Hadwiger Number and Chromatic Number for Near Regular Degree Sequences

Neil Robertson[1] and Zi-Xia Song[2]

[1]DEPARTMENT OF MATHEMATICS
THE OHIO STATE UNIVERSITY
COLUMBUS, OH 43210
E-mail: robertso@math.ohio-state.edu

[2]DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CENTRAL FLORIDA
ORLANDO, FL 32816
E-mail: zsong@mail.ucf.edu

Abstract: We consider a problem related to Hadwiger's Conjecture. Let $D = (d_1, d_2, \ldots, d_n)$ be a graphic sequence with $0 \leq d_1 \leq d_2 \leq \cdots \leq d_n \leq n-1$. Any simple graph G with D its degree sequence is called a realization of D. Let $R[D]$ denote the set of all realizations of D. Define $h(D) = \max\{h(G) : G \in R[D]\}$ and $\chi(D) = \max\{\chi(G) : G \in R[D]\}$, where $h(G)$ and $\chi(G)$ are Hadwiger number and chromatic number of a graph G, respectively. Hadwiger's Conjecture implies that $h(D) \geq \chi(D)$. In this paper, we establish the above inequality for near regular degree sequences. © 2009 Wiley Periodicals, Inc.
J Graph Theory 64: 175–183, 2010

1. INTRODUCTION

All graphs in this paper are finite and have no loops or multiple edges. A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. An *H minor* is a minor isomorphic to H. If H is a complete graph, we also say that G contains a *clique minor* of size $|H|$. For a graph G, the *Hadwiger number* $h(G)$ of G is the maximum integer k such that G contains a clique minor of size k. As usual we denote by $\chi(G)$ the chromatic number of G and by $\chi'(G)$ the edge chromatic number of G.

Our research is motivated by Hadwiger's Conjecture from 1943 which suggests a far-reaching generalization of the Four Color Theorem [1, 2, 7] and is considered by many as one of the deepest open problems in graph theory. Hadwiger's Conjecture states the following.

Conjecture 1.1.   *For every integer $k \geq 1$, every k-chromatic graph has a K_k minor.*

Conjecture 1.1 is trivially true for $k \leq 3$, and reasonably easy for $k = 4$, as shown by Dirac [5] and Hadwiger himself [6]. However, for $k \geq 5$, Conjecture 1.1 implies the Four Color Theorem. In 1937, Wagner [10] proved that the case $k = 5$ of Conjecture 1.1 is, in fact, equivalent to the Four Color Theorem. In 1993, Robertson, Seymour, and Thomas [8] proved that a minimal counterexample to the case $k = 6$ is a graph G which has a vertex v such that $G - v$ is planar. By the Four Color Theorem, this implies Conjecture 1.1 for $k = 6$. Hence the cases $k = 5, 6$ are each equivalent to the Four Color Theorem [1, 2, 7]. Conjecture 1.1 is open for $k \geq 7$. Note that Conjecture 1.1 also states that for every graph G, $h(G) \geq \chi(G)$.

In this paper, we consider a weaker version of Hadwiger's Conjecture. The goal is to establish more evidence for Conjecture 1.1. Let $D = (d_1, d_2, \ldots, d_n)$ be an integer sequence with $0 \leq d_1 \leq d_2 \leq \cdots \leq d_n \leq n-1$. We say that D is *graphic* if there is a graph G with $V(G) = \{u_1, u_2, \ldots, u_n\}$ such that $d_G(u_i) = d_i$. In those circumstances, we say that G is a *realization* of D. If a sequence D consists of the terms $d_1, \ldots, d_t$ with multiplicities $m_1, \ldots, m_t$, we may write $D = (d_1^{m_1}, \ldots, d_t^{m_t})$. For a graphic degree sequence D, let $R[D]$ denote the set of all realizations of D. Define $h(D) = \max\{h(G) : G \in R[D]\}$ and $\chi(D) = \max\{\chi(G) : G \in R[D]\}$. Hadwiger's Conjecture implies the following conjecture.

Hadwiger's Conjecture for Degree Sequences: For any graphic degree sequence D, $h(D) \geq \chi(D)$.

For a graphic sequence $D = (d_1, d_2, \ldots, d_n)$, we say that D is near regular if $D = ((k-1)^p, k^{n-p})$ for some integers $k \geq 1$ and p satisfying $0 \leq p \leq n-1$. The purpose of this paper is to prove that Hadwiger's Conjecture for Degree Sequences is true for near

regular degree sequences. Our proof technique is to construct a graph G in $R[D]$ with a clique minor of desired size.

We need to introduce more notation. Let G be a graph. The *complement* of G is denoted by $\overline{G}$. If $X \subseteq V(G)$, we denote the subgraph of G induced on X by $G[X]$. We use $G \backslash X$ denote the subgraph of G induced on $V(G) \backslash X$. If $A, B \subseteq V(G)$ are disjoint, we say that A is *complete* to B if every vertex in A is adjacent to every vertex in B, and A is *anticomplete* to B if no vertex in A is adjacent to a vertex in B. If $A = \{x\}$, we simply say x is complete to B or x is anticomplete to B. We use $G[A, B]$ to denote the bipartite graph obtained from $G[A \cup B]$ by deleting all edges with both ends in A or in B. If $F \subseteq E(G)$ and $M \subseteq E(\overline{G})$, then $G - F$ (resp. $G + M$) denotes the graph obtained from G by deleting the edges in F from G (reps. adding the edges in M to G). For two disjoint graphs G and H, $G + H$ is the *join* of G and H with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$. As usual we denote by K_n and $K_{n,m}$, respectively, the complete graph on n vertices and the complete bipartite graph such that one partite set has n vertices and the other partite set has m vertices.

The following well-known results [3, 9] will be used later in the proof of the main results. We list them below. A proof of Hall's Theorem and other notation not introduced here can be found in [4].

Theorem 1.2.   *Let G be a connected graph with maximum degree Δ. Suppose G is neither a complete graph nor an odd cycle. Then $\chi(G) \le \Delta$.*

Theorem 1.3.   *If G is a r-regular bipartite graph with $r \ge 1$, then G has r pairwise disjoint perfect matchings.*

Theorem 1.4.   *If G is simple graph with maximum degree r, then $\chi'(G) = r + 1$ or r.*

2.   NEAR REGULAR DEGREE SEQUENCES

In this section, we prove that Hadwiger's Conjecture for Degree Sequences is true for near regular degree sequences. We first prove some preliminary results.

Lemma 2.1.   *For integers $r \ge 2$ and $n \ge r + 1$, if nr is even, then there exists an r-regular graph G of order n such that G has at least two pairwise disjoint perfect matchings if n is even and one near perfect matching if n is odd.*

Proof.   If $r = n - 1$, then $G = K_n$ has the desired property. So we may assume that $r \le n - 2$. Assume that n is even. If $r \le \frac{n}{2}$, let G be an r-regular bipartite graph with each partite of size $\frac{n}{2}$. If $r \ge \frac{n}{2} + 1$, let G be a graph with $V(G)$ partitioned into A and B such that $G[A] = G[B] = K_{\frac{n}{2}}$ and $G[A, B]$ is s-regular, where $s = r - (\frac{n}{2} - 1) \ge 2$. In either case, by Theorem 1.3, G has at least two pairwise disjoint perfect matchings. If n is odd, then r and $n - 1$ must be even. Let H be the r-regular graph of order $n - 1$, as constructed above, with two disjoint perfect matchings, say M_1, M_2. Let $F \subseteq M_1$ with $|F| = \frac{r}{2}$. We may assume that $F = \{x_1 x_2, x_3 x_4, \ldots, x_{r-1} x_r\}$. Let G be obtained from $H - F$

by joining a new vertex w to x_i, $i=1,2,\ldots,r$. Clearly G is r-regular and M_2 is a near perfect matching of G.

This completes the proof of Lemma 2.1.   ∎

Lemma 2.2.   *Let $D=((k-1)^p,k^{n-p})$ be the degree sequence of a near k-regular graph on n vertices, where $0\leq p\leq n-1$. If $3n\geq 4k+4$, then $h(D)\geq k+1$ if $p=0$ and $h(D)\geq k$ if $p>0$.*

Proof.   The statement is trivially true if $k\leq 2$. So we may assume that $k\geq 3$. We consider the following two cases.

Case 1. n *is even.*

Since $\sigma(D)=nk-p$ is even, p must be an even integer. Suppose that $n\geq 2k+2$. If $n-p\geq k+1$, by Lemma 2.1, let H be a k-regular graph of order $n-k-1$ and let F be a matching of size $\frac{p}{2}$ in H. Then $K_{k+1}\cup G\in R[D]$, where $G=H-F$. Thus $h(D)\geq h(K_{k+1}\cup G)\geq k+1$. If $n-p\leq k$, by Lemma 2.1, let H be a $(k-1)$-regular graph of order $n-k$ and let F be a matching of size $\frac{n-p}{2}$ in $\overline{H}$. Then $K_k\cup G\in R[D]$, where $G=H+F$. Hence $h(D)\geq h(K_k\cup G)\geq k$. So we may assume that $n\leq 2k$. For integers r,s,t satisfying $n=2s+2t$ and $r=s+2t-1$, define $G(r,s,t)$ to be an r-regular graph of order n with its vertex set partitioned into A,B,C,D such that $G(r,s,t)[A]=G(r,s,t)[B]=K_s$, $G(r,s,t)[C]=G(r,s,t)[D]=K_t$, $G(r,s,t)[A,B]$ is a t-regular bipartite graph. Moreover, C is complete to $A\cup D$, D is complete to B, A is anticomplete to D, C is anticomplete to B.

If $p\leq 2(n-k-1)$, let $s=n-k-1$ and $t=k+1-\frac{n}{2}$. Then $\frac{k}{3}<s\leq k-1$, $1\leq t<\frac{k}{3}$, and $s-\frac{p}{2}\geq 0$. Observe that $2s+2t=n$, $s+2t=k+1$ and $t\leq s$ because $3n\geq 4k+4$. Let $G=G(k,s,t)$. Then G is a k-regular graph of order n. If $t=1$, then $n=2k$ and so $s+t=k$. Let M be a perfect matching of $G[A,B]$ and $F\subseteq M$ with $|F|=\frac{p}{2}$. Let $H=G-F$. Then $H\in R[D]$. By contracting each of the edges of $M-F$ into single vertices, we have $h(D)\geq h(H)\geq |A|+|C|+|D|=s+2t=k+1$ if $p=0$, and $h(D)\geq h(H)\geq |A|+|C|=s+t=k$ if $p>0$. So we may assume that $t\geq 2$. By Theorem 1.3, $G[A,B]$ contains two disjoint perfect matchings M_1 and M_2. Let $F\subset M_1$ with $|F|=\frac{p}{2}$ and let $H=G-F$. Then $H\in R[D]$. By contracting each of the edges of M_2 into single vertices, we see that $h(D)\geq h(H)\geq s+2t=k+1$.

So we may assume that $p\geq 2(n-k)$. Then $n-p\leq n-2(n-k)=2k-n<2(n-k)$ because $3n\geq 4k+4$. Note that $n-p$ is even. Let $s=n-k$ and $t=k-\frac{n}{2}$. Then $\frac{k}{3}<s\leq k$, $0\leq t<\frac{k}{3}$. Observe that $2s+2t=n$, $s+2t=k$ and $t<s$ because $3n\geq 4k+4$. Let $G=G(k-1,s,t)$. Then G is a $(k-1)$-regular graph of order n. Since $t<s$, by Theorem 1.3, $G[A,B]$ contains an anti-matching F of size $\frac{n-p}{2}$. Let $H=G+F$, then $H\in R[D]$. If $t=0$, then $n=2k$ and so $h(D)\geq h(H)\geq |A|=s=k$. If $t\geq 1$, let M be a perfect matching of $G[A,B]$. Clearly, M is also a matching of H. By contracting each of the edges of M into single vertices, we see that $h(D)\geq h(H)\geq s+2t=k$.

Case 2. n *is odd.*

In this case, since $\sigma(D)=nk-p$ is even, we have

(a) p and k are either both even or both odd.

By (a), both $p-k$ and $n-p-k-1$ are even. Suppose that $n\geq 2k+1$. If $n-p\leq k$, then $p-k\leq 2(n-k)$. By Lemma 2.1, let H be a k-regular graph of order $n-k$ and let F be a matching of size $\frac{p-k}{2}$ in H. Then $K_k\cup G\in R[D]$, where $G=H-F$. Hence $h(D)\geq h(K_k\cup G)\geq k$. We may assume that $n-p\geq k+1$. If $n\geq 2k+3$, by Lemma 2.1, let H be a $(k-1)$-regular graph of order $n-k-1\geq k+2$ and let F be a matching of size $\frac{n-p-k-1}{2}$ of $\overline{H}$. Then $K_{k+1}\cup G\in R[D]$, where $G=H+F$. Thus $h(D)\geq h(K_{k+1}\cup G)\geq k+1$. We may assume that $n=2k+1$. Then $p\leq n-k-1=k$. Let $F=\{x_1x_2,\ldots,x_{2q-1}x_{2q}\}$ be a matching of size q of K_{k+1}, where $q=\frac{n-p-k-1}{2}=\frac{k-p}{2}$. Let $y_1,y_2,\ldots,y_{2q}$ be $2q$ distinct vertices in K_k. Let G be the graph obtained from $K_{k+1}-F$ and K_k by joining y_i to x_i, $i=1,2,\ldots,2q$. Then $G\in R[D]$. By contracting $G[\{y_{2i-1},y_{2i}\}]$ onto x_{2i-1}, $i=1,2,\ldots,q$, we see that $h(D)\geq h(G)\geq k+1$.

So we may assume that $n\leq 2k-1$. For integers r, s, t satisfying $n=2s+2t+1$ and $r=s+2t$, define $I(r,s,t)$ to be a graph of order n with its vertex set partitioned into A, B, C, D such that $I(r,s,t)[A]=K_s$, $I(r,s,t)[B]=K_s-E$, $I(r,s,t)[C]=K_t$, $I(r,s,t)[D]=K_{t+1}$, $I(r,s,t)[A,B]$ is a $(t+1)$-regular bipartite graph, where E is matching of size $[\frac{s}{2}]$ of K_s. Moreover, C is complete to $A\cup D$, D is complete to B, A is anticomplete to D, C is anticomplete to B. By (a), we consider the following two subcases.

Case 2.1. p and k are even.

If $p\leq 2(n-k-1)$, let $s=n-k-1$ and $t=k+1-\frac{n+1}{2}$. Then $\frac{k}{3}<s\leq k-2$, $1\leq t<\frac{k}{3}$. Observe that s is even, $s+2t=k$ and $t\leq s$. Let $G=I(k,s,t)$. Then G is a k-regular graph of order n. Since $t\geq 1$, by Theorem 1.3, let M_1 and M_2 be two perfect matchings of $G[A,B]$. Let $F\subseteq M_1$ with $|F|=\frac{p}{2}$. Let $H=G-F$. Then $H\in R[D]$. By contracting each of the edges of M_2 into single vertices, we see that $h(H)\geq s+2t+1=k+1$. Hence $h(D)\geq k+1$.

So we may assume that $p\geq 2(n-k)$. Then $n-p\leq n-2(n-k)=2k-n<2(n-k)$ and $n-p$ is odd. Let $s=n-k$ and $t=k-\frac{n+1}{2}$. Then $\frac{k}{3}\leq s\leq k-1$, $0\leq t<\frac{k}{3}$. Observe that s is odd, $s+2t=k-1$ and $t<s$ because $3n\geq 4k+4$. Let $G=I(k-1,s,t)$. Then $DS(G)=(k^1,(k-1)^{n-1})$. Let w be the vertex of degree $s-1$ in $G[B]$. Since $n-p<2s$, by Theorem 1.3, $G[A,B\backslash w]$ contains an anti-matching, say F, of size $\frac{n-p-1}{2}$. Let $H=G+F$. Then $H\in R[D]$. Let M be a perfect matching of $G[A,B]$. By contracting each of the edges of M into single vertices, we see that $h(D)\geq h(H)\geq s+2t+1=k$.

Case 2.2. p and k are odd.

In this case, $n-p$ is even. If $n-p\leq 2(n-k)$, let $s=n-k$ and $t=k-\frac{n+1}{2}$. Then $\frac{k}{3}<s\leq k-1$, $0\leq t<\frac{k}{3}$. Observe that s is even, $s+2t+1=k$ and $t\leq s$. Let $G=I(k-1,s,t)$. Then G is a $(k-1)$-regular graph of order n. By Theorem 1.3, let M be a perfect matching and F be an anti-matching of size $\frac{n-p}{2}$ of $G[A,B]$, respectively. Let $H=G+F$. Then $H\in R[D]$. By contracting each of the edges of M into single vertices, we see that $h(H)\geq s+2t+1=k$. Hence $h(D)\geq k$.

So we may assume that $n-p\geq 2(n-k)+2$. Then $p\leq n-2(n-k)-2=2k-n-2<2(n-k)$. Let $s=n-k-1$ and $t=k+1-\frac{n+1}{2}$. Then $\frac{k}{3}<s\leq k-2$, $1\leq t<\frac{k}{3}$. Observe that s is odd, $s+2t+1=k+1$ and $t<s$ because $3n\geq 4k+4$. Let J be an Hamiltonian cycle of K_s and let E be a matching of size $\frac{s-1}{2}$ in J. Now let G be a graph of order n with $V(G)$ partitioned into A,B,C,D such that $G[A]=K_s$, $G[B]=K_s-E(J)+E$, $G[C]=K_t$, $G[D]=K_{t+1}$, $G[A,B]$ is a $(t+1)$-regular bipartite graph. Moreover, C is complete to $A\cup D$, D is complete to B, A is anticomplete to D, C is anticomplete to B. Let w be the vertex of degree $s-3$ in $G[B]$. Since $t\geq 1$, by Theorem 1.3, $G[A,B]$ contains two perfect matchings M_1 and M_2. Since $p<2s$, there exists $F\subset M_1$ with $|F|=\frac{p-1}{2}$ and $w\notin V(F)$. Let $H=G-F$. Then $H\in R[D]$. By contracting each of the edges of M_2 into single vertices, we see that $h(D)\geq h(H)\geq s+2t+1=k+1$.

This completes the proof of Lemma 2.2. ∎

Theorem 2.3. *Let $D=((k-1)^p,k^{n-p})$ be the degree sequence of a near k-regular graph on n vertices, where $0\leq p\leq n-1$. Then*

$$
h(D)\geq
\begin{cases}
k+1 & \text{if } p=0 \text{ and } n=k+1 \\[2mm]
k+1 & \text{if } p=0 \text{ and } n\geq \dfrac{4k+4}{3} \\[2mm]
k & \text{if } p>0 \text{ and } n\geq \dfrac{4k+4}{3} \\[2mm]
k+1-\left\lceil \dfrac{p}{4}\right\rceil & \text{if } p>0 \text{ and } n=k+1 \\[2mm]
\left\lfloor \dfrac{3n}{4}\right\rfloor & \text{if } k+2\leq n<\dfrac{4k+4}{3}
\end{cases}
$$

Proof. If $n=k+1$ and $p=0$, then $R[D]=\{K_{k+1}\}$ and $h(D)=k+1$. If $n=k+1$ and $p>0$, then p must be even and D has a unique realization $K_{k+1}-M$, where M is a matching of size $\frac{p}{2}$ of K_{k+1}. Clearly, $h(D)\geq (n-p)+\frac{p}{2}+\lfloor\frac{p}{4}\rfloor\geq k+1-\lceil\frac{p}{4}\rceil$. So we may assume that $n\geq k+2$. If $n=k+2$ and $p=0$, then k must be even and D has a unique realization K_n-M, where M is a perfect matching of K_n. It can be easily checked that $h(D)=\lfloor\frac{3n}{4}\rfloor$. Suppose $n=k+2$ and $p>0$. If k is odd, then n and p must be odd. Let J be a Hamilton cycle of K_n and let F be a matching of size $\frac{n-p}{2}$ of J. Let $G=K_n-E(J)+F$. Then $G\in R[D]$. Let $\{x_1,x_2,\ldots,x_n\}$ be the vertices of J in order. Let $M=\{x_3x_n,x_5x_{n-2},\ldots,x_{\frac{n+1}{2}}x_{\frac{n+1}{2}+2}\}$ if $\frac{n-1}{2}$ is even; and $M=\{x_1x_{\frac{n+1}{2}+1},x_3x_n,x_5x_{n-2},\ldots,x_{\frac{n-1}{2}}x_{\frac{n+1}{2+3}}\}$ if $\frac{n-1}{2}$ is odd. By contracting each of the edges of M into single vertices, we see that $h(D)\geq h(G)\geq\lfloor\frac{3n}{4}\rfloor$. If k is even, then n and p must be even. Let $x_1,x_2,\ldots,x_{\frac{n}{2}},y_1,y_2,\ldots,y_{\frac{n}{2}}$ be the n vertices of K_n. Let $G=K_n-E$, where $E=\{x_1y_1,x_2y_2,\ldots,x_{\frac{n}{2}}y_{\frac{n}{2}},y_1x_2,y_2x_3,\ldots,y_{\frac{p}{2}}x_{\frac{p}{2}+1}\}$. By contracting each

of the edges $y_i y_{\lfloor \frac{n}{4} \rfloor + i}$, where $i = 1, 2, \ldots, \lfloor \frac{n}{4} \rfloor$, into single vertices, we see that $h(D) \geq h(G) \geq \lfloor \frac{3n}{4} \rfloor$. Thus we may assume that $n \geq k + 3$.

By Lemma 2.2, we may assume that $k + 3 \leq n < \frac{4k+4}{3}$. We next show that $h(D) \geq \lfloor \frac{3n}{4} \rfloor$. Note that $\overline{D}$ is the degree sequence of a near r-regular graph on n vertices, where $2 \leq r = n - 1 - k < \frac{k+1}{3}$. Since $3n < 4k + 4$, we have $4r = 4(n - k - 1) < n$. Let m, s be nonnegative integers so that $n = 4m + s$, where $0 \leq s \leq 3$. Clearly, $m \geq r$ because $n > 4r$. Assume that $p = 0$. For an integer $t > 0$, we denote by $B_{t,t}^r$ an r-regular bipartite graph with each partite of size t, and let M be a matching of $B_{t,t}^r$ of size $\lceil \frac{r}{2} \rceil$. Let B^* be obtained from $B_{m,m}^r - M$ by adding a new vertex v joining to each vertex of $V(M)$. Note that if n is odd, then r is even. Let

$$\overline{G} = \begin{cases} K_{m,m}^r \cup K_{m,m}^r & \text{if } s = 0 \\ K_{m,m}^r \cup B^* & \text{if } s = 1 \\ K_{m,m}^r \cup K_{m+1,m+1}^r & \text{if } s = 2 \\ K_{m+1,m+1}^r \cup B^* & \text{if } s = 3 \end{cases}$$

It can be easily checked that $G \in R[D]$ and G contains a clique minor of size at least $\lfloor \frac{3n}{4} \rfloor$. Thus $h(D) \geq \lfloor \frac{3n}{4} \rfloor$, as desired. So we may assume that $p > 0$. We consider the following two cases.

Case 1. n is even.

Then $n = 4m$ or $4m + 2$, and p must be even. Since $p \leq n - 1$, we have $p \leq n - 2$. Let H be a k-regular graph of order n with $V(H)$ partitioned into A, B, C, D such that $H[A] = K_m$, $H[B] = K_{\frac{n-2m}{2}}$, $H[C] = K_{\frac{n-2m}{2}}$, $H[D] = K_m$, $A \cup D$ is complete to $B \cup C$, $H[A, D]$ and $H[B, C]$ are $(m - r)$-regular and $(m + 1 - r)$-regular bipartite graphs, respectively. If $r < m$, let M and M' be perfect matchings of $H[A, D]$ and $H[B, C]$, respectively. Let $F \subseteq M \cup M'$ with $|F| = \frac{p}{2}$. If $r = m$ and $p \leq 2m + 2$, then $n = 4m + 2$. Thus $|B| = |C| = m + 1$. Let M be a perfect matching of $G[B, C]$. Let $F \subseteq M$ with $|F| = \frac{p}{2}$. In both cases, let $H^* = H - F$. Then $H^* \in R[D]$. By contracting each of the edges of a matching of size m in $H[A, B]$ into single vertices, we see that $h(D) \geq h(H^*) \geq \lfloor \frac{3n}{4} \rfloor$. So we may assume that $r = m$ and $p \geq 2m + 4$. Since $r = m$, we have $n = 4m + 2$ and so $k = 3m + 1$. Note that $n - p \leq (4m + 2) - (2m + 4) = 2m - 2$. Let $H = G(3m, m + 1, m)$, as defined in the proof of Lemma 2.2 (see Case 1). Let A and B be as given in the definition of $G(3m, m + 1, m)$. Now let M and M' be a perfect matching and an anti-matching of size $\frac{n-p}{2}$ of $G(3m, m + 1, m)[A, B]$, respectively. Let $H^* = H + M'$. Then $H^* \in R[D]$. By contracting each of the edges of M into single vertices, we have $h(D) \geq h(H^*) \geq \lfloor \frac{3n}{4} \rfloor$.

Case 2. n is odd.

In this case, $n = 4m + 1$ or $4m + 3$. Since $\sigma(D) = nk - p$ is even, we have

(b) p and k are either both even or both odd.

Let $s = k$ if k is even and $s = k - 1$ if k is odd. Then $n - 1 - s$ is even and let $r' = (n - 1) - 1 - s$. Then $r' = r - 1$ if $s = k$ and $r' = r$ if $s = k - 1$. Define $H(s)$ to be an $(s - 1)$-regular graph of order $n - 1$ with $V(H)$ partitioned into A, B, C, D such that $H(s)[A] = K_m$,

$H(s)[B] = K_{\frac{n-1-2m}{2}}$, $H(s)[C] = K_{\frac{n-1-2m}{2}}$, $H(s)[D] = K_m$, $A \cup D$ is complete to $B \cup C$, $H(s)[A,D]$ and $H(s)[B,C]$ are $(m-r'-1)$-regular and $(\frac{n-1-2m}{2} - r' - 1)$-regular bipartite graphs, respectively. Note that $|B| = |C| = m$ if $n = 4m+1$ and $|B| = |C| = m+1$ if $n = 4m+3$. Let F be an anti-matching of size $\frac{n-1-s}{2}$ of $H(s)[A,D]$. Let $H^*(s)$ be the graph obtained from $H(s)+F$ by joining a new vertex w to $V(H) \setminus V(F)$. Then $H^*(s)$ is s-regular.

If k is odd, by (b), $n-p$ is even. Note that $s = k-1$ and $r' = r \geq 2$. Let $X = H^*(s)$, as defined above. Then X is $(k-1)$-regular. Let M and M' be two perfect anti-matchings of $H(s)[A,D]$ and $H(s)[B,C]$, respectively. Let $F \subseteq M \cup M'$ with $|F| = \frac{n-p}{2}$. Then $X+F \in R[D]$. If k is even, by (b), p is even. Note that $s = k$ and $r' = r-1$. Thus $m - r' = m - (r-1) = m+1-r \geq 1$ and $\frac{n-1-2m}{2} - r' \geq m - r' \geq 1$. Let $Y = H^*(s)$, as defined above. Then Y is k-regular. Let M and M' be perfect matchings of $H(s)[A,D]$ and $H(s)[B,C]$, respectively. Let $F \subseteq M \cup M'$ with $|F| = \frac{p}{2}$. One can easily check that $Y-F \in R[D]$. In both cases, let M^* be a matching of size m of $H^*(s)[A,B]$. Let $w' \in B \setminus V(M^*)$ if $|B| = m+1$. Now by contracting each of the edges of M^* (and $w'w$ if $|B| = m+1$) into single vertices, we see that $h(D) \geq \lfloor \frac{3n}{4} \rfloor$.

This completes the proof of Theorem 2.3. ∎

We are now ready to prove the main result.

Theorem 2.4. *Let $D = ((k-1)^p, k^{n-p})$ be the degree sequence of a near k-regular graph on n vertices, where $0 \leq p \leq n-1$. Then $h(D) \geq \chi(D)$.*

Proof. By Theorem 1.2 and Theorem 2.3, we may assume that $n = k+1$ and $h(D) \geq k+1 - \lceil \frac{p}{4} \rceil$ or $k+2 \leq n < \frac{4k+4}{3}$ and $h(D) \geq \lfloor \frac{3n}{4} \rfloor$. In the first case, p is even and D has a unique realization $K_{k+1} - M$, where M is a matching of size $\frac{p}{2}$. Clearly, $\chi(D) = k+1 - \frac{p}{2} \leq h(D)$. It remains to show that $\chi(D) \leq \lfloor \frac{3n}{4} \rfloor$ when $k+2 \leq n < \frac{4k+4}{3}$. Let $G \in R[D]$ and let M be a maximum anti-matching of G. Then $G \setminus V(M)$ is a complete subgraph. Thus $\chi(G) \leq |G \setminus V(M)| + |M| = (n - 2|M|) + |M| = n - |M|$. It suffices to show that $|M| \geq \frac{n}{4}$. If $p = 0$, then G is k-regular and $\overline{G}$ is r-regular, where $r = n-1-k \geq 1$. By Theorem 1.4, $\chi'(\overline{G}) \leq r+1$, we have $\frac{rn}{2} = |E(\overline{G})| \leq (r+1)|M|$, which yields $|M| \geq \frac{n}{4}$. Assume that $p > 0$. Then $\overline{G}$ contains $n-p$ vertices of degree $r = n-1-(k-1) = n-k$ and p vertices of degree $n-1-k$. Let $r = n-k$. Then $r \geq 2$. If $r \geq 3$, by Theorem 1.4 again, $\chi'(\overline{G}) \leq r+1$, we have $\frac{(r-1)n}{2} < |E(\overline{G})| \leq (r+1)|M|$. It follows that $|M| \geq \frac{n}{4}$. So we may assume that $r = 2$. Then $\overline{G}$ consists of disjoint unions of cycles and paths. One can easily check that $\overline{G}$ contains a matching of size at least $\frac{n}{3}$. Thus $|M| \geq \frac{n}{4}$ and so $\chi(G) \leq n - |M| \leq \frac{3n}{4}$, as desired. Consequently, $\chi(D) \leq \lfloor \frac{3n}{4} \rfloor$ by the arbitrary choice of $G \in R[D]$.

This completes the proof of Theorem 2.4. ∎

ACKNOWLEDGMENTS

REFERENCES

[1] K. Appel and W. Haken, Every planar map is four colorable, Part I. Discharging, Illinois J Math 21 (1977), 429–490.

[2] K. Appel, W. Haken, and J. Koch, Every planar map is four colorable, Part II. Reducibility, Illinois J Math 21 (1977), 491–567.

[3] R. L. Brooks, On colouring the notes of a network, Proc Cambridge Phil Soc 37 (1941), 194–197.

[4] R. Diestel, Graph Theory, Graduate texts in mathematics 173, Springer, 1997.

[5] G. A. Dirac, A property of 4-chromatic graphs and some remarks on critical graphs, J London Math Soc 27 (1952), 85–92.

[6] H. Hadwiger, Über eine Klassifikation der Streckenkomplexe, Vierteljahrsschr Naturforsch Ges Zürich 88 (1943), 133–142.

[7] N. Robertson, D. P. Sanders, P. D. Seymour, and R. Thomas, The four-color theorem, J Combin Theory Ser B 70 (1997), 2–44.

[8] N. Robertson, P. D. Seymour, and R. Thomas, Hadwiger's conjecture for K_6-free graphs, Combinatorica 13 (1993), 279–361.

[9] V. G. Vizing, On an estimate of the chromatic class of a p-graph, Diskret Analiz 3 (1964), 25–30.

[10] K. Wagner, Über eine Eigenschaft der ebenen Komplexe. Math Ann 114 (1937), 570–590.