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ORIGINAL PAPER

On the Size of Graphs of Class 2 Whose Cores have Maximum Degree Two

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Abstract The core G_Δ of a graph G is the subgraph of G induced by the vertices of maximum degree $\Delta(G)$. In this paper, we show that if G is a connected graph with $\Delta(G_\Delta) \leq 2$ and $\Delta(G) \geq \frac{1}{2}(|V(G)| - 1)$, then G is of class 2 if and only if G is overfull. Our result generalizes several results of Hilton and Zhao.

Keywords Edge coloring · Chromatic index · Overfull

1 Introduction

Throughout this paper, all graphs we deal with are finite, simple, and undirected. We use $V(G)$, $|G|$, $E(G)$, $e(G)$, $\Delta(G)$, and $\delta(G)$ to denote, respectively, the vertex set, order, edge set, size, maximum degree, and minimum degree of a graph G. We also use K_n, O_n, C_n, $G \cup H$, and rG to denote, respectively, the complete graph of order n, null graph of order n, cycle of order n, union of two vertex-disjoint graphs G and H, and vertex-disjoint union of r copies of a graph G. The *join* $G + H$ of two vertex-disjoint graphs G and H is the graph with the vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy | x \in V(G), y \in V(H)\}$. Vertices of maximum degree in G are called *major vertices* and others are called *minor vertices*. If $x \in V(G)$, we use $N_G(x)$ (or simply $N(x)$) to denote the neighborhood of x, $d_G(x)$ (or simply $d(x)$) the degree of x, and $d_\Delta(x)$ the number of major vertices of G adjacent to x.

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Let $N[x] = N(x) \cup \{x\}$. If $A \subseteq V(G)$, we use $G - A$ (or simply $G - x$ if $A = \{x\}$) to denote the graph obtained by deleting the set of vertices A from G, and use $G[A]$ (or simply $G[x_1, x_2, \ldots, x_k]$ if $A = \{x_1, x_2, \ldots, x_k\}$) to denote the subgraph of G induced by A. If A and B are disjoint subsets of $V(G)$, we use $e_G(A, B)$ (or simply $e_G(x, B)$ if $A = \{x\}$) to denote the number of edges joining A with B. If $F \subseteq E(G)$, we use $G - F$ to denote the graph obtained by deleting F from G.

An *edge-coloring* of a graph G is a map $\pi : E(G) \to C$, where C is a set of colors, such that $\pi(e) \neq \pi(f)$ if e and f are adjacent. The *chromatic index* $\chi'(G)$ of G is the least value of $|C|$ for which an edge-coloring $\pi : E(G) \to C$ exists. A well-known theorem of Vizing [12] states that, for any graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$. A graph G is said to be of *class i, i = 1,2* if $\chi'(G) = \Delta(G) + i - 1$. If G is a connected class 2 graph with $\Delta(G) = \Delta$ and $\chi'(G - e) < \chi'(G)$ for each edge $e \in E(G)$, then G is said to be Δ-*critical*. A graph G is *overfull* if $e(G) \geq \Delta(G)\lfloor \frac{|G|}{2} \rfloor + 1$. It is easy to see that if G is overfull, then G is of class 2.

The *core* G_Δ of a graph G is the subgraph of G induced by the major vertices of G. Fournier [5] showed that if G_Δ is a forest, then G is of class 1 (this result also follows easily by Vizing's original coloring argument [12] or from the Vizing's Adjacency Lemma [13]). Thus if G is of class 2, then its core must contain cycles. The simplest non-trivial case when G_Δ contains cycles is that G_Δ consists of vertex-disjoint cycles. Hilton and Zhao [6–8] considered this case and proved the following results.

Theorem A *Suppose G is a connected graph with $\Delta(G_\Delta) \leq 2$ and $\Delta \geq \frac{1}{2}(|G| + 3)$. Then G is of class 2 if and only if G is overfull.*

Theorem B *Suppose G is a connected graph with $\Delta(G_\Delta) \leq 2$ and $\Delta \geq |G| - |G_\Delta| + 1$. Then G is of class 2 if and only if G is overfull.*

Let P^* be the graph obtained from the Petersen graph by removing one vertex.

Theorem C *Suppose G is a connected graph such that $G \neq P^*$, $\Delta(G_\Delta) \leq 2$, and $\Delta \geq |G| - |G_\Delta|$. Then G is of class 2 if and only if G is overfull.*

Theorem D *Let G be a connected graph such that $G \neq P^*$, $\Delta(G_\Delta) \leq 2$, and*

$$|G| \geq \begin{cases} 2k^2 + \frac{32}{3}k + \frac{47}{3} & if \quad |G| \quad is \quad even \\ 3k^2 + 12k + 16 & if \quad |G| \quad is \quad odd \end{cases}$$

where $k \geq 0$. If $\Delta = |G| - |G_\Delta| - k \geq k + 5$, then G is of class 2 if and only if G is overfull.

In this paper, we shall first improve Theorem A by retaining the result but lowering the lower bound for Δ from $\frac{1}{2}(|G|+3)$ to $\frac{1}{2}(|G|-1)$ (see Theorem 3.3) and then show that Theorems B and C follow immediately from this improvement. We finally confine ourselves to the family of graphs H which are not odd cycles and with $\Delta(H_\Delta) \leq 2$. By deriving information on $\Delta(H)$ when H is either overfull or of class 2, we eventually show that any graph G which satisfies the assumption stated in Theorem D or the inequalities that $\Delta(G_\Delta) \leq 2$ and $\frac{|G|-1}{2} \leq \Delta(G) < \frac{|G|+3}{2}$ can never be of class 2.

2 Some Useful Results

In this section, we state without proof some basic results on Δ-critical graphs, which shall be used in the sequel. Proofs of the first four lemmas can be found, for instance, in [15]. Alternative/shorter proofs of Lemmas 2.8, 2.9 and 2.10, and a proof of Lemma 2.13 can be found in [14].

Lemma 2.1 (Vizing [12]) *For any graph* G, $\chi'(G) \leq \Delta(G) + 1$.

Lemma 2.2 (Vizing's Adjacency Lemma (VAL) [13])
Suppose G *is a* Δ*-critical graph and* $vw \in E(G)$, *where* $d(v) = k$. *Then*

(i) $d_\Delta(w) \geq \Delta - k + 1$ *if* $k < \Delta$;
(ii) $d_\Delta(w) \geq 2$ *if* $k = \Delta$;
(iii) $|G_\Delta| \geq \max\{3, \Delta - \delta(G) + 2\}$.

Lemma 2.3 (Vizing [12]) *Let* G *be a class 2 graph. Then* G *contains a* k*-critical subgraph for each* k *satisfying* $2 \leq k \leq \Delta(G)$.

Lemma 2.4 *Let* G *be a* Δ*-critical graph with* $|G|$ *odd. Then* $e(G) \leq \Delta\frac{|G|-1}{2} + 1$.

Lemma 2.5 (Chetwynd and Hilton [2]) *Let* G *be a graph and let* $u \in V(G)$ *such that* $d_\Delta(u) \leq 1$. *If* $\Delta(G - u) = \Delta(G)$, *then* $\chi'(G - u) = \chi'(G)$.

Lemma 2.6 (Chetwynd and Hilton [2]) *Let* G *be a graph and let* $uv = e \in E(G)$ *such that* $d_\Delta(u) + d(v) \leq \Delta(G)$. *If* $\Delta(G - e) = \Delta(G)$, *then* $\chi'(G - e) = \chi'(G)$.

Lemma 2.7 (Hilton and Zhao [6]) *Let* G *be a connected class 2 graph with* $\Delta(G_\Delta) \leq 2$. *Then*

(i) G *is* Δ*-critical;*
(ii) $\delta(G_\Delta) = 2$;
(iii) $\delta(G) = \Delta - 1$ *unless* G *is an odd cycle.*

Lemma 2.8 (Chetwynd and Hilton [2]) *Let* G *be a connected graph of order* n *with* $\Delta = \Delta(G) \geq 3$. *Suppose* $|G_\Delta| = 3$. *Then* G *is of class 2 if and only if* $G \cong (n-2)^{n-3}(n-1)^3$ *(and thus* n *is odd).*

Lemma 2.9 (Chetwynd and Hilton [3]) *There does not exist any* Δ*-critical graph* G *of even order with* $|G_\Delta| = 4$.

Lemma 2.10 (Chetwynd and Hilton [3]) *Let* G *be a* Δ*-critical graph of order* $2n + 1$ *with* $|G_\Delta| = 4$. *Then either*
(i) $G \cong (2n-2)^{2n-3}(2n-1)^4$ *or (ii)* $G \cong (2n-2)(2n-1)^{2n-4}(2n)^4$.
In particular, $e(G) = n\Delta + 1$.

Lemma 2.11 (Song and Yap [14]) *There does not exist any* Δ*-critical graph* G *of even order with* $|G_\Delta| = 5$.

Lemma 2.12 (Song [9]) *Let* G *be a* Δ*-critical graph of order* $2n + 1$ *with* $|G_\Delta| = 5$. *Then* $e(G) = n\Delta + 1$.

Let J_s be a graph of order s and $G_0 = J_s + O_{s+2}$, and let G'_0 be a spanning subgraph of G_0 such that each vertex of O_{s+2} is joined to at least $s-1$ vertices of J_s and at least one vertex of O_{s+2} is joined to exactly $s-1$ vertices of J_s.

Lemma 2.13 (Yap and Song [14]) *A connected graph G of order $2n$ has a 1-factor if*

(i) $\delta(G) \geq n-1$ *except when* $G = G_0$;
(ii) $\delta(G) = n-2$ *except when* $G = G'_0$ *or* $G = 3K_3 + K_1$.

Lemma 2.14 (Tutte [11]) *A graph G has a 1-factor if and only if $o(G-S) \leq |S|$ for all $S \subset V(G)$, where $o(G-S)$ is the number of odd components of $G-S$.*

3 Main Results

We shall establish our main results in this section. First of all, we shall prove two lemmas, where the first one is an extension of Lemma 2.13.

Let G''_0 denote a spanning subgraph of G_0 such that each vertex of O_{s+2} is adjacent to at least $s-2$ vertices of J_s. Let H be a spanning subgraph of $O_{s+4} + J_s$ such that each vertex of O_{s+4} is adjacent to at least $s-1$ vertices of J_s. Let J'_s be a connected graph of order s. Let H' be a spanning subgraph of $(O_{s+1} \cup J'_3) + J_s$ such that each vertex of O_{s+1} is adjacent to at least $s-1$ vertices of J_s and each vertex of J'_3 is adjacent to at least $s-3$ vertices of J_s.

Lemma 3.1 *Let G be a connected graph of order $2n$ with $\delta(G) = n-3 \geq 1$. Then G has a 1-factor except when $G \in \{G''_0 H, H', O_{s+6}+J_s, 3K_3+K_1, 4K_3+J_2, 3K_5+K_1\}$ or G is a connected spanning subgraph of some graph in $\{3K_3 + K_1, (2K_3 \cup J'_5) + K_1, (O_{s+3} \cup J'_3) + J_s, (O_{s+1} \cup J'_5) + J_s, (O_s \cup J'_3 \cup J'_3) + J_s\}$.*

Proof Suppose G has no 1-factor. Then by Lemma 2.14, there exists $S \subset V(G)$ such that $o(G-S) > |S| = s \geq 1$. Since $|G|$ is even, $o(G-S)$ and $|S|$ have the same parity. Thus $o(G-S) \geq s+2$ and so

$$s + (s+2) \leq s + o(G-S) \leq |G| = 2n. \tag{1}$$

It follows that $n \geq s+1$. Suppose $n = s+1$. By (1), $o(G-S) = s+2$, and so $G = G_0''$. Suppose $n = s+2$. By (1) again, either $o(G-S) = s+4$ or $o(G-S) = s+2$, and so $G \in \{H, H'\}$. Suppose $n = s+3$. Then by (1), $o(G-S) = s+6$, $o(G-S) = s+4$ or $o(G-S) = s+2$, and so $G = O_{s+6} + J_s$ or G is a connected spanning subgraph of $(O_{s+3} \cup J'_3) + J_s$, $(O_{s+1} \cup J'_5) + J_s$, or $(O_s \cup J'_3 \cup J'_3) + J'_3$, where each vertex in O_{s+3}, O_{s+1}, O_s is adjacent to every vertex in J_s and each vertex in J'_3 (resp. J'_5) is adjacent to at least $s-2$ (resp. $s-4$) vertices in J_s.

Hence we assume that

$$n \geq s+4. \tag{2}$$

Among all the odd components of $G-S$, let G_1 be one with minimum order and let $x \in V(G_1)$ be such that $d_G(x) = \min\{d_G(v) : v \in V(G_1)\}$. Then $|G_1| \leq \frac{2n-s}{s+2}$

and so $\delta(G) \leq d(x) \leq (|G_1| - 1) + s \leq \frac{2n-s}{s+2} - 1 + s$. Suppose $n - 3 < d(x)$ or $d(x) < \frac{2n-s}{s+2} - 1 + s$. Then

$$n - 2 \leq d(x) \leq \frac{2n - s}{s + 2} - 1 + s \tag{3}$$

or

$$n - 3 \leq d(x) < \frac{2n - s}{s + 2} - 1 + s. \tag{4}$$

From (3), it follows that

$$ns \leq s^2 + 2s + 2. \tag{5}$$

Clearly, (5) does not hold for $n \geq s + 5$ because $s \geq 1$. By (2), $n = s + 4$. Suppose $n = s+4$. By (5), $s = 1$ and so $n = s+4 = 5$. By (3), $d(x) = 3$. Thus $G = 3K_3 + K_1$.

From (4), it follows that

$$ns < s^2 + 3s + 4. \tag{6}$$

Clearly, (6) does not hold for $n \geq s + 7$ because $s \geq 1$. By (2), we have $s + 4 \leq n \leq s + 6$. Suppose $n = s + 6$. Then (6) implies that $s = 1$ and so $n = s + 6 = 7$. By (4), $d(x) = 4$, and we have $|G_1| \geq d(x) + 1 - s = 4$. Since $|G_1|$ is odd, it follows that $|G_1| \geq 5$, which contradicts the fact that $|G_1| \leq \frac{2n-s}{s+2} = \frac{13}{3}$. Suppose $n = s + 5$. Then (6) implies that $s = 1$ and so $n = s + 5 = 6$. By (4), $d(x) = 3$, and we have $|G_1| \geq d(x) + 1 - s = 3$ and so G is a spanning subgraph of $(2K_3 \cup J'_5) + K_1$, where the vertex in K_1 is adjacent to each vertex in $2K_3$ and at least one vertex in J'_5. Suppose $n = s+4$. Then (6) implies that $1 \leq s \leq 3$. If $2 \leq s \leq 3$, then $|G_1| \geq d(x)+1-s \geq (n-3)+1-s = (s+4-3)+1-s = 2$. Since $|G_1|$ is odd, it follows that $|G_1| = 3$, which contradicts the fact that $|G_1| \leq \frac{2n-s}{s+2} = \frac{2(s+4)-s}{s+2} = 1 + \frac{6}{s+2} \leq \frac{10}{4}$. If $s = 1$, then $n = s+4 = 5$. By (4), $d(x) = 2$, and so G_1 is a connected spanning subgraph of $3K_3 + K_1$, where the vertex in K_1 is adjacent to at least one vertex in each copy of K_3.

So we may assume that

$$n - 3 = d(x) = \frac{2n - s}{s + 2} - 1 + s. \tag{7}$$

However, from (7), we have $|G_1| = \frac{2n-s}{s+2}$ and

$$ns = s^2 + 3s + 4. \tag{8}$$

By (2), $n \geq s + 4$. Now (8) implies that $s \leq 4$. If $s = 4$, then it follows from (8) that $n = 8$. Hence $|G_1| = \frac{2n-s}{s+2} = \frac{16-4}{4+2} = 2$, which contradicts the fact that $|G_1|$ is odd. If $s = 3$, then it follows from (8) that $n = \frac{22}{3}$, which is impossible. If $s = 2$, then it follows from (8) that $n = 7$, $d(x) = n - 3 = 4$, and $|G_1| = \frac{2n-s}{s+2} = \frac{14-2}{2+2} = 3$, and thus $G = 4K_3 + J_2$. If $s = 1$, then from (8) again, we have $n = 8$, $d(x) = n - 3 = 5$, and $|G_1| = \frac{2n-s}{s+2} = \frac{16-1}{1+2} = 5$. Thus $G = 3K_5 + K_1$. □

Lemma 3.2 *Suppose G is a Δ-critical graph of order $2n+1$ with $\Delta(G_\Delta) = 2$, $\Delta \geq n \geq 5$, and $r = |G_\Delta| \geq 6$. Then for any minor vertex $u \in V(G)$ and major vertices $a_1, a_2 \in V(G)$, where $ua_1 \notin E(G)$ and $a_1a_2 \in E(G)$, $G - \{u, a_1, a_2\}$ has a 1-factor.*

Proof Let $A = V(G_\Delta) = \{a_1, a_2, \ldots, a_r\}$. As G is Δ-critical and $\Delta(G_\Delta) = 2$, by Lemma 2.7, $\delta = \delta(G) = \Delta - 1$ and G_Δ is the union of vertex-disjoint cycles. Let $(a_1, \ldots, a_t)$ be a cycle of G_Δ, where $3 \leq t \leq r$, and let $G' = G - \{u, a_1, a_2\}$.

Clearly, $\Delta(G') = \Delta$, $\delta(G') \geq \delta(G) - 3 \geq (\Delta - 1) - 3 \geq (n-1) - 3$. We claim that G' is connected. Suppose to the contrary that G' is disconnected. Then G' has at least two components (say H_1 and H_2, where $\Delta(H_1) = \Delta$ and $\Delta(H_2)$ is the maximum among all the components other than H_1). Then $|H_1| \geq \Delta + 1$ and $|H_2| \geq \delta(G') + 1 = \Delta - 3$. Observe that G' contains at least two vertices of degree Δ and for any $a \in \{a_3, \ldots, a_r\}$, $d_\Delta(a) = 2$ and $d_{G'}(a) \geq \Delta - 3$. If H_1 contains all the vertices $a_3, \ldots, a_r$, then $|H_1| \geq (r-2) + (\Delta - 2) \geq \Delta + 2$ (because $r \geq 6$). Thus $2n - 2 = |G'| \geq |H_1| + |H_2| \geq (\Delta + 2) + (\Delta - 3)$. It follows that $\Delta \leq n - 1$, which contradicts the fact that $\Delta \geq n$. If H_1 does not contain all the vertices in $\{a_3, \ldots, a_r\}$, then $\Delta(H_2) \geq \Delta - 3$ and so $|H_2| \geq \Delta(H_2) + 1 \geq \Delta - 2$. Therefore, $2n - 2 = |G'| \geq |H_1| + |H_2| \geq (\Delta + 1) + (\Delta - 2)$, which again implies that $\Delta \leq n - 1$, contradicting the fact that $\Delta \geq n$.

Now by Lemmas 2.13 and 3.1, G' has a 1-factor except when $G \in \{G_0, G'_0, G''_0, H, H', O_{s+6}+J_s, 3K_3+K_1, 4K_3+J_2, 3K_5+K_1\}$ or G is a connected spanning subgraph of some graph in $\{3K_3+K_1, (2K_3 \cup J'_5)+K_1, (O_{s+3} \cup J'_3)+J_s, (O_{s+1} \cup J'_5)+J_s, (O_s \cup J'_3 \cup J'_3)+J_s\}$. Since $\Delta(G') - \delta(G') \leq \Delta - (\Delta - 4) = 4$ and G' has at least two vertices of degree Δ (because $r \geq 6$), $G \notin \{4K_3 + J_2, 3K_5 + K_1, 3K_3 + K_1, O_{s+6} + J_s\}$ and G is not a connected spanning subgraph of $\{3K_3 + K_1, (2K_3 \cup J'_5) + K_1\}$.

Suppose G' is a connected spanning subgraph of $(O_{s+3} \cup J'_3) + J_s$ or $(O_s \cup J'_3 \cup J'_3) + J_s$. Then $2n - 2 = |G'| = 2s + 6$ and $\Delta - 4 \leq \delta(G') \leq s$. This, together with the fact that $\Delta \geq n$, implies that $\Delta = n = s + 4$. We next show that $\{a_3, \ldots, a_r\} \subset V(J_s)$. Observe that if $t = 3$, then $d_{G'}(a_3) \geq \Delta - 3 = s + 1$ and for any $v \in A - \{a_1, a_2, a_3\}$, $d_{G'}(v) = \Delta = s + 4$. If $t \geq 4$, then $d_{G'}(a_3) \geq \Delta - 2 \geq s + 2$, $d_{G'}(a_4) \geq \Delta - 2 \geq s + 2$, and for any $v \in A - \{a_1, a_2, a_3, a_t\}$, $d_{G'}(v) = \Delta = s + 4$. Therefore, $A \cap V(O_{s+3}) = \emptyset$, $A \cap V(O_s) = \emptyset$, $|A \cap V(J_s)| \geq r - 4$ and J'_3 contains at most two vertices, namely, a_3 and a_t of A. Suppose $a_3 \in V(J'_3)$. Since $d_\Delta(a_3) = 2$, a_3 is adjacent to $\Delta - 2 = s + 2$ minor vertices in G; that is, a_3 is adjacent to $s+2$ vertices of degree $\Delta - 1$ of $\{u, a_1, a_2\} \cup V(J_s) \cup V(J'_3)$ in G. However, $\{u, a_1, a_2\} \cup V(J_s) \cup V(J'_3)$ has at most $(s+6) - (r-1) (\leq s+1)$ vertices of degree $\Delta - 1$ in G. Thus $a_3 \in V(J_s)$. Similarly, $a_t \in V(J_s)$. Now let $Y = V(J_s) \cup \{u, a_1, a_2\}$ and $X = V(G') - Y$. Then $A \subset Y$. Since $d_\Delta(a_i) = 2$ for $i = 1, \ldots, r$, we have $e(G[Y]) \geq r$. Now by counting the number of edges joining X and Y in G, we have $s(\Delta - 1) + 6(\Delta - 1 - 2) \leq e_G(X, Y) \leq r\Delta + (s+3-r)(\Delta - 1) - 2e(G[Y]) \leq (s+3)\Delta - (s+r+3)$, from which it follows that $3\Delta \leq 15 - r$, which is impossible because $\Delta \geq 5$ and $r \geq 6$. Hence G' is not a connected spanning subgraph of $(O_{s+3} \cup J'_3) + J_s$ and $(O_s \cup J'_3 \cup J'_3) + J_s$.

Suppose G' is a spanning subgraph of $(O_{s+1} \cup J'_5) + J_s$. Then $2n - 2 = |G'| = 2s + 6$ and $\Delta - 4 = \delta(G') \leq s$. Thus $\Delta = n = s + 4$ and so u is adjacent to each vertex of $V(O_{s+1})$ in G and $d_\Delta(u) = 2$. Let $|A \cap V(J_5')| = p$. Then $s = |J_s| \geq |A - ((A \cap V(J_5')) \cup \{a_1, a_2\})| = r - p - 2$. Suppose there exists a vertex $a \in A \cap V(J_5')$ such that $ua \notin E(G)$. Since $d_\Delta(a) = 2, a$ is adjacent to $\Delta - 2 = s + 2$ vertices of $(V(J_s) \cup V(J_5')) - A$ in G, and we have $s + 2 \leq |(V(J_s) \cup V(J_5')) - A| = s + 5 - (r - 2) = s + 7 - r$, which is false because $r \geq 6$. Thus u is adjacent to each vertex of $A \cap V(J_5')$ in G. Since $d_\Delta(u) = 2$, we have $p \leq 2$. Let $Y = V(J_s) \cup \{u, a_1, a_2\}$ and $X = V(O_{s+1}) \cup V(J_5')$. If $p \leq 1$, then since $d_\Delta(a_i) = 2$ for $i = 1, \ldots, r$ and $d_\Delta(v) \geq 2$ for any $v \in V(G) - A$, we have $e(G[Y]) \geq (r - 2p) + (s + 3 - r + p)$ and $e(G[X]) = e(J_5') \leq 10$. Thus $(s + 6)\Delta + (-26 + p - s) \leq (s + 6 - p)(\Delta - 1) + p\Delta - 2e(G[X]) \leq e_G(X, Y) \leq (r - p)\Delta + (s + 3 - r + p)(\Delta - 1) - 2e(G[Y]) \leq (s + 3)\Delta - 2(r - 2p) - 3(s + 3 - r + p)$, from which it follows that $3(s + 4) = 3\Delta \leq 17 - 2s + r$, which contradicts the facts that $s \geq r - p - 2 \geq r - 3$ and $r \geq 6$. If $p = 2$, then $s \geq r - p - 2 \geq 2$ and $\Delta = s + 4 \geq 6$. Let $\{a_i, a_j\} = A \cap V(J_5')$. Suppose $a_i a_j \in E(G)$. Then $e(G[Y]) \geq r - 3$ and $e(G[X]) \leq 10$. Thus $(s + 6)\Delta + (-24 - s) \leq (s + 4)(\Delta - 1) + 2\Delta - 2e(G[X]) \leq e_G(X, Y) \leq (r - 2)\Delta + (s + 3 - r + 2)(\Delta - 1) - 2e(G[Y]) \leq (s + 3)\Delta - 2(r - 3) - (s + 3 - r + 2)$, from which it follows that $3\Delta \leq 25 - r$. This, together with the facts that $n = \Delta \geq 6$ and $r \geq 6$, implies that $n = \Delta = 6, s = \Delta - 4 = 2$ and $r = 6$. On the other hand, if $a_i a_j \notin E(G)$, then $e(G[Y]) \geq r - 4$ and $e(G[X]) = e(J_5') \leq 9$. Now $(s + 6)\Delta + (-22 - s) \leq (s + 4)(\Delta - 1) + 2\Delta - 2e(G[X]) \leq e_G(X, Y) \leq (r - 2)\Delta + (s + 3 - r + 2)(\Delta - 1) - 2e(G[Y]) \leq (s + 3)\Delta - 2(r - 4) - (s + 3 - r + 2)$. This, together with the facts that $n = \Delta \geq 6$ and $r \geq 6$, implies that $n = \Delta = 6, s = \Delta - 4 = 2$ and $r = 6$. However, in either case, X contains 2 vertices of degree $\Delta = 6$ and 6 vertices of degree $\Delta - 1 = 5$, and Y contains 4 vertices of degree $\Delta = 6$ and 1 vertex of degree $\Delta - 1 = 5$. Thus by counting the number of edges between X and Y in G, we have $6 \times 5 + 2 \times 6 - 2e(J_5') = e_G(X, Y) = 4 \times 6 + 1 \times 5 - 2e(G[Y])$, which is impossible because the left hand side is always even and the right hand side is always odd.

Suppose $G' \in \{G_0, G_0', G_0''\}$. Then $2s + 2 = 2n - 2$ and $\Delta - 4 \leq \delta(G') \leq s$. Thus $n = s + 2$ and $s + 2 = n \leq \Delta \leq s + 4$. We claim that $\{a_3, \ldots, a_r\} \subseteq V(J_s)$. Let $C = A - \{a_1, a_2, a_3, a_t\}$. Then $|C| = r - 4 \geq 6 - 4 = 2$. Observe that for any $v \in C, d_{G'}(v) \geq \Delta - 1 \geq (s + 2) - 1 = s + 1$, which implies that $C \subset V(J_s)$. Suppose $t \geq 4$. Then for any $v \in \{a_3, a_t\}$, v is adjacent to exactly one vertex in C and $d_{G'}(v) \geq \Delta - 2 \geq (s + 2) - 2 = s$. This, together with the inequality $|V(J_s) - C| \leq s - 2$, implies that $v \in V(J_s)$. On the other hand, suppose $t = 3$. Then $|C| = r - 3 \geq 6 - 3 = 3$ and a_3 is not adjacent to any vertex in C. As $d_{G'}(a_3) \geq \Delta - 3 \geq (s + 2) - 3 = s - 1$ and $|V(J_s) - C| \leq s - 3$, it follows that $a_3 \in V(J_s)$. Thus, in both cases, $\{a_3, \ldots, a_r\} \subseteq V(J_s)$ as claimed. It follows that $r - 2 = |\{a_3, \ldots, a_r\}| \leq |J_s| = s$. Let $X = V(O_{s+2})$ and $Y = G - X$. Then $A \subset V(Y)$ and $|Y| = s + 3$. Since $d_\Delta(v) = 2$ for any $v \in A$ and $d_\Delta(v) \geq 2$ for any $v \in V(Y) - A$, we have $e(Y) \geq e(G_\Delta) + 2|V(Y) - A| = r + 2(s + 3 - r) = 2s - r + 6$. Now by counting the number of edges joining X and $V(Y)$ in two different ways, we have $(s + 2)(\Delta - 1) = e_G(X, V(Y)) = r\Delta + (s + 3 - r)(\Delta - 1) - 2e(Y) \leq (s + 3)\Delta - (s + 3 - r) - 2(2s - r + 6)$, and it follows that $4s + 13 \leq \Delta + 3r$, which is false because $\Delta \leq s + 4$ and $r - 2 \leq s$. Hence $G' \notin \{G_0, G_0', G_0''\}$.

Next, suppose $G' = H$. Then $2s + 4 = 2n - 2$ and $\Delta - 4 \leq s$. Thus $n = s + 3$ and $s + 3 = n \leq \Delta \leq s + 4$. Again, we claim that $\{a_3, \ldots, a_r\} \subseteq V(J_s)$. If $t \geq 4$, then for any $v \in \{a_3, \ldots, a_r\}$, $d_{G'}(v) \geq \Delta - 2 \geq (s+3) - 2 = s+1$, and so $\{a_3, \ldots, a_r\} \subseteq V(J_s)$. On the other hand, if $t = 3$, then $d_{G'}(v) \geq \Delta - 1 \geq (s+3) - 1 = s+2$ for any $v \in \{a_4, \ldots, a_r\}$. Thus $\{a_4, \ldots, a_r\} \subset V(J_s)$. Observe that a_3 is not adjacent to any vertex in $\{a_4, \ldots, a_r\}$ and $d_{G'}(a_3) \geq \Delta - 3 \geq (s+3) - 3 = s$. This, together with the inequality $|V(J_s) - \{a_4, \ldots, a_r\}| = s - (r-3) \leq s - 3$, implies that $a_3 \in V(J_s)$. Thus, in both cases, $\{a_3, \ldots, a_r\} \subseteq V(J_s)$ as claimed. It follows that $r - 2 = |\{a_3, \ldots, a_r\}| \leq |J_s| = s$. Let $X = O_{s+4}$ and $Y = G - X$. Then $A \subset V(Y)$ and $|Y| = s+3$. Since $d_\Delta(v) = 2$ for any $v \in A$ and $d_\Delta(v) \geq 2$ for any $v \in V(Y) - A$, we have $e(Y) \geq r + 2(s+3-r) = 2s - r + 6$. Now by counting the number of edges joining X and $V(Y)$ in two different ways, we have $(s+4)(\Delta - 1) = e_G(X, V(Y)) = r\Delta + (s+3-r)(\Delta-1) - 2e(Y) \leq (s+3)\Delta - (s+3-r) - 2(2s-r+6)$, and it follows that $\Delta + 4s + 11 \leq 3r$, which is false because $r - 2 \leq s$ and $\Delta \geq n = s+3$. Hence $G' \neq H$.

Finally, suppose $G' = H'$. Then $2s+4 = 2n-2$ and $\Delta - 4 \leq s$. Thus $n = s+3$ and $s+3 = n \leq \Delta \leq s+4$. We first claim that $\{a_3, \ldots, a_r\} \subset (V(J_s) \cup V(J_3'))$. If $t \geq 4$, then for any $v \in \{a_3, \ldots, a_r\}$, $d_{G'}(v) \geq \Delta - 2 \geq (s+3) - 2 = s+1$, and so $v \in V(J_s)$. On the other hand, if $t = 3$, then for any $v \in \{a_4, \ldots, a_r\}$, $d_\Delta(v) \geq \Delta - 1 \geq s+2$, and so $v \in V(J_s)$. Suppose $a_3 \in V(O_{s+1})$. As a_3 is not adjacent to any vertex in $\{a_4, \ldots, a_r\}$ and $d_G(a_3) \geq \Delta \geq s+3$, it follows that a_3 is adjacent to all the vertices in $V(J_s) \cup \{u, a_1, a_2\}$. Hence $\Delta = d_G(a_3) = s+3$, $A \cap V(J_s) = \emptyset$, and $V(J_3') = \{a_4, a_5, a_6\}$ (because $r \geq 6$), and we have $\Delta + s(\Delta - 1) + 3(\Delta - 2) = e_G(V(O_{s+2} \cup J_3'), V(J_s) \cup \{u, a_1, a_2\}) \leq 2(\Delta-1) + (s+1)(\Delta-1)$. It follows from above that $\Delta \leq 3$, which contradicts the fact that $\Delta \geq n \geq 5$. Hence $a_3 \in V(J_s)$. In both cases, we have $\{a_3, \ldots, a_r\} \subset (V(J_s) \cup V(J_3'))$ as required. Let $X = V(O_{s+1})$ and $Y = G - X$. Then $A \subset V(Y)$ and $|Y| = s+6$. We next show that $e(Y) \geq r + 3\Delta - 7$. If J_3' contains no major vertex of G, then $e(J_3') + e_G(V(Y) - V(J_3'), V(J_3')) \geq 3 + 3(\Delta - 3) = 3\Delta - 6$, and so $e(Y) \geq e(G_\Delta) + (e(J_3') + e_G(V(Y) - V(J_3'), V(J_3'))) \geq r + 3\Delta - 6$. If J_3' contains exactly one major vertex of G, then $e(J_3') + e_G(V(Y) - V(J_3'), V(J_3')) \geq 3 + (\Delta - 4) + 2(\Delta - 3) = 3\Delta - 7$, and $e(Y) \geq e(G_\Delta) + (e(J_3') + e_G(V(Y) - V(J_3'), V(J_3'))) \geq r + 3\Delta - 7$. If J_3' contains exactly two major vertices of G, then $e(J_3') + e_G(V(Y) - V(J_3'), V(J_3')) \geq 3 + 3(\Delta - 3) = 3\Delta - 6$ and $e(Y) \geq e(G_\Delta) + (e(J_3') + e_G(V(Y) - V(J_3'), V(J_3')) - 1) \geq r + 3\Delta - 7$. Finally, if $V(J_3')$ contains exactly three major vertices of G, then $e(J_3') + e_G(V(Y) - V(J_3'), V(J_3')) \geq 3 + 3(\Delta - 2) = 3\Delta - 3 e(Y) \geq e(G_\Delta) + (e(J_3') + e_G(V(Y) - V(J_3'), V(J_3')) - 3) \geq r + 3\Delta - 6$. Thus, in either case, $e(Y) \geq r + 3\Delta - 7$ as required. Now by counting the number of edges joining X and $V(Y)$ in two different ways, we have $(s+1)(\Delta - 1) = e_G(X, V(Y)) = r\Delta + (s+6-r)(\Delta-1) - 2e(Y) \leq r\Delta + (s+6-r)(\Delta-1) - 2(r + 3\Delta - 7)$. It follows that $\Delta + r \leq 9$, which is false because $r \geq 6$ and $\Delta \geq n \geq 5$. Hence $G' \neq H'$. □

We are now in a position to prove the following main result.

Theorem 3.3 *Suppose G is a connected graph with $\Delta(G_\Delta) \leq 2$ and $\Delta \geq \frac{1}{2}(|G| - 1)$. Then G is of class 2 if and only if G is overfull.*

Proof It suffices to show that any graph G of class 2 is overfull under the assumption. Let $r = |G_\Delta|$, $A = V(G_\Delta) = \{a_1, a_2, \ldots, a_r\}$, and $B = V(G) - A$. By Lemmas 2.8, 2.9, 2.10, 2.11, and 2.12, it follows that the above statement is true for $r = 3, 4$, and 5. From now on, we assume that $r \geq 6$. Beineke and Fiorini [1] proved that $e(G) = n\Delta + 1$ for all Δ-critical graphs G of order $2n + 1 \leq 7$ and there are no Δ-critical graphs of even order ≤ 10 (for alternative proofs, see also [10]). Chetwynd and Yap [4] proved that $e(G) = n\Delta + 1$ for any Δ-critical graph G with $|G| = 2n + 1 = 9$ except when $G = P^*$. Hence we assume that $|G| \geq 11$.

Since G is of class 2 and $\Delta(G_\Delta) \leq 2$, by Lemma 2.7, $\delta = \delta(G) = \Delta - 1$ and G_Δ is the union of vertex-disjoint cycles. Let $(a_1, \ldots, a_t)$ be a cycle of G_Δ, where $3 \leq t \leq r$. We first show that $|G|$ is odd.

Suppose, otherwise, that $|G| = 2n$. As $\Delta \geq \frac{1}{2}(|G| - 1)$, it follows that $\Delta \geq n$. Let $G' = G - \{a_1, a_2\}$. Then $\Delta(G') = \Delta$ and $\delta(G') \geq \delta(G) - 2 = (\Delta - 1) - 2 \geq (n - 1) - 2$. We claim that G' is connected. Suppose otherwise. Let H_1 and H_2 be two of the components of G', where $\Delta(H_1) = \Delta$ and $\Delta(H_2) \geq \delta(G') \geq \Delta - 3$. Then $|H_1| \geq \Delta + 1$ and $|H_2| \geq \Delta - 2$. Thus $2n - 2 = |G'| \geq |H_1| + |H_2| \geq (\Delta + 1) + (\Delta - 2)$. It follows that $\Delta \leq n - 1$, which contradicts the fact that $\Delta \geq n$. Hence G' is connected, as claimed. Now, by Lemma 2.13, G' has a 1-factor F except when $G' \in \{G_0, G_0', 3K_3 + K_1\}$. Since $\Delta(3K_3 + K_1) - \delta(3K_3 + K_1) = 6 > 3 = \Delta - (\Delta - 3) \geq \Delta(G') - \delta(G')$, $G' \neq 3K_3 + K_1$. Suppose $G' \in \{G_0, G_0'\}$. Then $2s + 2 = 2n - 2$ and so $n = s + 2$. We next show that $\{a_3, \ldots, a_r\} \subseteq V(J_s)$. Observe that if $t \geq 4$, then for any $v \in \{a_3, \ldots, a_r\}$, $d_{G'}(v) \geq \Delta - 1 \geq n - 1 = (s + 2) - 1 = s + 1$, and so $\{a_3, \ldots, a_r\} \subseteq V(J_s)$. If $t = 3$, then for any $v \in \{a_4, \ldots, a_r\}$, $d_{G'}(v) = \Delta \geq n = s + 2$, and so $\{a_4, \ldots, a_r\} \subseteq V(J_s)$. Suppose $a_3 \in V(O_{s+2})$. Since a_3 is not adjacent to any vertex in $\{a_4, \ldots, a_r\} \subset V(J_s)$, we have $\Delta - 2 = d_{G'}(a_3) \leq |J_s - \{a_4, \cdots, a_r\}| = s - (r - 3)$, and it follows that $\Delta \leq s - r + 5$, which however contradicts the fact that $\Delta \geq n = s + 2$ (because $r \geq 6$). Thus, in both cases, $\{a_3, \ldots, a_r\} \subseteq V(J_s)$ as desired. It follows that $r - 2 = |\{a_3, \ldots, a_r\}| \leq |J_s| = s$. Let $X = V(O_{s+2})$ and $Y = G - X$. Then $A \subseteq V(Y)$ and $|Y| = s + 2$. Since $d_\Delta(v) = 2$ for any $v \in A$ and $d_\Delta(v) \geq 2$ for any $v \in V(Y) - A$, we have $e(Y) \geq e(G_\Delta) + 2|V(Y) - A| = r + 2(s + 2 - r) = 2s - r + 4$. Now by counting the number of edges joining X and $V(Y)$ in two different ways, we have $(s + 2)(\Delta - 1) = e_G(X, V(Y)) = r\Delta + (s + 2 - r)(\Delta - 1) - 2e(Y) \leq (s + 2)\Delta - (s + 2 - r) - 2(2s - r + 4)$, and it follows that $4s + 8 \leq 3r$, which is false because $r - 2 \leq s$. Hence $G' \notin \{G_0, G_0'\}$ and G' has a 1-factor F. Let $G^* = G - F \cup \{a_1a_2\}$. Then G^* is of class 2 and the core of G^* has maximum degree two. By Lemma 2.7, the core of G^* is the union of vertex-disjoint cycles. However, from the choice of F, a_1 is adjacent to only one major vertex (namely a_t) in G^*, which is a contradiction.

We thus conclude that $|G| \geq 11$ is odd. Let $|G| = 2n + 1$. Then $n \geq 5$ and $\Delta \geq \frac{|G|-1}{2} \geq n \geq 5$. By Lemma 2.4, $(|G| - r)(\Delta - 1) + r\Delta = 2e(G) \leq (|G| - 1)\Delta + 2$, which implies that $\Delta \leq |G| - r + 2$. Hence G is overfull if and only if $\Delta = |G| - r + 2$. Suppose now that $\Delta < |G| - r + 2$. Then there exists a minor vertex $u \in B$ such that $ua_1 \notin E(G)$. Let $G' = G - \{a_1, a_2, u\}$. By Lemma 3.2, G' has a 1-factor F' and $F = F' \cup \{a_1a_2\}$ is a 1-factor of $G - u$. Since $r \geq 6$, we consider the following two cases.

Case 1. $r \geq 7$. Let $G^* = G - F$. Then G^* is of class 2. Observe that in G^*, $|N_{G^*}(a_1) - A| + |N_{G^*}(a_i) - A| \geq (\Delta - 2) + (\Delta - 3) = 2\Delta - 5 > (2n + 1) - r$, where $i = 2, \ldots, r$. Thus $N_{G^*}(a_1) - A$ and $N_{G^*}(a_i) - A$ have at least one vertex, say x_i, in common in

G^*. Since a_1 is adjacent to only one major vertex a_t in G^* and $\Delta(G^* - a_1) = \Delta(G^*)$, by Lemma 2.5, $G^* - a_1$ is of class 2. We now consider two subcases separately.

Subcase 1.1. u is adjacent to some vertex in $\{a_1, \ldots, a_t\}$.

We may assume that $ua_2 \in E(G)$. Observe that $d_{G^*-a_1}(x_i) = \Delta - 3$. Since a_2 is adjacent to only two major vertices u, a_3 in $G^* - a_1$ and $\Delta(G^* - a_1 - x_2a_2) = \Delta(G^* - a_1)$, by Lemma 2.6, $G_1 = G^* - a_1 - x_2a_2$ is of class 2. Now a_3 is adjacent to at most two major vertices (namely, u if $t = 3$ and u, a_t if $t \geq 4$) in G_1, by Lemma 2.6 again, $G_2 = G_1 - x_3a_3$ is of class 2. Now a_2 is adjacent to only one major vertex u in G_2. Since $r \geq 7$, we have $\Delta(G_2 - a_2) = \Delta(G_2)$, and by Lemma 2.5, $G_3 = G_2 - a_2$ is of class 2. Note that G_3 is a subgraph of $G^* - a_2$ with $\Delta(G_3) = \Delta(G^* - a_2)$. Thus $G^* - a_2$ is of class 2 and the core of $G^* - a_2$ has maximum degree two. Since $N_{G^*}(a_1) - A$ and $N_{G^*}(a_i) - A$ $(i = 2, \ldots, r)$ have at least one vertex x_i in common in G^* and $\Delta \geq n$, it follows that $G^* - a_2$ is connected. Consequently, by Lemma 2.7, $G^* - a_2$ is $(\Delta - 1)$-critical and the core of $G^* - a_2$ is the union of vertex-disjoint cycles, which contradicts the fact that a_1 is adjacent to only one major vertex a_t in $G^* - a_2$.

Subcase 1.2. u is not adjacent to any vertex in $\{a_1, \ldots, a_t\}$.

In this case, a_2 is adjacent to only one major vertex a_3 in $G^* - a_1$ and $\Delta(G^* - \{a_1, a_2\}) = \Delta(G^* - a_1)$. By Lemma 2.5, $G_1 = G^* - \{a_1, a_2\}$ is of class 2. Now consider x_{t+1}, x_{t+2}, and x_{t+3}. We may assume that $x_{t+j}a_i \in E(G)$ for any $1 \leq i \leq t$ and $1 \leq j \leq 3$ (otherwise if $x_{t+j}a_i \notin E(G)$, we may take $a_1 = a_i$ and $u = x_{t+j}$, which, similar to Subcase 1.1, would lead to a contradiction). Note that $d_{G_1}(x_{t+j}) = \Delta - 4$ for $1 \leq j \leq 3$ and a_{t+j} is adjacent to at most three major vertices in G_1. Since $\Delta(G_1 - \{x_{t+1}a_{t+1}, x_{t+2}a_{t+2}, x_{t+3}a_{t+3}\}) = \Delta(G_1)$, by Lemma 2.6, $G_{j+1} = G_j - x_{t+j}a_{t+j}$ is of class 2 for $j = 1, 2, 3$. By VAL, G_4 has at least three major vertices. Note that a_{t+2} is adjacent to only one major vertex (namely u) in G_4 and $\Delta(G_4 - a_{t+2}) = \Delta(G_4)$ (because $r \geq 7$). By Lemma 2.5, $G_5 = G_4 - a_{t+2}$ is of class 2. Note that G_5 is a subgraph of $G^* - a_{t+2}$ with $\Delta(G_5) = \Delta(G^* - a_{t+2})$. Thus, $G^* - a_{t+2}$ is of class 2 and the core of $G^* - a_{t+2}$ has maximum degree two. As $\Delta \geq n$ and $N_{G^*}(a_1) - A$ and $N_{G^*}(a_i) - A$ have at least one vertex in common in G^*, where $i = 2, 3, \ldots, r$, it follows that $G^* - a_{t+2}$ is connected. By Lemma 2.7, the core of $G^* - a_{t+2}$ is the union of vertex-disjoint cycles, which contradicts the fact that a_1 is adjacent to only one major vertex a_t in $G^* - a_{t+2}$.

Case 2. $r = 6$.

In this case, either $G_\Delta \cong C_6$ or $G_\Delta \cong C_3 \cup C_3$. We consider these two subcases separately.

Subcase 2.1. $G_\Delta \cong C_6$.

Let $G_\Delta \cong (a_1, a_2, \ldots, a_6)$. As $d_\Delta(u) \geq 2$, we may assume that $ua_2 \in E(G)$. Let $G^* = G - F$. Then G^* is of class 2. Since a_1 is adjacent to only one major vertex a_6 in G^* and $\Delta(G^* - a_1) = \Delta(G^*)$, by Lemma 2.5, $G^* - a_1$ is of class 2 with five major vertices a_2, a_3, a_4, a_5, u. Note that $|N_{G^*}(a_1) - A| + |N_{G^*}(a_2) - A| = (\Delta - 2) + (\Delta - 2) \geq 2n - 4 > |G| - r = (2n + 1) - 6 = 2n - 5$. Thus $N_{G^*}(a_1) - A$ and $N_{G^*}(a_2) - A$ have at least one vertex, say x, in common. Since $d_{G^*-a_1}(x) = \Delta - 3$, a_2 is adjacent to exactly two major vertices (namely u, a_3) in $G^* - a_1$ and $\Delta(G^* - \{a_1, xa_2\}) = \Delta(G^* - a_1)$.

It follows by Lemma 2.6 that $G_1 = G^* - \{a_1, xa_2\}$ is of class 2 with four major vertices u, a_3, a_4, a_5. By Lemma 2.3, G_1 contains a $(\Delta - 1)$-subgraph H with at most four major vertices u, a_3, a_4, a_5. By Lemma 2.8 and Lemma 2.9, $|H|$ is odd. Clearly, u is a major vertex in H (otherwise the core of H is a path, which, by VAL, is false). Since $ua_2 \in E(G_1), a_2 \in V(H)$. Note that $a_2a_4, a_2a_5 \notin E(G)$. Thus $\delta(H) \leq d_H(a_2) \leq |V(H) - \{a_2, a_4, a_5\}| = |H| - 3 = \Delta(H) - 2$. By Lemmas 2.8 and 2.10, H has exactly four major vertices u, a_3, a_4, a_5 and $\Delta(H) = |H| - 1$. However, as $a_2a_4 \notin E(H)$ and a_4 is a major vertex in H, it follows that $\Delta(H) = d_H(a_4) \leq |V(H) - \{a_2, a_4\}| = |H| - 2$, which is a contradiction.

Subcase 2.2 $G_\Delta \cong C_3 \cup C_3$.

Assume that $G_\Delta \cong (a_1, a_2, a_3) \cup (a_4, a_5, a_6)$. From the choice of F, we know that a_1 and u are arbitrarily chosen. In this case, we first choose a_1 and u such that $d_\Delta(u) = \min\{d_\Delta(v) : v \in B\}$ and $ua_1 \notin E(G)$. Let $G^* = G - F$. Then G^* is of class 2. Since a_1 is adjacent to only one major vertex a_3 in G^* and $\Delta(G^* - a_1) = \Delta(G^*)$, by Lemma 2.5, $G^* - a_1$ is of class 2. Now a_2 is adjacent to at most one major vertex u in $G^* - a_1$, a_3 is adjacent to at most one major vertex u in $G^* - \{a_1, a_2\}$, and $\Delta(G^* - \{a_1, a_2\}) = \Delta(G^* - a_1)$. By Lemma 2.5 again, $G_1 = G^* - \{a_1, a_2, a_3\}$ is of class 2 with at most four major vertices u, a_4, a_5, a_6. By Lemma 2.3, G_1 contains a $(\Delta - 1)$-critical subgraph H with at most four major vertices. If u is adjacent to some vertex in $\{a_2, a_3\}$, say a_2, then u is a minor vertex in G_1 and so H has exactly three major vertices a_4, a_5, a_6. By Lemma 2.8, $\delta(H) = \Delta(H) - 1$ and $|H| = \Delta(H) + 1 = \Delta$. Observe that for any $v \in N_G(a_1) - \{a_2\}, d_{G_1}(v) \leq \Delta - 3 = \Delta(H) - 2$. Thus $N_G[a_1] \cap V(H) = \emptyset$, and so $(2n+1) - (\Delta + 1) = |G| - |N_G[a_1]| \geq |H| = \Delta$. This, together with the inequality $\Delta \geq n$, implies that $\Delta = n$ and $V(H) = V(G) - N_G[a_1]$. Therefore, $u \in V(H)$. By Lemma 2.8, $ua_i \in E(H), i = 4, 5, 6$. Since $ua_2 \in E(G)$, we have $d_\Delta(u) \geq |\{a_2, a_4, a_5, a_6\}| = 4$ and so $d_\Delta(v) \geq 4$ for any $v \in V(G) - A$ (because $d_\Delta(u) = \min\{d_\Delta(v) : v \in B\}$). By VAL, $4(2n + 1 - 6) \leq e_G(B, A) = 6(\Delta - 2)$, and it follows that $\Delta \geq n + 1$ (because $n \geq 5$), which contradicts the fact that $\Delta = n$. Hence we assume that $ua_i \notin E(G)$ for $i = 1, 2, 3$ and thus $ua_j \in E(G)$ for any $j = 4, 5, 6$ (otherwise if $ua_j \notin E(G)$, we may take $a_1 = a_j$, which, similar to the above, would lead to a contradiction). Therefore $d_\Delta(u) = 3$ and $d_\Delta(v) \geq 3$ for any $v \in V(G) - A$ (because $d_\Delta(u) = \min\{d_\Delta(v) : v \in B\}$).

Since H has at most four major vertices, by Lemmas 2.8, 2.9, and 2.10, $|H|$ is odd and $\delta(H) \geq \Delta(H) - 2$. Let $w \in B$ be such that $d_\Delta(u) = \max\{d_\Delta(v) : v \in B\}$. Suppose $d_\Delta(w) = 6$. Then $d_{G_1}(w) \leq \Delta - 4 = \Delta(H) - 3$, and so $w \notin V(H)$ (because $\delta(H) \geq \Delta(H) - 2$). Thus $G_1 - w$ has at most two major vertices, and so H has at most two major vertices, which, by VAL, is false. Hence $d_\Delta(w) \leq 5$. On the other hand, as $\Delta \geq n$ and $d_\Delta(v) \geq 3$ for any $v \in V(G) - A$, by VAL, we have $d_\Delta(w) \geq 4$. Thus $4 \leq d_\Delta(w) \leq 5$. Without loss of generality, we may assume that $wa_1 \notin E(G)$ and $wa_2 \in E(G)$. From the choice of F, we know that u and a_1 are arbitrarily chosen. We now choose u to be w. Then $d_\Delta(u) = \max\{d_\Delta(v) : v \in B\}$ and u is a minor vertex in G_1 (because $ua_2 \in E(G)$). Thus H has exactly three major vertices a_4, a_5, a_6. By Lemma 2.8, $\delta(H) = \Delta(H) - 1$ and $|H| = \Delta(H) + 1 = \Delta$. Observe that for any $v \in N_G(a_1) - \{a_2\}, d_{G_1}(v) \leq \Delta - 3 = \Delta(H) - 2$. Thus $N_G[a_1] \cap V(H) = \emptyset$, and so $(2n+1) - (\Delta + 1) = |G| - |N_G[a_1]| \geq |H| = \Delta$. This, together with the inequality $\Delta \geq n$, implies that $\Delta = n$ and $V(H) = V(G) - N_G[a_1]$. Therefore, $u \in V(H)$, and by Lemma 2.8, $d_H(u) = \Delta(H) - 1 = \Delta - 2$. Thus u is not adjacent to a_3 in G. By Lemma 2.8 again, each a_j $(j = 4, 5, 6)$ is adjacent to every other vertex in H. In particular, $ua_j \in E(H)$. Thus $d_\Delta(u) = 4$ and from the assumption that $d_\Delta(u) = \max\{d_\Delta(v) : v \in B\}$, we have $3 \leq d_\Delta(v) \leq 4$ for any $v \in V(G) - A$. Since $\Delta = n$, by VAL, G has exactly three minor vertices each of which is adjacent to exactly four major vertices in G. This, together with the fact that $d_\Delta(u) = 4$ and each a_j $(j = 4, 5, 6)$ is adjacent to every other vertex in H, implies that there exist $u_1, u_2, u_3 \in N_G(a_1)$ such that $u_1a_4, u_2a_5, u_3a_6 \in F$, and at least one of u_1, u_2, u_3, say u_1, is such that $d_\Delta(u_1) = 3$. Observe that $u_1a_4 \in E(G)$ and $d_{G-V(H)}(u_1) = \Delta - 2 = n - 2$ imply that u_1 is adjacent to one of u_2 and u_3, say u_2. Now replacing F by $F^* = (F - \{u_1a_4, u_2a_5\}) \cup \{u_1u_2, a_4a_5\}$, we see that $G - F^* - \{a_1, a_2, a_3\}$ is of class 2 and the core of $G - F^* - \{a_1, a_2, a_3\}$ is a path, which, by VAL, is false. □

Corollary 3.4 (Theorems B and C) *Suppose G is a connected graph such that $G \neq P^*$, $\Delta(G_\Delta) \leq 2$, and $\Delta \geq |G| - |G_\Delta| + k$, where $k \in \{0, 1\}$. Then G is of class 2 if and only if G is overfull.*

Proof It follows from the results in [1,4] that the statement is true for $|G| \leq 10$ except when $G = P^*$. Hence we assume that $|G| \geq 11$. Suppose G is of class 2. By Lemma 2.7, G is Δ-critical and $\delta(G) = \Delta - 1$. Let $r = |G_\Delta|$. Then by counting the number of edges joining $V(G_\Delta)$ and $V(G) - V(G_\Delta)$ in two different ways, we have $r(\Delta - 2) \leq (|G| - r)(\Delta - 1)$. This, together with the inequality $\Delta \geq |G| - r + k$, where $k \in \{0, 1\}$, implies that

$$|G| - \Delta \leq |G| - \Delta + k \leq r \leq \Delta + 1 + \frac{2}{\Delta - 2}. \tag{9}$$

If $\Delta \leq 4$, then (9) implies that $|G| \leq 10$, which is false. Thus $\Delta \geq 5$, and (9) again implies that $\Delta \geq \frac{|G|-1}{2}$. By Theorem 3.3, G is overfull. □

To end this paper, we shall show that any connected graph G with $\Delta(G_\Delta) \leq 2$ and $\frac{|G|-1}{2} \leq \Delta(G) < \frac{|G|+3}{2}$ must be of class 1, and any connected graph that satisfies the assumption stated in Theorem D can never be of class 2. To get to these, we first prove the following two lemmas.

Lemma 3.5 *Let G be a connected graph such that $G \neq C_{2n+1}$ and $\Delta(G_\Delta) \leq 2$. If G is overfull, then*

(i) $\Delta = |G| - |G_\Delta| + 2$;
(ii) $\Delta \geq \frac{1}{2}(|G| + 3)$.

Proof Since G is overfull, G is class 2, $|G|$ is odd, and $2e(G) \geq \Delta(|G| - 1) + 2$. By Lemma 2.7, G is critical, and so by Lemma 2.4, $2e(G) \leq \Delta(|G| - 1) + 2$. Therefore $2e(G) = \Delta(|G| - 1) + 2$. By Lemma 2.7 again, $\delta(G) = \Delta - 1$, and so $\Delta(|G| - 1) + 2 = \sum_{v \in V(G)} d(v) = 2e(G) = \Delta|G_\Delta| + (\Delta - 1)(|G| - |G_\Delta|)$. It follows that $\Delta = |G| - |G_\Delta| + 2$, establishing (i).

Next, since $\Delta(G_\Delta) \leq 2$, by counting the number of edges joining $V(G_\Delta)$ and $V(G) - V(G_\Delta)$ in two different ways, we have $(\Delta - 2)|G_\Delta| \leq (|G| - |G_\Delta|)(\Delta - 1)$. This, together with (i), implies that $\Delta \geq \frac{1}{2}(|G| + 3)$, which is (ii). □

By applying Theorem 3.3, Corollary 3.4 and Lemma 3.5 together with Theorem C and Theorem D, we finally obtain the following results.

Corollary 3.6 *Let G be a connected graph with* $\Delta(G) = \Delta$. *If* $\Delta(G_\Delta) \leq 2$ *and* $\frac{|G|-1}{2} \leq \Delta < \frac{|G|+3}{2}$, *then G is of class 1.*

Corollary 3.7 *Let G be a connected graph such that* $G \neq P^*$, $\Delta(G_\Delta) \leq 2$, *and*

$$|G| \geq \begin{cases} 2k^2 + \frac{32}{3}k + \frac{47}{3} & if \quad |G| \quad is \quad even \\ 3k^2 + 12k + 16 & if \quad |G| \quad is \quad odd \end{cases}$$

where $k \geq 0$. *If* $\Delta = |G| - |G_\Delta| - k \geq k + 5$, *then G is of class 1.*

Proof Suppose G is of class 2. Then by Theorem D, G is overfull, and so by Lemma 3.5(i), $\Delta = |G| - |G_\Delta| + 2$, which contradicts the assumption that $\Delta = |G| - |G_\Delta| - k$, where $k \geq 0$. Thus G is of class 1. □

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