Multicolor Gallai-Ramsey numbers of C_9 and C_{11}

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Abstract

A *Gallai coloring* is a coloring of the edges of a complete graph without rainbow triangles, and a *Gallai k-coloring* is a Gallai coloring that uses k colors. We study Ramsey-type problems in Gallai colorings. Given an integer $k \geq 1$ and a graph H, the Gallai-Ramsey number $GR_k(H)$ is the least positive integer n such that every Gallai k-coloring of the complete graph on n vertices contains a monochromatic copy of H. It turns out that $GR_k(H)$ is more well-behaved than the classical Ramsey number $R_k(H)$. However, finding exact values of $GR_k(H)$ is far from trivial. In this paper, we study Gallai-Ramsey numbers of odd cycles. We prove that for $n \in \{4, 5\}$ and all $k \geq 1$, $GR_k(C_{2n+1}) = n \cdot 2^k + 1$. This new result provides partial evidence for the first two open cases of the Triple Odd Cycle Conjecture of Bondy and Erdős from 1973. Our technique relies heavily on the structural result of Gallai on Gallai colorings of complete graphs. We believe the method we developed can be used to determine the exact values of $GR_k(C_{2n+1})$ for all $n \geq 6$.

Keywords: Gallai coloring; Gallai-Ramsey number; Rainbow triangle
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1 Introduction

All graphs in this paper are finite and simple; that is, they have no loops or parallel edges. Given a graph G and a set $A \subseteq V(G)$, we use $|G|$ to denote the number of vertices of G, and $G[A]$ to denote the subgraph of G obtained from G by deleting all vertices in $V(G) \setminus A$. A graph H is an *induced subgraph* of G if $H = G[A]$ for some $A \subseteq V(G)$. We use K_n and C_n to denote the complete graph and cycle on n vertices, respectively. For any positive integer k, we write $[k]$ for the set $\{1, 2, \ldots, k\}$. We use the convention "$A :=$" to mean that A is defined to be the right-hand side of the relation.

Given an integer $k \geq 1$ and a graph H, the classical Ramsey number $R(H)$ is the least integer n such that every k-coloring of the edges of K_n contains a monochromatic copy of H. Ramsey numbers are notoriously difficult to compute in general. In this paper, we study Ramsey numbers

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of graphs in Gallai colorings, where a *Gallai coloring* is a coloring of the edges of a complete graph without rainbow triangles (that is, a triangle with all its edges colored differently). Gallai colorings naturally arise in several areas including: information theory [18]; the study of partially ordered sets, as in Gallai's original paper [12] (his result was restated in [15] in the terminology of graphs); and the study of perfect graphs [4]. There are now a variety of papers which consider Ramsey-type problems in Gallai colorings (see, e.g., [6, 10, 13, 14, 16, 3, 21, 22]). These works mainly focus on finding various monochromatic subgraphs in such colorings. More information on this topic can be found in [9, 11].

A *Gallai k-coloring* is a Gallai coloring that uses k colors. Given an integer $k \geq 1$ and a graph H, the *Gallai-Ramsey number* $GR_k(H)$ is the least integer n such that every Gallai k-coloring of K_n contains a monochromatic copy of H. Clearly, $GR_k(H) \leq R_k(H)$ for all $k \geq 1$ and $GR_2(H) = R_2(H)$. In 2010, Gyárfás, Sárközy, Sebő and Selkow [14] proved the general behavior of $GR_k(H)$.

Theorem 1.1 ([14]) *Let H be a fixed graph with no isolated vertices and let $k \geq 1$ be an integer. Then $GR_k(H)$ is exponential in k if H is not bipartite, linear in k if H is bipartite but not a star, and constant (does not depend on k) when H is a star.*

It turns out that for some graphs H (e.g., when $H = C_3$), $GR_k(H)$ behaves nicely, while the order of magnitude of $R_k(H)$ seems hopelessly difficult to determine. It is worth noting that finding exact values of $GR_k(H)$ is far from trivial, even when $|H|$ is small. We will utilize the following important structural result of Gallai [12] on Gallai colorings of complete graphs.

Theorem 1.2 ([12]) *For any Gallai-coloring c of a complete graph G, $V(G)$ can be partitioned into nonempty sets $V_1, V_2, \ldots, V_p$ with $p > 1$ so that at most two colors are used on the edges in $E(G) \setminus (E(V_1) \cup \cdots \cup E(V_p))$ and only one color is used on the edges between any fixed pair (V_i, V_j) under c, where $E(V_i)$ denotes the set of edges in $G[V_i]$ for all $i \in [p]$.*

The partition given in Theorem 1.2 is a *Gallai-partition* of the complete graph G under c. Given a Gallai-partition $V_1, V_2, \ldots, V_p$ of the complete graph G under c, let $v_i \in V_i$ for all $i \in [p]$ and let $\mathcal{R} := G[\{v_1, v_2, \ldots, v_p\}]$. Then $\mathcal{R}$ is the *reduced graph* of G corresponding to the given Gallai-partition under c. Clearly, $\mathcal{R}$ is isomorphic to K_p. By Theorem 1.2, all edges in $\mathcal{R}$ are colored by at most two colors under c. One can see that any monochromatic H in $\mathcal{R}$ under c will result in a monochromatic H in G under c. It is not surprising that Gallai-Ramsey numbers $GR_k(H)$ are related to the classical Ramsey numbers $R_2(H)$. Recently, Fox, Grinshpun and Pach posed the following conjecture on $GR_k(H)$ when H is a complete graph.

Conjecture 1.3 ([9]) *For all integers $k \geq 1$ and $t \geq 3$,*

$$GR_k(K_t) = \begin{cases} (R_2(K_t) - 1)^{k/2} + 1 & \text{if } k \text{ is even} \\ (t-1)(R_2(K_t) - 1)^{(k-1)/2} + 1 & \text{if } k \text{ is odd.} \end{cases}$$

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The first case of Conjecture 1.3 follows from a result of Chung and Graham [6] in 1983. The next open case when $t = 4$ was recently settled in [19]. In this paper, we study Gallai-Ramsey numbers of odd cycles. Using the same construction given by Erdős, Faudree, Rousseau and Schelp in 1976 (see Section 2 in [8]) for classical Ramsey numbers of odd cycles, we see that $GR_k(C_{2n+1}) \geq n \cdot 2^k + 1$ for all $k \geq 1$ and $n \geq 2$. General upper bounds for $GR_k(C_{2n+1})$ were first studied in [10] and later improved in [16].

Theorem 1.4 ([16]) *For all $k \geq 1$ and $n \geq 2$,*

$$n \cdot 2^k + 1 \leq GR_k(C_{2n+1}) \leq (2^{k+3} - 3)n \ln n.$$

Theorem 1.5 and Theorem 1.6 below determine the exact values of $GR_k(C_3)$ and $GR_k(C_5)$, respectively. A simpler proof of Theorem 1.5 can be found in [14].

Theorem 1.5 ([6]) *For all $k \geq 1$,* $GR_k(C_3) = \begin{cases} 5^{k/2} + 1 & \text{if } k \text{ is even} \\ 2 \cdot 5^{(k-1)/2} + 1 & \text{if } k \text{ is odd.} \end{cases}$

Theorem 1.6 ([10]) *For all $k \geq 1$,* $GR_k(C_5) = 2 \cdot 2^k + 1$.

Recently, Bruce and Song [3] considered the next step and determined the exact values of $GR_k(C_7)$ for all integers $k \geq 1$.

Theorem 1.7 ([3]) *For every integer $k \geq 1$,* $GR_k(C_7) = 3 \cdot 2^k + 1$.

We continue to study the Gallai-Ramsey numbers of odd cycles in this paper. We determine the exact values of Gallai-Ramsey numbers of C_9 and C_{11} in this paper by showing that the lower bound in Theorem 1.4 is also the desired upper bound. That is, we prove that $GR_k(C_{2n+1}) \leq n \cdot 2^k + 1$ for all integers $n \in \{4, 5\}$ and $k \geq 1$. Jointly with Bosse and Zhang [2], we are currently working on the Gallai-Ramsey numbers of C_{13} and C_{15}, using the key ideas developed in this paper. We believe the method we developed in this paper and [2] will be helpful in determining the exact values of Gallai-Ramsey numbers of C_{2n+1} for all $n \geq 8$. Theorem 1.8 is our main result.

Theorem 1.8 *For all integers $n \in \{4, 5\}$ and $k \geq 1$,* $GR_k(C_{2n+1}) = n \cdot 2^k + 1$.

It is worth mentioning that Theorem 1.8 also provides partial evidence for the first two open cases of the Triple Odd Cycle Conjecture due to Bondy and Erdős [1], which states that $R_3(C_{2n+1}) = 8n + 1$ for all integers $n \geq 2$. Łuczak [20] showed that $R_3(C_{2n+1}) = 8n + o(n)$, as $n \to \infty$, and Kohayakawa, Simonovits and Skokan [17] announced a proof in 2005 that the Triple Odd Cycle Conjecture holds when n is sufficiently large.

We shall make use of the following known results in the proof of Theorem 1.8.

Theorem 1.9 ([1]) *For all $n \geq 2$,* $R_2(C_{2n+1}) = 4n + 1$.

Proposition 1.10 ([5]) $R_2(C_4) = 6$ *and* $R_2(C_6) = 8$.

Finally, we need to introduce more notation. For positive integers n, k and a complete graph G, let c be any Gallai k-coloring of G with color classes $E_1, \ldots, E_k$. Then c is *bad* if G contains no monochromatic C_{2n+1} under c. For any $W \subseteq V(G)$ and any color $i \in [k]$, $E := E_i \cap E(G[W])$ is an *induced matching* in $G[W]$ if E is a matching in $G[W]$. For two disjoint sets $A, B \subseteq V(G)$, A is *mc-complete* to B under the coloring c if all the edges between A and B in G are colored the same color under c; and we simply say A is *j-complete* to B if all the edges between A and B in G are colored by some color $j \in [k]$ under c; and A is *blue-complete* to B if all the edges between A and B in G are colored blue under c. For convenience, we use $A \setminus B$ to denote $A - B$; and $A \setminus b$ to denote $A - \{b\}$ when $B = \{b\}$. We conclude this section with two useful lemmas.

Lemma 1.11 *For all integers $n \geq 3$ and $k \geq 1$, let c be a k-coloring of the edges of a complete graph G on at least $2n + 1$ vertices. Let $Y, Z \subseteq V(G)$ be two disjoint sets with $|Y| \geq n$ and $|Z| \geq n$. If Y is mc-complete, say blue-complete, to Z under the coloring c, then no vertex in $V(G) \setminus (Y \cup Z)$ is blue-complete to $Y \cup Z$ in G. Moreover, if $|Z| \geq n + 1$, then $G[Z]$ has no blue edges. Similarly, if $|Y| \geq n + 1$, then $G[Y]$ has no blue edges.*

Proof. Suppose there exists a vertex $x \in V(G) \setminus (Y \cup Z)$ such that x is blue-complete to $Y \cup Z$ in G. Let $Y = \{y_1, \ldots, y_{|Y|}\}$ and $Z = \{z_1, \ldots, z_{|Z|}\}$. We may further assume that $z_1 z_2$ is colored blue under c if $|Z| \geq n + 1$ and $G[Z]$ has a blue edge. We then obtain a blue C_{2n+1} with vertices $y_1, x, z_1, y_2, z_2, \ldots, y_n, z_n$ in order when $|Y| \geq n, |Z| \geq n$ or vertices $y_1, z_1, z_2, y_2, z_3, \ldots, y_n, z_{n+1}$ in order when $|Z| \geq n + 1$ and $G[Z]$ has a blue edge $z_1 z_2$, a contradiction. Thus no vertex in $V(G) \setminus (Y \cup Z)$ is blue-complete to $Y \cup Z$ in G; and if $|Z| \geq n + 1$, then $G[Z]$ has no blue edges. Similarly, one can prove that if $|Y| \geq n + 1$, then $G[Y]$ has no blue edges. ∎

Lemma 1.12 *For all integers $\ell \geq 3$ and $n \geq 1$, let $n_1, n_2, \ldots, n_\ell$ be positive integers such that $n_i \leq n$ for all $i \in [\ell]$ and $n_1 + n_2 + \cdots + n_\ell \geq 2n + 1$. Then the complete multipartite graph $K_{n_1, n_2, \ldots, n_\ell}$ has a cycle of length $2n + 1$.*

Proof. Let $G := K_{n_1', n_2', \ldots, n_{\ell'}'}$ be an induced subgraph of $K_{n_1, n_2, \ldots, n_\ell}$ with $\ell' \geq 3$, $n_1' + n_2' + \cdots + n_{\ell'}' = 2n + 1$ and for all $i \in [\ell']$, $1 \leq n_i' \leq n$. Then $\delta(G) \geq n + 1 \geq |G|/2$. By a well-known theorem of Dirac [7], G has a Hamilton cycle, and so $K_{n_1, n_2, \ldots, n_\ell}$ has a cycle of length $2n + 1$. ∎

2   Proof of Theorem 1.8

Let $n \in \{4, 5\}$. By the construction given by Erdős, Faudree, Rousseau and Schelp in 1976 (see Section 2 in [8]) for classical Ramsey numbers of odd cycles, $GR_k(C_{2n+1}) \geq n \cdot 2^k + 1$ for all $k \geq 1$. We next show that $GR_k(C_{2n+1}) \leq n \cdot 2^k + 1$ for all $k \geq 1$. This is trivially true for $k = 1$. By Theorem 1.9 and the fact that $GR_2(C_{2n+1}) = R_2(C_{2n+1})$, we may assume that $k \geq 3$. Let $G := K_{n \cdot 2^k + 1}$ and

let c be any Gallai k-coloring of G. We next show that G contains a monochromatic copy of C_{2n+1} under the coloring c.

Suppose that G does not contain any monochromatic C_{2n+1} under c. Then c is bad. Among all complete graphs on $n \cdot 2^k + 1$ vertices with a bad Gallai k-coloring, we choose G with k minimum. We next prove a series of claims.

Claim 2.1 *Let $W \subseteq V(G)$ and let $\ell \geq 3$ be an integer. Let $x_1, \ldots, x_\ell \in V(G) \setminus W$ such that $\{x_1, \ldots, x_\ell\}$ is mc-complete, say blue-complete, to W under c. Let $q \in \{0, 1, \ldots, k-1\}$ be the number of colors, other than blue, missing on $G[W]$ under c.*

(i) If $\ell \geq n$, then $|W| \leq n \cdot 2^{k-1-q}$.

(ii) If $\ell = n - 1$, then $|W| \leq n \cdot 2^{k-1-q} + 2$.

(iii) If $\ell = n - 2$, then $n = 5$ and $|W| \leq 8 \cdot 2^{k-1-q} - 1$.

Proof. The statement in each of (i), (ii) and (iii) is trivially true if $|W| < \max\{2n + 1 - \ell, n + 1\}$. So we may assume that $|W| \geq \max\{2n + 1 - \ell, n + 1\}$. We may further assume that $G[W]$ contains at least one blue edge, else, by minimality of k, $|W| \leq n \cdot 2^{k-1-q}$, giving the result. Note that $q \leq k - 1$. If $q = k - 1$, then all the edges of $G[W]$ are colored only blue. Since $\{x_1, \ldots, x_\ell\}$ is blue-complete to W and $|W| \geq \max\{2n + 1 - \ell, n + 1\}$, we see that $G[W \cup \{x_1, \ldots, x_\ell\}]$ contains a blue C_{2n+1}, a contradiction. Thus $q \leq k - 2$. Since $|W| \geq n + 1$ and $G[W]$ contains at least one blue edge, by Lemma 1.11, $\ell \leq n - 1$. Let W^* be a minimal set of vertices in W such that $G[W \setminus W^*]$ has no blue edges. By minimality of k, $|W \setminus W^*| \leq n \cdot 2^{k-1-q}$.

We now consider the case when $\ell = n - 1$. Then $|W| \geq 2n + 1 - \ell = n + 2$. If $G[W]$ contains three blue edges, say $u_1 v_1, u_2 v_2, u_3 v_3$, such that $u_1, u_2, u_3, v_1, v_2, v_3$ are all distinct, then we obtain a blue C_{2n+1} with vertices $x_1, u_1, v_1, x_2, u_2, v_2, x_3, u_3, v_3$ in order (when $n = 4$) and vertices $x_1, u_1, v_1, x_2, u_2, v_2, x_3, u_3, v_3, x_4, u$ in order (when $n = 5$, where $u \in W \setminus \{u_1, u_2, u_3, v_1, v_2, v_3\}$), a contradiction. Thus $|W^*| \leq 2$, and so $|W| \leq n \cdot 2^{k-1-q} + 2$.

It remains to consider the case when $3 \leq \ell \leq n - 2$. Then $n = 5$ and $\ell = n - 2 = 3$. Note that $|W| \geq 2n + 1 - \ell \geq 8$. Let P be a longest blue path in $G[W]$ with vertices $v_1, \ldots, v_{|P|}$ in order. Since $\{x_1, x_2, x_3\}$ is blue-complete to W, we see that $|P| \leq 5$, else we obtain a blue C_{11} with vertices $x_1, v_1, \ldots, v_6, x_2, u_1, x_3, u_2$ in order, where $u_1, u_2 \in W \setminus \{v_1, \ldots, v_6\}$, a contradiction. Assume first that $|W^*| \leq 4$. Then,

$$|W| = |W \setminus W^*| + |W^*| \leq n \cdot 2^{k-1-q} + 4 < 8 \cdot 2^{k-1-q} - 1,$$

because $q \leq k - 2$ and $k \geq 3$. So we may assume that $|W^*| \geq 5$. By the choice of W^*, we see that $|P| \in \{2, 3\}$, else we obtain a blue C_{11}. Furthermore, if $|P| = 3$, then $G[W \setminus V(P)]$ has no blue path on three vertices. Thus all the blue edges in $G[W \setminus V(P)]$ induce a blue matching. Let $m := |W^* \setminus V(P)|$ and let $u_2 w_2, \ldots, u_{m+1} w_{m+1}$ be all the blue edges in $G[W \setminus V(P)]$,

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where $u_2, \ldots, u_{m+1}, w_2, \ldots, w_{m+1}$ are all distinct. By the choice of W^*, we may assume that $u_2, \ldots, u_{m+1} \in W^*$. Let $u_1 = v_1$ and $w_1 = v_2$, and $A := W \setminus (V(P) \cup \{u_2, \ldots, u_{m+1}, w_2, \ldots, w_{m+1}\})$. Let $B := \{u_1, u_2, \ldots, u_{m+1}\}$ when $|A| \leq 1$ and let $B := \{u_1, u_2, \ldots, u_{m+1}\} \cup \{a_1, a_2\}$ when $|A| \geq 2$ and $a_1, a_2 \in A$ with $a_1 \neq a_2$. We claim that $|B| \leq 3 \cdot 2^{k-1-q}$. Suppose $|B| \geq 3 \cdot 2^{k-1-q} + 1$. By Theorem 1.7, $G[B]$ has a monochromatic, say green, C_7. Then $|V(C_7) \cap \{u_1, u_2, \ldots, u_{m+1}\}| \geq 5$ and so $C_7 \setminus \{a_1, a_2\}$ has a matching of size two. We may assume that $u_2 u_3, u_4 u_5 \in E(C_7)$. Since G has no rainbow triangles under the coloring c, we see that for any $i \in \{2, 4\}$, $\{u_i, w_i\}$ is green-complete to $\{u_{i+1}, w_{i+1}\}$. Thus we obtain a green C_{11} from the C_7 by replacing the edge $u_2 u_3$ with the path $u_2 w_3 w_2 u_3$ and edge $u_4 u_5$ with the path $u_4 w_5 w_4 u_5$, a contradiction (see Figure 2.1). Thus $|B| \leq 3 \cdot 2^{k-1-q}$, as claimed.
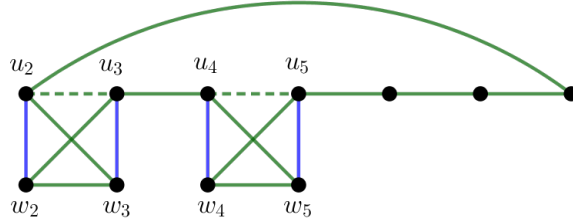

Figure 2.1: An example of a green C_{11} arising from the green C_7.

When $|A| \leq 1$, we have $|W| = |A| + 2|B| + |V(P) \setminus \{v_1, v_2\}| \leq 1 + 6 \cdot 2^{k-1-q} + 1 < 8 \cdot 2^{k-1-q} - 1$ because $q \leq k - 2$ and $k \geq 3$. When $|A| \geq 2$, since $G[A \cup \{w_1, w_2, \ldots, w_{m+1}\}]$ has no blue edges, by minimality of k, $|A \cup \{w_1, w_2, \ldots, w_{m+1}\}| \leq 5 \cdot 2^{k-1-q}$. Hence,

$$\begin{aligned} |W| &= |A \cup \{w_1, w_2, \ldots, w_{m+1}\}| + |B \setminus \{a_1, a_2\}| + |V(P) \setminus \{v_1, v_2\}| \\ &\leq 5 \cdot 2^{k-1-q} + (3 \cdot 2^{k-1-q} - 2) + 1 \\ &= 8 \cdot 2^{k-1-q} - 1. \end{aligned}$$

This completes the proof of Claim 2.1. ∎

Let $X_1, \ldots, X_m$ be a maximum sequence of disjoint subsets of $V(G)$ such that, for all $j \in [m]$, one of the following holds.

(a) $1 \leq |X_j| \leq 2$, and X_j is mc-complete to $V(G) \setminus \bigcup_{i \in [j]} X_i$ under c, or

(b) $3 \leq |X_j| \leq 4$, and X_j can be partitioned into two non-empty sets X_{j_1} and X_{j_2}, where $j_1, j_2 \in [k]$ are two distinct colors, such that for each $t \in \{1, 2\}$, $1 \leq |X_{j_t}| \leq 2$, X_{j_t} is j_t-complete to $V(G) \setminus \bigcup_{i \in [j]} X_i$ but not j_t-complete to $X_{j_{3-t}}$, and all the edges between X_{j_1} and X_{j_2} in G are colored using only the colors j_1 and j_2.

Note that such a sequence $X_1, \ldots, X_m$ may not exist. Let $X := \bigcup_{j \in [m]} X_j$. For each $x \in X$, let $c(x)$ be the unique color on the edges between x and $V(G) \setminus X$ under c. For all $i \in [k]$, let

$X_i^* := \{x \in X : c(x) = \text{color } i\}$. Then $X = \bigcup_{i \in [k]} X_i^*$. It is worth noting that for all $i \in [k]$, X_i^* is possibly empty. By abusing the notation, we use X_b^* to denote X_i^* when the color i is blue. Similarly, we use X_r^* to denote X_i^* when the color i is red.

Claim 2.2 *For all $i \in [k]$, $|X_i^*| \leq 2$.*

Proof. Suppose the statement is false. Then $m \geq 2$. When choosing $X_1, X_2, \ldots, X_m$, let $j \in [m-1]$ be the largest index such that $|X_p^* \cap (X_1 \cup X_2 \cup \cdots \cup X_j)| \leq 2$ for all $p \in [k]$. Then $3 \leq |X_i^* \cap (X_1 \cup X_2 \cup \cdots \cup X_j \cup X_{j+1})| \leq 4$ for some color $i \in [k]$ by the choice of j. Such a color i and an index j exist due to the assumption that the statement of Claim 2.2 is false. Let $A := X_1 \cup X_2 \cup \cdots \cup X_j \cup X_{j+1}$. By the choice of $X_1, X_2, \ldots, X_m$, there are at most two colors $i \in [k]$ such that $3 \leq |X_i^* \cap A| \leq 4$. We may assume that such a color i is either blue or red. Let $A_b := \{x \in A : c(x) \text{ is color blue}\}$ and $A_r := \{x \in A : c(x) \text{ is color red}\}$. It suffices to consider the worst case when $3 \leq |A_b| \leq 4$ and $3 \leq |A_r| \leq 4$. Then for any color $p \in [k]$ other than red and blue, $|X_p^* \cap A| \leq 2$. Thus by the choice of j, $|A \setminus (A_b \cup A_r)| \leq 2(k-2)$. We may assume that $|A_b| \geq |A_r|$. Note that $|A_b| \leq n$. If $|A_b| \geq n-1$, then by Claim 2.1(ii) applied to any $n-1$ vertices in A_b and $V(G) \setminus A$, we see that $|V(G) \setminus A| \leq n \cdot 2^{k-1} + 2$. Thus,

$$|G| = |A \setminus (A_b \cup A_r)| + |A_b| + |A_r| + |V(G) \setminus A| \leq 2(k-2) + n + n + (n \cdot 2^{k-1} + 2) < n \cdot 2^k + 1$$

for all $k \geq 3$ and $n \in \{4, 5\}$, a contradiction. Thus $3 \leq |A_b| \leq n-2$. Then $|A_b| = 3$ and $n = 5$. By Claim 2.1(iii) applied to A_b and $V(G) \setminus A$, we see that $|V(G) \setminus A| \leq 8 \cdot 2^{k-1} - 1$. Thus,

$$|G| = |A \setminus (A_b \cup A_r)| + |A_b| + |A_r| + |V(G) \setminus A| \leq 2(k-2) + 3 + 3 + (8 \cdot 2^{k-1} - 1) < 5 \cdot 2^k + 1$$

for all $k \geq 3$, a contradiction. ∎

By Claim 2.2, $|X| \leq 2k$. Let $X' \subseteq X$ be such that for all $i \in [k]$, $|X' \cap X_i^*| = 1$ when $X_i^* \neq \emptyset$. Let $X'' := X \setminus X'$. Now consider a Gallai partition $A_1, \ldots, A_p$ of $G \setminus X$ with $p \geq 2$. We may assume that $1 \leq |A_1| \leq \cdots \leq |A_s| < 3 \leq |A_{s+1}| \leq \cdots \leq |A_p|$, where $0 \leq s \leq p$. Let $\mathcal{R}$ be the reduced graph of $G \setminus X$ with vertices $a_1, a_2, \ldots, a_p$, where $a_i \in A_i$ for all $i \in [p]$. By Theorem 1.2, we may assume that the edges of $\mathcal{R}$ are colored red and blue. Note that any monochromatic C_{2n+1} in $\mathcal{R}$ would yield a monochromatic C_{2n+1} in G. Thus $\mathcal{R}$ has neither a red nor a blue C_{2n+1}. By Theorem 1.9, $p \leq 4n$. Then $|A_p| \geq 2$ because $|G \setminus X| \geq n \cdot 2^k + 1 - 2k \geq 8n - 5$. If $|A_p| = 2$, then $k = 3$. Thus $|A_{p-4n+8}| = 2$, else $|G| \leq 2(4n-8) + (p - (4n-8)) + |X| \leq 8n - 2 < n \cdot 2^3 + 1$, a contradiction. Since $R_2(C_{2n-3}) = 4n-7$ by Theorem 1.9, we see that $\mathcal{R}[\{a_{p-4n+8}, a_{p-4n+9}, \ldots, a_p\}]$ has a monochromatic, say blue, C_{2n-3}, and so $G[A_{p-4n+8} \cup A_{p-4n+9} \cup \cdots \cup A_p]$ has a blue C_{2n+1}, a contradiction. Thus $|A_p| \geq 3$ and so $p - s \geq 1$. Let

$$B := \{a_i \in \{a_1, \ldots, a_{p-1}\} \mid a_i a_1 \text{ is colored blue in } \mathcal{R}\}$$
$$R := \{a_j \in \{a_1, \ldots, a_{p-1}\} \mid a_j a_1 \text{ is colored red in } \mathcal{R}\}$$

Then $|B| + |R| = p - 1$. Let $B_G := \bigcup_{a_i \in B} A_i$ and $R_G := \bigcup_{a_j \in R} A_j$.

Claim 2.3 *If $|A_p| \geq n$ and $|B| \geq 3$ (resp. $|R| \geq 3$), then $|B_G| \leq 2n$ (resp. $|R_G| \leq 2n$).*

Proof. Suppose $|A_p| \geq n$ and $|B| \geq 3$ but $|B_G| \geq 2n + 1$. By Lemma 1.11, $G[B_G]$ has no blue edges and no vertex in X is blue-complete to $V(G) \setminus X$. Thus all the edges of $\mathcal{R}[B]$ are colored red in $\mathcal{R}$. Let $m := |B|$ and let $B := \{a_{i_1}, a_{i_2}, \ldots, a_{i_m}\}$ with $|A_{i_1}| \geq |A_{i_2}| \geq \cdots \geq |A_{i_m}|$. Then $G[B_G] - \bigcup_{j=1}^{m} E(G[A_{i_j}])$ is a complete multipartite graph with at least three parts. If $|A_{i_1}| \leq n$, then by Lemma 1.12 applied to $G[B_G] - \bigcup_{j=1}^{m} E(G[A_{i_j}])$, $G[B_G]$ has a red C_{2n+1}, a contradiction. Thus $|A_{i_1}| \geq n + 1$. Let $Q_b := \{v \in R_G : v$ is blue-complete to $A_{i_1}\}$, and $Q_r := \{v \in R_G : v$ is red-complete to $A_{i_1}\}$. Then $Q_b \cup Q_r = R_G$. Let $Q := (B_G \setminus A_{i_1}) \cup Q_r \cup X_r^*$. Then A_{i_1} is red-complete to Q and $G[Q]$ must contain red edges, because $|B| \geq 3$ and all the edges of $\mathcal{R}[B]$ are colored red. By Lemma 1.11 applied to A_{i_1} and Q, $|Q| \leq n$. Note that $|A_p \cup Q_b| \geq |A_p| \geq |A_{i_1}| \geq n + 1$ and $A_p \cup Q_b$ is blue-complete to A_{i_1}. By Lemma 1.11 applied to A_{i_1} and $A_p \cup Q_b$, $G[A_p \cup Q_b]$ has no blue edges. Since no vertex in X is blue-complete to $V(G) \setminus X$, we see that neither $G[A_p \cup Q_b \cup (X'' \setminus X_r^*)]$ nor $G[B_G \cup X']$ (and thus $G[A_{i_1} \cup (X' \setminus X_r^*)]$) has blue edges. By minimality of k, $|A_p \cup Q_b \cup (X'' \setminus X_r^*)| \leq n \cdot 2^{k-1}$. Suppose first that $Q_r \cup X_r^* = \emptyset$. Then $Q_b = R_G$, so that

$$|G| = |B_G \cup X'| + |A_p \cup Q_b \cup X''| \leq n \cdot 2^{k-1} + n \cdot 2^{k-1} < n \cdot 2^k + 1,$$

a contradiction. Thus $Q_r \cup X_r^* \neq \emptyset$. Since $|B| \geq 3$, we see that $|B_G \setminus A_{i_1}| \geq 2$. Thus $n \geq |Q| \geq 3$. Since $G[A_{i_1} \cup (X' \setminus X_r^*)]$ has no blue edges, by Claim 2.1 applied to Q and A_{i_1} we see that

$$|A_{i_1} \cup (X' \setminus X_r^*)| \leq \begin{cases} n \cdot 2^{k-2} + 2, & \text{if } |Q| \in \{n-1, n\} \\ 8 \cdot 2^{k-2} - 1, & \text{if } |Q| = n-2 \text{ and } n = 5. \end{cases}$$

But then

$$\begin{aligned} |G| &= |Q| + |A_{i_1} \cup (X' \setminus X_r^*)| + |A_p \cup Q_b \cup (X'' \setminus X_r^*)| \\ &\leq \begin{cases} n + (n \cdot 2^{k-2} + 2) + n \cdot 2^{k-1}, & \text{if } |Q| \in \{n-1, n\} \\ 3 + (8 \cdot 2^{k-2} - 1) + n \cdot 2^{k-1}, & \text{if } |Q| = n-2 \text{ and } n = 5. \end{cases} \\ &< n \cdot 2^k + 1 \end{aligned}$$

for all $k \geq 3$ and $n \in \{4, 5\}$, a contradiction. Hence, $|B_G| \leq 2n$. Similarly, one can prove that if $|A_p| \geq n$ and $|R| \geq 3$, then $|R_G| \leq 2n$. ∎

Claim 2.4 $p \leq 2n - 1$.

Proof. Suppose $p \geq 2n$. Then $|B| + |R| = p - 1 \geq 2n - 1$. We claim that $|A_p| \leq n - 1$. Suppose $|A_p| \geq n$. We may assume that $|B| \geq |R|$. Then $|B_G| \geq |B| \geq n > 3$. By Claim 2.3, $|B_G| \leq 2n$. If $|R_G| \geq n + 1$, then by Lemma 1.11 to A_p and R_G, $G[R_G]$ has no red edges, and no vertex in X is

red-complete to $V(G) \setminus X$. Then $|X''| \leq k - 1$ and $G[R_G \cup X']$ has no red edges. By minimality of k, $|R_G \cup X'| \leq n \cdot 2^{k-1}$. Then

$$|A_p| = |G| - |B_G| - |R_G \cup X'| - |X''| \geq n \cdot 2^k + 1 - 2n - n \cdot 2^{k-1} - (k-1) \geq 2n - 1,$$

for all $k \geq 3$. By Lemma 1.11 applied to A_p and B_G, $G[A_p]$ has no blue edges and no vertex in X is blue-complete to $V(G) \setminus X$. Thus $G[A_p \cup X'']$ has neither red nor blue edges, and so $|A_p \cup X''| \leq n \cdot 2^{k-2}$ by the choice of k. But then

$$|B_G| = |G| - |R_G \cup X'| - |A_p \cup X''| \geq n \cdot 2^k + 1 - n \cdot 2^{k-1} - n \cdot 2^{k-2} \geq 2n + 1,$$

contrary to Claim 2.3. This proves that $|R_G| \leq n$. Then

$$|A_p \cup X'| = |G| - |B_G| - |R_G| - |X''| \geq (n \cdot 2^k + 1) - 2n - n - k > n \cdot 2^{k-1} + 1.$$

By minimality of k, $G[A_p \cup X']$ must have blue edges. Since $|A_p| \geq n$ and $|B_G| \geq n$, by Lemma 1.11 applied to A_p and B_G, $|A_p| = n$ and no vertex in X is blue-complete to $V(G) \setminus X$. Thus $|X| \leq 2(k-1)$. But then

$$|G| = |B_G| + |R_G| + |A_p| + |X| \leq 2n + n + n + 2(k-1) < n \cdot 2^k + 1,$$

for all $k \geq 3$, a contradiction. This proves that $|A_p| \leq n - 1$, as claimed.

Since $|A_p| \geq 3$, we have $3 \leq |A_p| \leq n - 1$. Then $k = 3$ because $n \in \{4, 5\}$ and $|G| = n \cdot 2^k + 1$. It follows that $|G| = 8n + 1$ and $|X| \leq 6$. Therefore, $|B_G| + |R_G| = |G| - |A_p| - |X| \geq (8n+1) - (n-1) - 6 = 7n - 4$. We may thus assume that $|B_G| > 2n + 3$. We next prove that $|A_p| \leq n - 2$. Suppose $|A_p| = n - 1$. If $G[B_G]$ contains three blue edges u_1v_1, u_2v_2, u_3v_3 such that $u_1, u_2, u_3, v_1, v_2, v_3$ are all distinct, then we obtain a blue C_{2n+1} with vertices in $A_p \cup \{u_1, u_2, u_3, u_4, v_1, v_2, v_3\}$, where $u_4 \in B_G \setminus \{u_1, u_2, u_3, v_1, v_2, v_3\}$, a contradiction. Thus there exists $B^* \subseteq B_G$ such that $|B^*| \leq 2$ and $G[B_G \setminus B^*]$ has no blue edges. Then $|B_G \setminus B^*| > 2n + 1$, and so $|B \setminus B^*| \geq 3$ because $|A_i| \leq n - 1$ for all $i \in [p]$. By the choice of B^*, all the edges in $\mathcal{R}[B \setminus B^*]$ are colored red. But then by Lemma 1.12, $G[B_G \setminus B^*]$ has a red C_{2n+1}, a contradiction. This proves that $3 \leq |A_p| \leq n - 2$. Then $|A_p| = 3$, $n = 5$, $|G| = 41$, and $p \leq 20$. If $|A_{p-7}| = 3$ or $|A_{p-12}| \geq 2$, then $\mathcal{R}[\{a_{p-8}, a_{p-7}, \ldots, a_p\}]$ has a monochromatic C_5, or $\mathcal{R}[\{a_{p-12}, a_{p-11}, \ldots, a_p\}]$ has a monochromatic C_7 because $R_2(C_5) = 9$ and $R_2(C_7) = 13$. In either case, we see that G has a monochromatic C_{11}, a contradiction. Thus $|A_{p-7}| \leq 2$ and $|A_{p-12}| \leq 1$. Then $|A_{p-7}| = 2$, else $|G| \leq 7 \cdot 3 + 13 \cdot 1 + 6 < 41$, a contradiction. Since $R_2(C_6) = 8$, we see that $\mathcal{R}[\{a_{p-7}, a_{p-6}, \ldots, a_p\}]$ has a monochromatic, say blue, C_6, and so $G \setminus X$ has a blue C_{10}. Thus no vertex in X is blue-complete to $G \setminus X$ and so $|X| \leq 2(k-1) = 4$. Furthermore, if $|A_{p-8}| = 2$, then $|A_{p-4}| = 2$, else $\mathcal{R}[\{a_{p-8}, a_{p-7}, \ldots, a_p\}]$ has a monochromatic C_5, and so G has a monochromatic C_{11}, a contradiction. But then $|G| \leq 4 \cdot 3 + 8 \cdot 2 + (p-12) \cdot 1 + |X| \leq 40 < 41$ when $|A_{p-8}| = 2$; and $|G| \leq 7 \cdot 3 + 2 + (p-8) \cdot 1 + |X| \leq 39 < 41$ when $|A_{p-8}| \leq 1$. In both cases, we obtain a contradiction. $\blacksquare$

Claim 2.5 $|A_p| \geq n + 1$.

Proof. Suppose $|A_p| \leq n$. By Claim 2.4, $p \leq 2n - 1$. We may assume that $a_p a_{p-1}$ is colored blue in $\mathcal{R}$. Then $|A_p \cup A_{p-1} \cup X| \leq 2n + 2(k-1)$, else we obtain a blue C_{2n+1}. If $|A_{p-4}| \geq n - 1$, then $\mathcal{R}[\{a_{p-4}, a_{p-3}, \ldots, a_p\}]$ has a monochromatic C_3 or C_5, and so G contains a monochromatic C_{2n+1}, a contradiction. Thus $|A_{p-4}| \leq n - 2$. But then

$$|G| \leq (2n + 2(k-1)) + 2n + (p-4)(n-2) \leq 4n + (2n-5)(n-2) + 2k - 2 < n \cdot 2^k + 1.$$

for all $n \in \{4, 5\}$ and $k \geq 3$, a contradiction. ∎

For the remainder of the proof, let $B_G^* := B_G \cup X_b^*$ and $R_G^* := R_G \cup X_r^*$.

Claim 2.6 $2 \leq p - s \leq 3n - 7$.

Proof. Suppose $p - s \geq 3n - 6$. Then $\mathcal{R}[\{a_{p-3n+7}, a_{p-3n+8}, \ldots, a_p\}]$ has a monochromatic C_{2n-5} because $R_2(C_{2n-5}) = 3n - 6$ when $n \in \{4, 5\}$. But then G would contain a monochromatic C_{2n+1}.

Next suppose $p - s \leq 1$. Then $p - s = 1$ because $p - s \geq 1$. Thus $|A_i| \leq 2$ for all $i \in [p-1]$ by the choice of p and s. By Claim 2.4, $p \leq 2n - 1$. Then $|B_G \cup R_G| \leq 2(p-1)$ and so $|B_G^* \cup R_G^*| \leq 2(p-1) + 2 + 2 = 2(p+1) \leq 4n$. We may assume that $|B_G^*| \geq |R_G^*|$. If $|R_G^*| \geq n$, then $|B_G^*| \geq n$. By Claim 2.5 and Lemma 1.11, $G[A_p]$ has neither blue nor red edges. By minimality of k, $|A_p| \leq n \cdot 2^{k-2}$. But then

$$|G| = |B_G^* \cup R_G^*| + |A_p| + |X \setminus (B_G^* \cup R_G^*)| \leq 4n + n \cdot 2^{k-2} + 2(k-2) < n \cdot 2^k + 1$$

for all $k \geq 3$, a contradiction. Thus $|R_G^*| \leq n - 1$. We claim that $|B_G^*| \leq 2n + 2$. This is trivially true if $|B| \leq n$. If $|B| \geq n + 1$, then $|B_G| \leq 2n$ by Claim 2.3. Thus $|B_G^*| \leq 2n + 2$, as claimed. If $|B_G^*| \geq n - 1$, then applying Claim 2.1(i,ii) to B_G^* and A_p implies that

$$|B_G^*| + |A_p| \leq \begin{cases} (n-1) + (n \cdot 2^{k-1} + 2), & \text{if } |B_G^*| = n - 1 \\ (2n+2) + n \cdot 2^{k-1}, & \text{if } |B_G^*| \geq n. \end{cases}$$

In either case, $|B_G^*| + |A_p| \leq 2n + n \cdot 2^{k-1} + 2$. But then

$$|G| = |R_G^*| + |B_G^*| + |A_p| + |X \setminus (B_G^* \cup R_G^*)| \leq (n-1) + (2n + n \cdot 2^{k-1} + 2) + 2(k-2) < n \cdot 2^k + 1,$$

for all $k \geq 3$ and $n \in \{4, 5\}$, a contradiction. Thus $n - 2 \geq |B_G^*| \geq |R_G^*|$. If $|B_G^*| = 3$, then $n = 5$. By Claim 2.1(iii) applied to B_G^* and A_p, $|A_p| \leq 8 \cdot 2^{k-1} - 1$. But then,

$$|G| = |B_G^*| + |R_G^*| + |A_p| + |X \setminus (B_G^* \cup R_G^*)| \leq 3 + 3 + (8 \cdot 2^{k-1} - 1) + 2(k-2) < 5 \cdot 2^k + 1$$

for all $k \geq 3$, a contradiction. Thus $2 \geq |B_G^*| \geq |R_G^*|$. Since $p \geq 2$, we see that $B \neq \emptyset$ or $R \neq \emptyset$. Then by maximality of m (see condition (a) when choosing $X_1, X_2, \ldots, X_m$), $B^* \neq \emptyset$, $R^* \neq \emptyset$, and B_G^* is neither blue- nor red-complete to R_G^* in G. But then, by maximality of m again (see condition (b) when choosing $X_1, X_2, \ldots, X_m$), $B_G^* = \emptyset$ and $R_G^* = \emptyset$, contrary to $p \geq 2$. ∎

Claim 2.7 $|A_{p-2}| \le n - 1$.

Proof. Suppose $|A_{p-2}| \ge n$. Then $n \le |A_{p-2}| \le |A_{p-1}| \le |A_p|$ and so $\mathcal{R}[\{a_{p-2}, a_{p-1}, a_p\}]$ is not a monochromatic triangle in $\mathcal{R}$ (else we obtain a monochromatic C_{2n+1}). Let B_1, B_2, B_3 be a permutation of A_{p-2}, A_{p-1}, A_p such that B_2 is, say blue-complete, to $B_1 \cup B_3$ in G. Then B_1 must be red-complete to B_3 in G. We may assume that $|B_1| \ge |B_3|$. By Lemma 1.11, no vertex in X is blue- or red-complete to $V(G) \setminus X$. Let $A := V(G) \setminus (B_1 \cup B_2 \cup B_3 \cup X)$. Then by Lemma 1.11, no vertex in A is red-complete to $B_1 \cup B_3$ in G, and no vertex in A is blue-complete to $B_1 \cup B_2$ or $B_2 \cup B_3$ in G. This implies that A must be red-complete to B_2 in G. We next show that $G[A]$ has no blue edges. Suppose that $G[A]$ has a blue edge, say, uv. Let

$$B_1^* := \{b \in A \mid b \text{ is blue-complete to } B_1 \text{ only in } G\}$$
$$B_2^* := \{b \in A \mid b \text{ is blue-complete to both } B_1 \text{ and } B_3 \text{ in } G\}$$
$$B_3^* := \{b \in A \mid b \text{ is blue-complete to } B_3 \text{ only in } G\}.$$

Then $A = B_1^* \cup B_2^* \cup B_3^*$. Note that B_1^*, B_2^*, B_3^* are pairwise disjoint and possibly empty. Let $b_1, \ldots, b_{n-1} \in B_1$, $b_n, \ldots, b_{2n-2} \in B_2$, and $b_{2n-1} \in B_3$. If uv is an edge in $G[B_1^* \cup B_2^*]$, then we obtain a blue C_{2n+1} with vertices $b_1, u, v, b_2, b_n, b_{2n-1}, b_{n+1}, b_3, b_{n+2}, \ldots, b_{n-1}, b_{2n-2}$ in order, a contradiction. Similarly, uv is not an edge in $G[B_2^* \cup B_3^*]$. Thus uv must be an edge in $G[B_1^* \cup B_3^*]$ with one end in B_1^* and the other in B_3^*. We may assume that $u \in B_1^*$ and $v \in B_3^*$. Then we obtain a blue C_{2n+1} with vertices $b_1, u, v, b_{2n-1}, b_n, b_2, b_{n+1}, \ldots, b_{n-1}, b_{2n-2}$ in order, a contradiction. This proves that $G[A]$ has no blue edges. By minimality of k, $|A| \le n \cdot 2^{k-1}$.

We next show that $|B_2 \cup A \cup X'| \le n \cdot 2^{k-1}$. Suppose $|B_2 \cup A \cup X'| \ge n \cdot 2^{k-1} + 1$. Then by minimality of k, $G[B_2 \cup A \cup X']$ must contain blue edges. Since $G[A]$ has no blue edges, A is red-complete to B_2, and no vertex in X is blue-complete to $V(G) \setminus X$, we see that $G[B_2]$ must contain blue edges. By Lemma 1.11, $|B_2| = n$. Then $B_2 \ne A_p$. We may assume that $B_1 = A_p$. By Lemma 1.11, $G[B_1]$ has neither blue nor red edges and so $G[B_1 \cup X']$ has neither blue nor red edges. By minimality of k, $|B_1 \cup X'| \le n \cdot 2^{k-2}$ and so $|B_3 \cup X''| \le |B_1 \cup X'| \le n \cdot 2^{k-2}$. Note that $A = \emptyset$, else, let $v \in A$. Then $G[B_2 \cup \{v\}]$ has blue edges and $B_2 \cup \{v\}$ is blue-complete to either B_1 or B_3, contrary to Lemma 1.11. But then

$$|G| = |B_1 \cup X'| + |B_2| + |B_3 \cup X''| \le n \cdot 2^{k-2} + n + n \cdot 2^{k-2} < n \cdot 2^k + 1,$$

for all $k \ge 3$, a contradiction. This proves that $|B_2 \cup A \cup X'| \le n \cdot 2^{k-1}$.

Since $|B_1| \ge |B_3|$ and $|B_1| + |B_3| = |G| - |B_2 \cup A \cup X'| - |X''| \ge n \cdot 2^{k-1} + 1 - (k-2) \ge 2n + 1$, we see that $|B_1| \ge n + 1$. Note that $|B_2| \ge n$ and $|B_3| \ge n$. By Lemma 1.11, $G[B_1]$ has neither red nor blue edges. Since each vertex in X is neither red- nor blue-complete to B_1, $G[B_1 \cup X'']$ has neither red nor blue edges. By minimality of k, $|B_1 \cup X''| \le n \cdot 2^{k-2}$ and so $|B_3| \le |B_1| \le n \cdot 2^{k-2}$. But then

$$|G| = |B_2 \cup A \cup X'| + |B_1 \cup X''| + |B_3| \le n \cdot 2^{k-1} + n \cdot 2^{k-2} + n \cdot 2^{k-2} = n \cdot 2^k,$$

a contradiction. ∎

By Claim 2.6, $2 \le p - s \le 3n - 7$ and so $|A_{p-1}| \ge 3$. We may now assume that $a_p a_{p-1}$ is colored blue in $\mathcal{R}$. Then $a_{p-1} \in B$ and so $A_{p-1} \subseteq B_G$. Thus $|B_G| \ge |A_{p-1}| \ge 3$.

Claim 2.8 $|R_G^*| \le 2n$.

Proof. Suppose $|R_G^*| \ge 2n + 1$. By Claim 2.5, $|A_p| \ge n + 1$. By Lemma 1.11, $G[R_G^*]$ has no red edges. Thus $|R_G^*| = |R_G|$ and so no vertex in X is red-complete to $V(G) \setminus X$. In particular, all the edges in $\mathcal{R}[R]$ are colored blue. By Claim 2.3, $|R| \le 2$. By Claim 2.7, $|A_{p-2}| \le n - 1$. Since $A_{p-1} \cap R_G = \emptyset$ and $|R_G| \ge 2n + 1$, we see that $|R| \ge 3$, a contradiction. ∎

Claim 2.9 $|A_{p-1}| \le n$.

Proof. Suppose $|A_{p-1}| \ge n + 1$. Then $|B_G| \ge |A_{p-1}| \ge n + 1$. By Lemma 1.11, neither $G[A_p]$ nor $G[B_G]$ has blue edges, and no vertex in X is blue-complete to $V(G) \setminus X$. Thus $|X| \le 2(k-1)$. By the choice of k, $|B_G \cup X''| \le n \cdot 2^{k-1}$ and $|A_p \cup X'| \le n \cdot 2^{k-1}$. We claim that $G[R_G]$ has blue edges. Suppose $G[R_G]$ has no blue edges. Then $G[A_p \cup R_G \cup X']$ has no blue edges. By the choice of k, $|A_p \cup R_G \cup X'| \le n \cdot 2^{k-1}$. But then $|B_G \cup X''| = |G| - |A_p \cup R_G \cup X'| \ge n \cdot 2^{k-1} + 1$, a contradiction. Thus $G[R_G]$ has blue edges, as claimed. Then $|R_G| \ge 2$. By Claim 2.8, $2 \le |R_G| \le |R_G^*| \le 2n$.

We first consider the case when $|R_G^*| \ge n - 1$. We claim that $|A_p \cup (X' \setminus R_G^*)| + |R_G^*| \le n \cdot 2^{k-2} + \max\{2n, k + n - 1\}$. If $|R_G^*| \ge n$, then by Lemma 1.11, $G[A_p]$ has no red edges and so $G[A_p \cup (X' \setminus R_G^*)]$ has no red edges. By the choice of k, $|A_p \cup (X' \setminus R_G^*)| \le n \cdot 2^{k-2}$ and so $|A_p \cup (X' \setminus R_G^*)| + |R_G^*| \le n \cdot 2^{k-2} + 2n$. If $|R_G^*| = n - 1$, then applying Claim 2.1(ii) to R_G^* and A_p, $|A_p| \le n \cdot 2^{k-2} + 2$. Thus $|A_p \cup (X' \setminus R_G^*)| + |R_G^*| \le n \cdot 2^{k-2} + 2 + (k-2) + (n-1) = n \cdot 2^{k-2} + k + n - 1$. Thus $|A_p \cup (X' \setminus R_G^*)| + |R_G^*| \le n \cdot 2^{k-2} + \max\{2n, k + n - 1\}$, as claimed. But then

$$|G| = |A_p \cup (X' \setminus R_G^*)| + |R_G^*| + |B_G \cup (X'' \setminus R_G^*)| \le (n \cdot 2^{k-2} + \max\{2n, k+n-1\}) + n \cdot 2^{k-1} < n \cdot 2^k + 1,$$

for all $k \ge 3$, a contradiction.

It remains to consider the case $2 \le |R_G| \le |R_G^*| \le n - 2$. If $|R_G^*| = 3$, then $n = 5$. By applying Claim 2.1(iii) to R_G^* and A_p, $|A_p| \le 8 \cdot 2^{k-2} - 1$. But then

$$|G| \le |A_p| + |B_G \cup X''| + |R_G^*| + |X' \setminus R_G^*| \le (8 \cdot 2^{k-2} - 1) + 5 \cdot 2^{k-1} + 3 + (k-2) < 5 \cdot 2^k + 1,$$

for all $k \ge 3$, a contradiction. Thus $|R_G^*| = |R_G| = 2$. Then no vertex in X is red-complete to $V(G) \setminus X$. Thus $|X''| \le k - 2$. Let $R_G = \{a, b\}$. Then ab must be colored blue under c because $G[R_G]$ has blue edges. If a or b, say b, is red-complete to B_G in G, then neither $G[A_p \cup \{a\} \cup X']$ nor $G[B_G \cup \{b\} \cup X'']$ has blue edges. By minimality of k, $|A_p \cup \{a\} \cup X'| \le n \cdot 2^{k-1}$ and $|B_G \cup \{b\} \cup X''| \le n \cdot 2^{k-1}$. But then $|G| = |A_p \cup \{a\} \cup X'| + |B_G \cup \{b\} \cup X''| \le n \cdot 2^{k-1} + n \cdot 2^{k-1} <$

$n \cdot 2^k + 1$ for all $k \geq 3$, a contradiction. Thus neither a nor b is red-complete to B_G in G. Let $a', b' \in B_G$ be such that aa' and bb' are colored blue under c. Then $a' = b'$, else we obtain a blue C_{2n+1} in G with vertices $a', a, b, b', x_1, y_1, x_2, \ldots, y_{n-2}, x_{n-1}$ in order, where $x_1, \ldots, x_{n-1} \in A_p$ and $y_1, \ldots, y_{n-2} \in B_G \setminus \{a', b'\}$, a contradiction. Thus $\{a, b\}$ is red-complete to $B_G \setminus a'$ in G. Then there exists $i \in [s]$ such that $A_i = \{a'\}$. Since $G[B_G]$ has no blue edges, we see that $\{a, b, a'\}$ must be red-complete to $B_G \setminus a'$ in G. By Claim 2.1(ii,iii) applied to the three vertices a, b, a' and $B_G \setminus a'$, we see that $|B_G \setminus a'| \leq 4 \cdot 2^{k-2} + 2$ when $n = 4$ and $|B_G \setminus a'| \leq 8 \cdot 2^{k-2} - 1$ when $n = 5$. But then

$$
\begin{aligned}
|G| &= |A_p \cup X'| + |B_G \setminus a'| + |\{a, b, a'\}| + |X''| \\
&\leq \begin{cases} 4 \cdot 2^{k-1} + (4 \cdot 2^{k-2} + 2) + 3 + (k-2), & \text{when } n = 4 \\ 5 \cdot 2^{k-1} + (8 \cdot 2^{k-2} - 1) + 3 + (k-2), & \text{when } n = 5 \end{cases} \\
&< n \cdot 2^k + 1
\end{aligned}
$$

for all $k \geq 3$, a contradiction. Hence, $|A_{p-1}| \leq n$. ∎

By Claim 2.8, $|R_G| \leq |R_G^*| \leq 2n$. We first consider the case when $|R_G| \geq n$. Since $|A_p| \geq n+1$, by Lemma 1.11, $G[A_p]$ has no red edges and no vertex in X is red-complete to $V(G) \setminus X$. Thus $|X| \leq 2(k-1)$. We first claim that $|B_G| \geq n$. Suppose $|B_G| \leq n-1$. If $|B_G| = n-1$, then $|A_p| \leq n \cdot 2^{k-2} + 2$ by Claim 2.1(ii) applied to B_G and A_p. But then

$$
|G| = |A_p| + |B_G| + |R_G| + |X| \leq (n \cdot 2^{k-2} + 2) + (n-1) + 2n + 2(k-1) < n \cdot 2^k + 1,
$$

for all $k \geq 3$, a contradiction. Thus $3 \leq |B_G| \leq n-2$. Then $n = 5$ and $|B_G| = 3$. By Claim 2.1(iii) applied to B_G and A_p, $|A_p| \leq 8 \cdot 2^{k-2} - 1$. But then

$$
|G| = |A_p| + |B_G| + |R_G| + |X| \leq (8 \cdot 2^{k-2} - 1) + 3 + 10 + 2(k-1) < 5 \cdot 2^k + 1,
$$

for all $k \geq 3$, a contradiction. Thus $|B_G| \geq n$, as claimed. By Lemma 1.11, $G[A_p]$ has no blue edges and no vertex in X is blue-complete to A_p in G. Since $G[A_p \cup X']$ has neither red nor blue edges, and no vertex in X is red- or blue-complete to A_p in G, it follows that $|X''| \leq k - 2$ and $|A_p \cup X'| \leq n \cdot 2^{k-2}$ by minimality of k. Then $|B_G| \geq n+1$, else

$$
|G| = |A_p \cup X'| + |X''| + (|B_G| + |R_G|) \leq n \cdot 2^{k-2} + (k-2) + (n + 2n) < n \cdot 2^k + 1,
$$

for all $k \geq 3$, a contradiction. By Lemma 1.11, $G[B_G]$ has no blue edges and so $G[B_G \cup X'']$ has no blue edges. By minimality of k, $|B_G \cup X''| \leq n \cdot 2^{k-1}$. But then

$$
|G| = |A_p \cup X'| + |B_G \cup X''| + |R_G| \leq n \cdot 2^{k-2} + n \cdot 2^{k-1} + 2n < n \cdot 2^k + 1,
$$

for all $k \geq 3$, a contradiction.

It remains to consider the case when $|R_G| \leq n-1$. Suppose first that $|B_G| \geq 2n+1$. By Lemma 1.11, $G[B_G]$ has no blue edges. Thus all the edges in $\mathcal{R}[B]$ are colored red. Since $|A_{p-1}| \leq n$ by

Claim 2.9, we see that $|B| \geq 3$, contrary to Claim 2.3. Thus $3 \leq |A_{p-1}| \leq |B_G| \leq 2n$. If $|B_G| \geq n-1$, by Claim 2.1(i,ii) applied to B_G and A_p (and Lemma 1.11 applied to B_G and A_p to obtain $|X| \leq 2(k-1)$ when $|B_G| \geq n$), we have

$$|A_p| + |B_G| + |X| \leq \begin{cases} (n \cdot 2^{k-1} + 2) + (n-1) + 2k, & \text{if } |B_G| = n-1 \\ n \cdot 2^{k-1} + 2n + 2(k-1), & \text{if } |B_G| \geq n. \end{cases}$$

Thus in either case, $|A_p| + |B_G| + |X| \leq n \cdot 2^{k-1} + 2n + 2k - 2$. But then

$$|G| = (|A_p| + |B_G| + |X|) + |R_G| \leq (n \cdot 2^{k-1} + 2n + 2k - 2) + (n-1) < n \cdot 2^k + 1,$$

for all $k \geq 3$, a contradiction. Thus $3 \leq |B_G| \leq n-2$. Then $|B_G| = 3$ and $n = 5$. If $|R_G^*| \geq 4$ or $|B_G^*| \geq 4$, by applying Claim 2.1(ii) to any four vertices in R_G^* or B_G^* and A_p, we have $|A_p| \leq 5 \cdot 2^{k-1} + 2$. But then

$$|G| = |A_p| + |B_G| + |R_G| + |X| \leq (5 \cdot 2^{k-1} + 2) + 3 + 4 + 2k < 5 \cdot 2^k + 1,$$

for all $k \geq 3$, a contradiction. Thus $|B_G| = |B_G^*| = 3$ and $|R_G| \leq |R_G^*| \leq 3$. Then no vertex in X is blue-complete to $V(G) \setminus X$. Thus $|X \setminus R_G^*| \leq 2(k-2)$. By Claim 2.1(iii) applied to B_G and A_p, $|A_p| \leq 8 \cdot 2^{k-1} - 1$. But then

$$|G| = |A_p| + |B_G| + |R_G^*| + |X \setminus R_G^*| \leq (8 \cdot 2^{k-1} - 1) + 3 + 3 + 2(k-2) < 5 \cdot 2^k + 1,$$

for all $k \geq 3$, a contradiction.

This completes the proof of Theorem 1.8. ∎

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