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Note

On the potentially P_k-graphic sequences[1]

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Abstract

A nonincreasing sequence π of n nonnegative integers is said to be graphic if it is the degree sequence of a simple graph G of order n and G is called a realization of π. A graph G of order n is said to have property P_k if it contains a clique of size k as a subgraph. An n-term graphic sequence π is said to be potentially (res. forcibly)P_k-graphic if it has a realization having (res. all its realizations have) property P_k. It is well known that, if $t_{k-1}(n)$ is the Turán number, then $t_{k-1}(n)$ is the smallest number such that each graph G of order n with edge number $\varepsilon(G) \geqslant t_{k-1}(n)+1$ has property P_k. The Turán theorem states that $t_{k-1}(n) = \binom{n}{2} - t(n-k+1-r)/2$, where $n = t(k-1)+r, 0 \leqslant r < k-1$. In terms of graphic sequences, $2(t_{k-1}(n)+1)$ is the smallest even number such that each graphic sequence $\pi = (d_1, d_2, \ldots, d_n)$ with $\sigma(\pi) = d_1 + d_2 + \cdots + d_n \geqslant 2(t_{k-1}(n)+1)$ is forcibly P_k-graphic. In 1991, Erdös et al. [1] considered a variation of this classical extremal problem: determine the smallest even number $\sigma(k,n)$ such that each graphic sequence $\pi = (d_1, d_2, \cdots, d_n)$ with $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n \geqslant 1$ and $\sigma(\pi) \geqslant \sigma(k,n)$ is potentially P_k- graphic. They gave a lower bound of $\sigma(k,n)$, i.e., $\sigma(k,n) \geqslant (k-2)(2n-k+1)+2$ and conjectured that the lower bound is the exact value of $\sigma(k,n)$. In this paper, we prove the upper bound $\sigma(k,n) \leqslant 2n(k-2)+2$ for $n \geqslant 2k-1$. © 1999 Elsevier Science B.V. All rights reserved

1. Introduction

Let $\pi = (d_1, d_2, \ldots, d_n)$ be a nonincreasing sequence of n nonnegative integers. π is said to be a graphic sequence if it is the degree sequence of a simple graph G of order n and such graph is referred to a realization of π. A graph G is said to have property P_k if it contains a complete subgraph K_k of order k. A graphic sequence π is

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potentially (res. forcibly) P_k-graphic if it has a realization G having property P_k (res. all its realizations have property P_k). For a given graphic sequence $\pi = (d_1, d_2, \ldots, d_n)$, the degree sum of π is denoted by $\sigma(\pi)$, i.e., $\sigma(\pi) = d_1 + d_2 + \cdots + d_n$.

Extremal graph theory is considered to have begun in 1941 when Turán proposed and then solved the following problem: For given positive integers k and n, determine the smallest edge number $t_{k-1}(n) + 1$ such that every graph G of order n with edge number $\varepsilon(G) \geqslant t_{k-1}(n) + 1$ has property P_k. The number $t_{k-1}(n)$ is called the Turán number. The Turán Theorem states that $t_{k-1}(n) = \binom{n}{2} - t(n - k + 1 + r)/2$, where $n = t(k-1) + r, 0 \leqslant r < k - 1$. In terms of graphic sequences, $2(t_{k-1}(n) + 1)$ is the smallest even number such that each n-term graphic sequence $\pi = (d_1, d_2, \ldots, d_n)$ with degree sum $\sigma(\pi) \geqslant 2(t_{k-1}(n) + 1)$ is forcibly P_k-graphic. In [1], Erdös et al. considered a variation of this extremal problem, and raised the following problem: determine the smallest degree sum $\sigma(k,n)$ so that each graphic sequence $\pi = (d_1, d_2, \ldots, d_n)$ with $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n \geqslant 1$ and $\sigma(\pi) \geqslant \sigma(k,n)$ is potentially P_k-graphic. By an example, they proved that $\sigma(k,n) \geqslant (k-2)(2n-k+1) + 2$, and asked whether the lower bound is the exact value of $\sigma(k,n)$. They proved that this is true for $k = 3$, viz, $\sigma(3,n) = 2n$, for $n \geqslant 6$. Their result can be stated as follows:

Theorem 1.1. *If $n \geqslant 6$ and the graphic sequence $\pi = (d_1, d_2, \ldots, d_n)$ satisfies $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n \geqslant 1$ and $\sigma(\pi) \geqslant 2n$, then π is potentially P_3-graphic. In other words, $\sigma(3,n) = 2n$ for $n \geqslant 6$.*

Recently, Li and Song [4] and Gould et al. [2] have proved independently that $\sigma(4,n) = 4n - 4$ for $n \geqslant 8$. Li and Song [5] further proved that $\sigma(5,n) = 6n - 10$ for $n \geqslant 10$. In other words, the lower bound is the exact value of $\sigma(k,n)$ for $k = 4, 5$.

At the end of [2], Gould et al. pointed out that as a weakening of this (conjecture), it would be nice to see that for n sufficiently large, this number (i.e., $\sigma(k,n)$) is linear in n, explaning the difference between this value and ex(K_k, n) (i.e., the number $2(t_{k-1}(n) + 1)$). In this paper, we will give an upper bound for $\sigma(k,n)$ as follows: If $k \geqslant 3$, then $\sigma(k,n) \leqslant 2n(k-2) + 2$ for $n \geqslant 2k - 1$. It is easy to see that, for given $k \geqslant 3$, this upper bound of $\sigma(k,n)$ is linear in n, and the number $\mathrm{ex}(K_k, n) = n(n-1) - t(n-k+1-r) + 2$ where $n = t(k-1) + r, 0 \leqslant r < k - 1$, is nonlinear in n. Moreover, clearly,

$$\lim_{n \to \infty} \frac{\{2n(k-2)+2\} - \{(k-2)(2n-k+1)+2\}}{(k-2)(2n-k+1)+2} = 0.$$

To prove our main theorem, we need the following results.

For a nonincreasing sequence of n nonnegative integers, denote

$$\pi' = \begin{cases} (d_1 - 1, \ldots, d_{k-1} - 1, d_{k+1} - 1, \ldots, d_{d_k+1} - 1, d_{d_k+2}, \ldots, d_n) & \text{if } d_k \geqslant k, \\ (d_1 - 1, \ldots, d_{d_k} - 1, d_{d_k+1}, \ldots, d_{k-1}, d_{k+1}, \ldots, d_n) & \text{if } d_k \leqslant k - 1. \end{cases}$$

Then π' is the residual sequence obtained by laying off d_k from π. The following is a theorem due to Kleitman and Wang [3].

Theorem 1.2. *A sequence π is graphic if and only if π' is graphic.*

Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let $\pi = (d_1, d_2, \ldots, d_n)$ be the degree sequence of G, where d_i is the degree of v_i and π is not necessarily nonincreasing. We say that G has property A_k if the subgraph induced by $V = \{v_1, v_2, \ldots, v_k\}$ is K_k. An n-term (not necessarily nonincreasing) sequence π is said to be potentially A_k-graphic if it is the degree sequence of a graph G of order n having property A_k. Rao [6] proved the following:

Theorem 1.3. *A nonincreasing sequence π of n nonnegative integers is potentially P_k-graphic if and only if π is potentially A_k-graphic.*

Rao [7] further gave the following criteria for a sequence π to be potentially A_k-graphic.

Theorem 1.4. *Let $\pi = (d_1, d_2, \ldots, d_n)$ be a sequence of n nonnegative integers in which $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_k$ and $d_{k+1} \geqslant d_{k+2} \geqslant \cdots \geqslant d_n$. Then π is potentially A_k-graphic if and only if the following conditions hold:*

(1) $d_k \geqslant k-1$;

(2) $\sigma(\pi)$ *is even;*

(3) *for any integers s and t, $0 \leqslant s \leqslant k, 0 \leqslant t \leqslant n-k$,*

$$\sum_{i=1}^{s} d_i + \sum_{j=1}^{t} d_{k+j} \leqslant (s+t)(s+t-1) + \sum_{i=s+1}^{k} \min\{s+t, d_i + s - k + 1\} + \sum_{j=k+1+t}^{n} \min\{s+t, d_j\}.$$

2. Main result

We first prove the following:

Theorem 2.1. *If $n = 2k-1, k \geqslant 3$, then $\sigma(k,n) = 2n(k-2)+2$.*

Proof. Put $\pi = ((n-3)^n)$, i.e., π consists of n integers $n-3$. Clearly, π is graphic and $\sigma(\pi) = n(n-3) = 2n(k-2)+2$. Assume G is a realization of π. Then the degree sequence of the complementary graph G^c of G is $\pi^c = (2^n)$. Clearly, G^c is the union of disjoint cycles. Since n is odd, G^c has at least an odd cycle. Hence any independent vertex set of G^c has at least $k-1$ vertices. In other words, the clique in G has at most $k-1$ vertices. Hence π is not potentially P_k-graphic. Thus $\sigma(k,n) \geqslant 2n(k-2)+2$.

Now, suppose that $\pi = (d_1, d_2, \ldots, d_n)$ is graphic and $\sigma(\pi) \geqslant 2n(k-2)+2$. Then any realization G of π has at least $\sigma(\pi)/2 \geqslant n(k-2)+1$ edges. Hence its complementary graph G^c has at most $\binom{n}{2} - n(k-2) - 1 = n - 1 = 2k - 2$ edges. Assume that G is a realization of π such that G^c has a minimum number of connected components and $S_1, S_2, \cdots, S_t$ are all connected components of G^c. Taking one vertex from each connected components of G^c, we obtain an independent set of t vertices in G^c. Hence G has a clique of size t as a subgraph. If $t \geqslant k$, then π is potentially P_k-graphic. So we may assume $t < k$. Then G^c has at least one nontrivial component. We consider the following cases:

Case 1: G^c has at least two nontrivial components. Let S_1 and S_2 be nontrivial components of G^c, and let $x_1 y_1$ and $x_2 y_2$ be the edges of S_1 and S_2, respectively. The graph obtained by deleting $x_1 y_1$ and $x_2 y_2$ from G^c and then adding the edges $x_1 x_2$ and $y_1 y_2$ is denoted by G'^c. Clearly $G' = (G'^c)^c$ is a realization of π, and the number of components of G^c and G'^c are the same. This shows that $x_i y_i$ is a bridge of S_i, for $i = 1, 2$. Thus each nontrivial component of G^c is a tree, Therefore, G^c is a forest. Hence G^c has an independent set with $\lceil (2k-1)/2 \rceil = k$ vertices.Thus, G has a clique of size k as a subgraph. In other words, π is potentially P_k-graphic.

Case 2: G^c has just one nontrivial component. Assume that S_1 is a nontrivial component with n_1 vertices and $S_2, \ldots, S_t$ are all remained components, where $t < k$. Then $n_1 = 2k - t \geqslant k + 1$ and S_1 has at most $2k - 2$ edges. Let T be a spanning tree of S_1 and let $S_1 \backslash T$ be the graph obtained by deleting all edges of T from S_1. Then $\varepsilon(S_1 \backslash T_1) = \varepsilon(S_1) - \varepsilon(T_1) \leqslant 2k - 2 - (n_1 - 1) = t - 1$, where $\varepsilon(G)$ is the edge number of G. Consider the bipartition $V_1 \cup V_2 = V(T) = V(S_1)$ of the spanning tree T. Let $s_i = |\{xy \in S_1 \backslash T \colon x, y \in V_i\}|$. Then $s_1 + s_2 = \varepsilon(S_1 \backslash T) \leqslant t - 1$. Obviously, V_i contains an independent set A_i of S_1 with $|A_i| \geqslant |V_i| - s_i$ for $i = 1, 2$. Indeed, from every vertex of V_i not in a maximum independent set $A_i \subset V_i$, there is an edge to A_i, hence $|V_i \backslash A_i| \leqslant s_i$, or equivalently, $|A_i| \geqslant |V_i| - s_i$ follows. Hence $|A_1| + |A_2| \geqslant |V(S_1)| - (s_1 + s_2) \geqslant n_1 - (t-1) = 2k - 2t + 1$. In other words, G^c has an independent set with $(2k - 2t + 1) + (t - 1) = 2k - t$ vertices. Because $2k - t \geqslant k + 1$, G has a clique of size $k + 1$ as a subgraph. Thus π is potentially P_k–graphic.

This shows that $\sigma(k, n) \leqslant 2n(k-2) + 2$. The proof is completed. □

Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphic sequence, where $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n \geqslant 1$ and $n \geqslant 2k$. For any given integers s and t, $0 \leqslant s \leqslant k$ and $0 \leqslant t \leqslant n - k$, let us denote

$$L(s,t) = \sum_{i=1}^{s} d_i + \sum_{j=1}^{t} d_{k+j}$$

and

$$R(s,t) = (s+t)(s+t-1) + \sum_{i=s+1}^{k} \min\{s+t, d_i + s - k + 1\} + \sum_{j=k+1+t}^{n} \min\{s+t, d_j\}.$$

We now prove the following:

Theorem 2.2. *Let $n \geqslant 2k$ and let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphic sequence. If*

$$n-2 \geqslant d_1 \geqslant d_2 \geqslant \cdots \geqslant d_{k-1} = d_k = \cdots = d_{d_1+2} \geqslant d_{d_1+3} \geqslant \cdots \geqslant d_n \geqslant k-1,$$

then π is potentially P_k-graphic.

Proof. It is enough to verify by Theorem 1.4 that, for any integers s and t, $0 \leqslant s \leqslant k$ and $0 \leqslant t \leqslant n-k$, $R(s,t) \geqslant L(s,t)$. We consider two cases as follows.

Case 1: $d_{k-1} \leqslant s+t-1$.

If $s \geqslant k-1$, then $d_i + s - k + 1 \geqslant (d_{k-1} - k + 1) + s \geqslant s \geqslant k-1$, for $1 \leqslant i \leqslant k$. Hence $\min\{s+t, d_i + s - k + 1\} \geqslant s \geqslant k-1$ for $s+1 \leqslant i \leqslant k$. In addition, $s+t > d_{k-1} \geqslant \cdots \geqslant d_n \geqslant k-1$. So we have

$$\begin{aligned} R(s,t) &\geqslant (s+t)(s+t-1) + (k-s)(k-1) + (n-k-t)(k-1) \\ &= (k-2)(n-2) + (s+t-k+2)(s+t-1) + (n+k-s-t) \\ &\geqslant (k-2)(n-2) + (s-k+2)(s+t-1) + t(s+t-1) \\ &\geqslant (k-2)d_1 + (s-k+2)d_{k-1} + td_{k-1} \\ &\geqslant L(s,t). \end{aligned}$$

If $s \leqslant k-2$, then $d_i + s - k + 1 \geqslant (d_{k-1} - k + 1) + s \geqslant s$ for $1 \leqslant i \leqslant k+1$. Moreover, $s+t > d_{k-1} \geqslant \cdots \geqslant k-1 > s$. Hence,

$$\begin{aligned} R(s,t) &\geqslant (s+t)(s+t-1) + (k-s)s + (n-k-t)s \\ &= s(n-1) + t(s+t-1) \\ &> sd_1 + td_{k-1} \\ &\geqslant L(s,t). \end{aligned}$$

Thus, $L(s,t) \leqslant R(s,t)$ for $d_{k-1} \leqslant s+t-1$.

Case 2: $d_{k-1} \geqslant s+t$.

If $d_{k-1} \geqslant t+k-1$, then

$$d_{k-1} = d_k = \cdots = d_{t+k+1} = \cdots = d_{d_1+2} \geqslant s+t.$$

Moreover, $d_i + s - k + 1 \geqslant d_{k-1} + s - k + 1 \geqslant s+t$ for $1 \leqslant i \leqslant k$. Hence,

$$\begin{aligned} R(s,t) &\geqslant (s+t)(s+t-1) + \sum_{i=s+1}^{k} \min\{s+t, d_i + s - k + 1\} + \sum_{j=k+1+t}^{d_1+2} \min\{s+t, d_j\} \\ &= (s+t)(s+t-1) + (k-s)(s+t) + (d_1 + 2 - k - t)(s+t) \\ &= (s+t)(d_1+1) > (s+t)d_1 \\ &\geqslant L(s,t). \end{aligned}$$

Now assume that $s+t \leqslant d_{k-1} \leqslant t+k-2$. Then $s \leqslant k-2$ and $d_{k-1}+s-k+1 \leqslant (t+k-2)+(s-k+1) \leqslant s+t-1$. In addition, $d_{k-1}+s-k+1 \leqslant d_i+s-k+1$ for $1 \leqslant i \leqslant k$. Hence, $\min\{s+t, d_i+s-k+1\} \geqslant d_{k-1}+s-k+1$ for $1 \leqslant i \leqslant k+1$. Denote $d_{k-1}=t+m$, where $0 \leqslant s \leqslant m \leqslant k-2$. Then $s+t=d_{k-1}-(m-s) \geqslant k-1-(m-s)$. Also, $d_n \geqslant k-1 \geqslant k-1-(m-s)$. Therefore,

$$\begin{aligned} R(s,t) &\geqslant (s+t)(s+t-1)+(k-s)(d_{k-1}+s-k+1) \\ &\quad +(k-1-m+s)(n-k-t) \\ &= s(n-2)+td_{k-1}+(k-1-m)(n-2k)+s(k-m) \\ &\geqslant s(n-2)+td_{k-1} \\ &\geqslant sd_1+td_{k-1} \\ &\geqslant L(s,t). \end{aligned}$$

This shows that $L(s,t) \leqslant R(s,t)$ holds for $d_{k-1} \geqslant s+t$. □

The following is our main result.

Theorem 2.3. *If $k \geqslant 3, n \geqslant 2k-1$ and $\pi=(d_1,d_2,\ldots,d_n)$ is a graphic sequence which has no zero term and satisfies $\sigma(\pi) \geqslant 2n(k-2)+2$, then π is potentially P_k-graphic. In other words, if $k \geqslant 3$, then $\sigma(k,n) \leqslant 2n(k-2)+2$ for $n \geqslant 2k-1$.*

Proof. We use induction on k. If $k=3$, then by Theorems 2.1 and 1.1, the conclusion holds. Now suppose $k>3$ and the conclusion on $k-1$ holds for $n \geqslant 2k-3$. We will prove by induction on n that the conclusion holds for k. By Theorem 2.1, the conclusion on k holds for $n=2k-1$. Let $n \geqslant 2k$ and assume that the conclusion on k holds for $n-1$ and $\pi=(d_1,d_2,\ldots,d_n)$ is a graphic sequence without zero term and $\sigma(\pi) \geqslant 2n(k-2)+2$. If $d_n \leqslant k-2$, then by Theorem 1.2,

$$\pi' = (d_1-1, d_2-1, \cdots, d_{d_n}-1, d_{d_n+1}, \cdots, d_{n-1})$$

is graphic and has no zero term. Since

$$\begin{aligned} \sigma(\pi') = \sigma(\pi)-2d_n &\geqslant 2n(k-2)+2-2(k-2) \\ &= 2(n-1)(k-2)+2, \end{aligned}$$

π' is potentially P_k-graphic by induction on n. Hence π is potentially P_k-graphic. So we may assume $d_n \geqslant k-1$. Denote

$$\pi'' = (d_2-1, d_3-1, \ldots, d_{d_1+1}-1, d_{d_1+2}, \ldots, d_n).$$

If $d_1=n-1$, then

$$\sigma(\pi'') = \sigma(\pi)-2(n-1) \geqslant 2n(k-3)+4.$$

By induction on k, π'' is potentially P_{k-1}-graphic. Hence, π is potentially P_k-graphic. So we may assume that $d_1 \leqslant n-2$. If there exists an integer t, $k \leqslant t \leqslant d_1+1$, such that $d_t > d_{t+1}$, then the $k-1$ largest numbers in π'' is in front of term d_{d_1+1}. Since

$$\sigma(\pi'') = \sigma(\pi) - 2d_1 \geqslant 2n(k-2)+2-2(n-2)$$
$$\geqslant 2(n-1)(k-3)+2,$$

we have by induction on k that π'' is potentially P_{k-1}-graphic. By Theorem 1.3, π'' is potentially A_{k-1}-graphic. Hence π is potentially P_k-graphic. So we may assume that

$$n-2 \geqslant d_1 \geqslant d_2 \geqslant \cdots \geqslant d_k = d_{k+1} = \cdots = d_{d_1+2} \geqslant d_{d_1+3} \geqslant \cdots \geqslant d_n \geqslant k-1.$$

If $d_{k-1} > d_k$, then $l = d_{d_1+2} \leqslant d_1 - 1$, and by Theorem 1.2,

$$\pi''' = (d_1-1, d_2-1, \ldots, d_l-1, d_{l+1}, \ldots, d_{d_1+1}, d_{d_1+3}, \ldots, d_n)$$

is a positive degree sequence with

$$\sigma(\pi''') = \sigma(\pi) - 2l \geqslant 2n(k-2)+2-2(n-2)$$
$$\geqslant 2(n-1)(k-3)+2.$$

Therefore, π''' is potentially P_{k-1}-graphic by induction on k. By Theorem 1.3, π''' is potentially A_k-graphic. Consequently, π is potentially P_k-graphic. So we may assume that

$$n-2 \geqslant d_1 \geqslant d_2 \geqslant \cdots \geqslant d_{k-1} = d_k = \cdots = d_{d_1+2} \geqslant d_{d_1+3} \geqslant \cdots \geqslant d_n \geqslant k-1.$$

By Theorem 2.2, π is potentially P_k- graphic. □

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