) = 3$ and $n \geq 9$ such that $h(G) < n/4$. Then $K_{\lceil n/4 \rceil} \npreceq G$.

Since $\alpha(G) = 3$ and G does not have a complete subgraph of size $n/3$, G contains at least one induced 2-edge path. Let $P_1, P_2, \ldots, P_t$ be pairwise disjoint-induced 2-edge paths in G with t maximum. By the choice of t, each component of $G - \cup_{k=1}^{t} P_k$ is complete.

(1) $t < n/4$.

Proof. Since $\alpha(G) = 3$, there is an edge between each pair of $P_i, P_j, (i \neq j)$. Thus, by contracting each P_k $(1 \leq k \leq t)$ into a single vertex, we see that $K_t \preceq G$. Thus $t < n/4$, as desired. ■

Since $\alpha(G) = 3$, $G - \cup_{k=1}^{t} P_k$ has at most three components. Let $V(G - \cup_{k=1}^{t} P_k) = A \cup B \cup C$ (where A, B, C could be empty) such that $G[A]$, $G[B]$, $G[C]$ are components of $G - \cup_{k=1}^{t} P_k$. Let $V(P_k) = \{x_k, y_k, z_k\}$, where $1 \leq k \leq t$, and $x_k y_k, y_k z_k \in E(G)$ but $x_k z_k \notin E(G)$. Since each of $G[A]$, $G[B]$, $G[C]$ is complete and $G \nsucceq K_{\lceil n/4 \rceil}$, we have $|A|, |B|, |C| < n/4$. By (1), we may assume that

(2) $A, B \neq \emptyset$.

We may further assume that $|A|$ is maximum among $|A|$, $|B|$, and $|C|$. Then $|A| \geq \frac{n-3t}{3}$ and so

(3) $t + |A| \geq n/3$.

Since $\alpha(G) = 3$, we have:

(4) If there exists a vertex $a \in A \cup B \cup C$ and P_k such that $ax_k, ay_k, az_k \notin E(G)$ (in this case, we say that P_k is *bad* for the component containing a), then for any $b \in A \cup B \cup C$ such that $ab \notin E(G)$, $bx_k \in E(G)$, or $bz_k \in E(G)$.

Let t_1 (resp. t_2) be the number of paths of $P_1, P_2, \ldots, P_t$ that are bad for A (resp. B). By (4), we may assume that $P_1, P_2, \ldots, P_{t_1}$ are bad for A and $P_{t_1+1}, \ldots, P_{t_1+t_2}$ are bad for B. By (4) again, every vertex of $B \cup C$ is adjacent to at least one vertex of each path $P_1, P_2, \ldots, P_{t_1}$, every vertex of $A \cup C$ is adjacent to at least one vertex of each path $P_{t_1+1}, \ldots, P_{t_1+t_2}$, and every vertex of $A \cup B$ is adjacent to at least one vertex of each path $P_{t_1+t_2+1}, \ldots, P_t$. Let $t_3 = t - t_1 - t_2$.

(5) $t_1 > 0$.

Proof. Suppose $t_1 = 0$. Then each vertex of A is adjacent to at least one vertex of P_i, where $1 \leq i \leq t$. By contracting each P_i into a single vertex, we see that $K_{t+|A|} \preceq G$. Thus $t + |A| < n/4$, contrary to (3). ■

(6) $C \neq \emptyset$.

Proof. By (2), $A, B \neq \emptyset$. Suppose $C = \emptyset$. Since $G \not\succeq K_{\lceil n/4 \rceil}$, we see that $t_2 + t_3 + |A| < n/4$ and $t_1 + |B| < n/4$. Summing up these two inequalities, we obtain that $t_1 + t_2 + t_3 + |A| + |B| < n/2$. Note that $t = t_1 + t_2 + t_3$ and $3(t_1 + t_2 + t_3) + |A| + |B| = n$. It follows that $2t = n - (t_1 + t_2 + t_3 + |A| + |B|) > n/2$, contrary to (1). ■

By (2) and (6), $A, B, C \neq \emptyset$. Since $G \not\succeq K_{\lceil n/4 \rceil}$, we see that

$$t_3 + t_2 + |A| < \frac{n}{4};$$
$$t_1 + t_3 + |B| < \frac{n}{4};$$
$$t_2 + t_1 + |C| < \frac{n}{4}.$$

Adding up the above three inequalities, we obtain $2(t_1 + t_2 + t_3) + |A| + |B| + |C| < 3n/4$. It follows that $t = t_1 + t_2 + t_3 = n - (2(t_1 + t_2 + t_3) + |A| + |B| + |C|) > n/4$, contrary to (1). This completes the proof of Theorem 1.5. ■

3. PROOF OF THEOREM 1.6

We shall prove that any graph G with $\alpha(G) \geq 3$ has a $K_{\lceil |G|/(2\alpha(G)-2) \rceil}$ minor. We need the following result of Kawarabayashi et al. [8].

Theorem 3.1. *Any graph G with $\alpha(G) \geq 2$ has a $K_{\lceil (|G|+\omega(G))/(2\alpha(G)-1)\rceil}$ minor.*

The proof of Theorem 1.6 is by induction on $\alpha(G)$. By Theorem 1.5, we may assume that $\alpha(G) \geq 4$. Throughout this section, we assume that G is a graph with $n = |G|$ and $\alpha = \alpha(G) \geq 4$ but $G \not\succeq K_{\lceil n/(2\alpha-2)\rceil)}$. Thus $n \geq 4\alpha - 4$, otherwise $G \succeq K_{\lceil n/(2\alpha-2)\rceil}$, a contradiction.

(1) G is connected.

Proof. Suppose G is disconnected. Let G_1 and G_2 be two subgraphs of G such that $G_1 \cup G_2 = G$ and $V(G_1) \cap V(G_2) = \emptyset$. Let $\alpha_i := \alpha(G_i)$ and $n_i := |G_i|$ for $i = 1, 2$. Then $n = n_1 + n_2$ and $\alpha = \alpha_1 + \alpha_2$. If $\alpha_1 = 1$ or $\alpha_2 = 1$, say the former, then $n_1 < \frac{n}{2\alpha-2}$ because $G \not\succeq K_{\lceil n/(2\alpha-2)\rceil}$. Thus $n_2 > \frac{n(2\alpha-3)}{2\alpha-2}$ and so $\frac{n_2}{2\alpha_2-2} > \frac{n(2\alpha-3)}{(2\alpha-2)(2\alpha-4)} \geq \frac{n}{2\alpha-2}$. Since $\alpha_2 \geq 3$ and $\alpha_2 < \alpha$, by induction $G \succeq G_2 \succeq K_{\lceil n_2/(2\alpha_2-2)\rceil} \succeq K_{\lceil n/(2\alpha-2)\rceil}$, a contradiction. Thus $\alpha_1 \geq 2$ and $\alpha_2 \geq 2$.

If $\alpha_1 \geq 3$ and $\alpha_2 \geq 3$, then by induction $G_i \succeq K_{\lceil n_i/(2\alpha_i-2)\rceil}$ for $i = 1, 2$. Since $G \not\succeq K_{\lceil n/(2\alpha-2)\rceil}$, we have

$$\frac{n_i}{2\alpha_i - 2} < \frac{n}{2\alpha - 2},$$

for $i = 1, 2$. It follows that $n_i < \frac{n(2\alpha_i-2)}{2\alpha-2}$ and thus $n = n_1 + n_2 < \frac{n(2\alpha-4)}{2\alpha-2}$, which is impossible.

It remains to prove the case when $\alpha_1 = 2$ or $\alpha_2 = 2$. By symmetry, we may assume that $\alpha_1 = 2$. Then $\alpha_2 = \alpha - 2$. If $\alpha_2 = 2$, then $\alpha = 4$ and so $n_1, n_2 < n/2$ (because $G_i \succeq K_{\lceil n_i/3\rceil}$ by Theorem 1.3), contrary to the fact that $n_1 + n_2 = n$. Thus $\alpha_2 \geq 3$. Since $\alpha_1 = 2$, by Theorem 1.3, we may assume that $n_1 < \frac{3n}{2\alpha-2}$. Then $n_2 > \frac{n(2\alpha-5)}{2\alpha-2}$. By induction, $G_2 \succeq K_{\lceil n_2/(2\alpha_2-2)\rceil}$. Since $G \not\succeq K_{\lceil n/(2\alpha-2)\rceil}$, we have

$$\frac{n}{2\alpha - 2} \geq \frac{n_2}{2\alpha_2 - 2} > \frac{n(2\alpha - 5)}{(2\alpha - 2)(2\alpha - 6)}$$

which is a contradiction. This completes the proof of (1). ∎

(2) G is claw-free.

Proof. Suppose G contains a claw, say C. We shall build a set $T \subseteq V(G)$ starting from the claw C by the method of Duchet and Meyniel. The obtained T satisfies:

(i) $V(C) \subseteq T \subseteq V(G)$;
(ii) $|T| \leq 2\alpha - 2$;
(iii) $G[T]$ is connected and $\alpha(G[T]) \geq 3$;
(iv) T is dominating in G, that is, every vertex in $G - T$ has a neighbor in T.

Let us sketch how to build T. The set T is built up stepwise from $V(C)$, in each step, adding two new adjacent vertices (if such two vertices exist), one from the neighborhood of T and one from the non-neighborhood of T. Since $\alpha(C) = 3$ and the independence number of $G[T]$ is increased by one in each step, the desired T

is obtained after at most $\alpha - 3$ steps. Clearly every vertex in $G - T$ has a neighbor in T, and T has at most $2\alpha - 2$ vertices. Note that $G - T$ may be disconnected.

From the above construction of T, we see that $|G - T| \geq n - 2\alpha + 2$. Suppose $\alpha(G - T) \geq 3$. Then by induction, $G - T \succeq K_{\lceil (n-2\alpha+2)/(2\alpha-2)\rceil}$ and so $G \succeq K_{\lceil n/(2\alpha-2)\rceil}$ by contracting T into a single vertex, a contradiction. Thus $\alpha(G - T) \leq 2$. By Theorem 1.3, $G - T \succeq K_{\lceil (n-2\alpha+2)/3\rceil}$. By contracting T into a single vertex, we see that $G \succeq K_{\lceil (n+5-2\alpha)/3\rceil} \succeq K_{\lceil n/(2\alpha-2)\rceil}$, because $n \geq 4\alpha - 4$ and $\frac{n+5-2\alpha}{3} \geq \frac{n}{6} \geq \frac{n}{2\alpha-2}$ when $\alpha \geq 4$, a contradiction. This completes the proof of (2). ■

Let $x \in V(G)$ be a vertex of maximum degree. Let $d := d_G(x)$ and let ω be the clique number of $G[N(x)]$. Since $n \geq 4\alpha - 4$, we have $d \geq 3$. Clearly $\alpha(G[N(x)]) \leq 2$ because G is claw-free by (2). By Theorem 3.1, $G[N(x)] \succeq K_{\lceil (d+\omega)/3\rceil}$ and so $G \succeq G[N[x]] \succeq K_{\lceil (d+\omega)/3\rceil+1}$. Since $G \not\succeq K_{\lceil n/(2\alpha-2)\rceil}$, we have

(3) $1 + \frac{d+\omega}{3} < \frac{n}{2\alpha-2}$.

Next, we claim that $\alpha(G[N(x)]) = 2$. Suppose $G[N(x)]$ is complete. Then $G \succeq K_{|N[x]|}$ and thus $d + 1 < \frac{n}{2\alpha-2}$. Now $|G - N[x]| > \frac{n(2\alpha-3)}{2\alpha-2}$ and $\alpha(G - N[x]) = \alpha - 1 \geq 3$. By induction, $G \succeq G - N[x] \succeq K_{\lceil |G-N[x]|/(2\alpha-4)\rceil} \succeq K_{\lceil n/(2\alpha-2)\rceil}$, a contradiction. Thus $\alpha(G[N(x)]) = 2$, as claimed. Since $\alpha(G[N(x)]) = 2$ and $|N(x)| = d \geq 3$, $G[N(x)]$ may contain induced 2-edge paths. Let $P_1, \ldots, P_t$ be pairwise disjoint-induced 2-edge paths in $G[N(x)]$ with $t \geq 0$ maximum. Then $G[N(x)] - \cup_{k=1}^{t} P_k$ has s complete components, where $0 \leq s \leq 2$, and each component has at most ω vertices. Note that every component of $G[N(x)] - \cup_{k=1}^{t} P_k$ is complete by the choice of t. Thus

(4) $t \geq \frac{d-s\omega}{3}$.

Now let G' be obtained from G by contracting each of the path P_k into a single vertex, where $1 \leq k \leq t$, and deleting the vertices in $N[x] - \cup_{k=1}^{t} V(P_k)$. Then $|G'| = n - d - 1 + t$. Moreover, $\alpha(G') \leq \alpha - 1$. Suppose $\alpha(G') \geq 3$. Then by induction, $G \succeq G' \succeq K_{\lceil (n-d-1+t)/(2\alpha-4)\rceil}$. Since $G \not\succeq K_{\lceil n/(2\alpha-2)\rceil}$, we have $\frac{n-d-1+t}{2\alpha-4} < \frac{n}{2\alpha-2}$. On the other hand, by (4), $n - d - 1 + t \geq n - 2(1 + \frac{1}{3}(d + w)) + \frac{(2-s)\omega}{3} + 1$. Thus,

$$\frac{n - 2(1 + \frac{1}{3}(d + w)) + \frac{(2-s)\omega}{3} + 1}{2\alpha - 4} < \frac{n}{2\alpha - 2},$$

from which it follows that $1 + \frac{d+w}{3} > \frac{n}{2\alpha-2} + \frac{(2-s)\omega}{6} + \frac{1}{2}$, contrary to (3) because $2 - s \geq 0$. Thus, $\alpha(G') \leq 2$. Then $\alpha = 4$ and by Theorem 1.3, $G' \succeq K_{\lceil |G'|/3\rceil}$. Since $G \not\succeq K_{\lceil n/6\rceil}$, we have

$$\frac{n - d - 1 + t}{3} < \frac{n}{6}.$$

It follows that $t < d + 1 - n/2$. This, together with (4), implies that $2d + 2\omega + 6 \geq 2d + s\omega + 6 > 3n/2 + 3$, contrary to (3). This completes the proof of Theorem 1.6. ■

4. CONCLUDING REMARKS

Recently, Chudnovsky and Seymour (private communication) gave a structure theorem for claw-free graphs, and used it to prove that every connected claw-free graph G with $\alpha(G) \geq 3$ has $\chi(G) \leq 2\omega(G)$, see [3]. Together with Theorem 3.1 and the argument of (2) in section 3, it is not hard to see that every graph G with $\alpha(G) \geq 3$ has a K_t minor, where $t \geq \frac{(2\alpha(G)+1}{2\alpha(G)-1)2\alpha(G)}|G|$. Note that Theorem 1.6 gives a better constant. Furthermore, Chudnovsky and Seymour (private communication) also proved Theorem 1.5 using their structure theorem on claw-free graphs. Our proof appears to be shorter.

ACKNOWLEDGMENT

The authors would like to thank the referees for helpful comments; in particular the authors are grateful to the referee who suggested a shortening of the proof of Theorem 1.5. The research work of K. K. was partly supported by the Japan Society for the Promotion of Science for Young Scientists, by Grant-in-Aid for Scientific Research, by Sumitomo Foundation, and by Inoue Research Award for Young Scientists.

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