Planar anti-Ramsey numbers of paths and cycles

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Abstract

Motivated by anti-Ramsey numbers introduced by Erdős, Simonovits and Sós in 1975, we study the anti-Ramsey problem when host graphs are plane triangulations. Given a positive integer n and a planar graph H, let $\mathcal{T}_n(H)$ be the family of all plane triangulations T on n vertices such that T contains a subgraph isomorphic to H. The *planar anti-Ramsey number of H*, denoted $ar_{\mathcal{P}}(n, H)$, is the maximum number of colors in an edge-coloring of a plane triangulation $T \in \mathcal{T}_n(H)$ such that T contains no rainbow copy of H. Analogous to anti-Ramsey numbers and Turán numbers, planar anti-Ramsey numbers are closely related to planar Turán numbers, where the *planar Turán number of H* is the maximum number of edges of a planar graph on n vertices without containing H as a subgraph. The study of $ar_{\mathcal{P}}(n, H)$ (under the name of rainbow numbers) was initiated by Horňák, Jendrol′, Schiermeyer and Soták [J Graph Theory 78 (2015) 248–257]. In this paper we study planar anti-Ramsey numbers for paths and cycles. We first establish lower bounds for $ar_{\mathcal{P}}(n, P_k)$ when $n \geq k \geq 8$. We then improve the existing lower bound for $ar_{\mathcal{P}}(n, C_k)$ when $k \geq 5$ and $n \geq k^2 - k$. Finally, using the main ideas in the above-mentioned paper, we obtain upper bounds for $ar_{\mathcal{P}}(n, C_6)$ when $n \geq 8$ and $ar_{\mathcal{P}}(n, C_7)$ when $n \geq 13$, respectively.

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1 Introduction

All graphs considered in this paper are finite and simple. Motivated by anti-Ramsey numbers introduced by Erdős, Simonovits and Sós [4] in 1975, we study the anti-Ramsey problem when host graphs are plane triangulations. A subgraph of an edge-colored graph is *rainbow* if all of its edges have different colors. Let $\mathcal{F}$ be a family of planar graphs. For the purpose of this paper, we call an edge-coloring that contains no rainbow copy of any graph in $\mathcal{F}$ an *$\mathcal{F}$-free edge-coloring*. A graph G is *$\mathcal{F}$-free* if no subgraph of G is isomorphic to any graph in $\mathcal{F}$. Let $n_{\mathcal{F}}$ be the smallest integer such that for any $n \geq n_{\mathcal{F}}$, there exists a plane triangulation on n vertices that is not $\mathcal{F}$-free. Such an integer $n_{\mathcal{F}}$ is well-defined, because for any $F \in \mathcal{F}$, we can obtain a plane triangulation from a plane drawing of F by adding new edges. When $\mathcal{F} = \{F\}$, then $n_{\mathcal{F}} = |F|$. For any integer $n \geq n_{\mathcal{F}}$, let $\mathcal{T}_n(\mathcal{F})$ be the family of all plane triangulations T on n vertices such that T is not $\mathcal{F}$-free. The *planar anti-Ramsey number of $\mathcal{F}$*, denoted $ar_{\mathcal{P}}(n, \mathcal{F})$, is the maximum number of colors in an $\mathcal{F}$-free edge-coloring of a plane triangulation in $\mathcal{T}_n(\mathcal{F})$. Clearly, $ar_{\mathcal{P}}(n, \mathcal{F}) < 3n - 6$. It is worth noting that this problem becomes trivial if the host plane triangulation on n vertices is $\mathcal{F}$-free, because $3n-6$ colors can be used.

Analogous to the relation between anti-Ramsey numbers and Turán numbers proved in [4], planar anti-Ramsey numbers are closely related to planar Turán numbers [3]. The *planar Turán number of $\mathcal{F}$*, denoted $ex_{\mathcal{P}}(n, \mathcal{F})$, is the maximum number of edges of an $\mathcal{F}$-free planar graph on n vertices. Given an edge-coloring c of a host graph T in $\mathcal{T}_n(\mathcal{F})$, we define a *representing graph* of c to be a spanning subgraph R of T obtained by taking one edge of each color under the coloring c (where R may contain isolated vertices). Clearly, if c is an $\mathcal{F}$-free edge-coloring of T, then R is $\mathcal{F}$-free. Thus $ar_{\mathcal{P}}(n, \mathcal{F}) \leq ex_{\mathcal{P}}(n, \mathcal{F})$ for any $n \geq n_{\mathcal{F}}$. When $\mathcal{F}$ consists of a single graph H, we write $ar_{\mathcal{P}}(n, H)$ and $ex_{\mathcal{P}}(n, H)$ instead of $ar_{\mathcal{P}}(n, \{H\})$ and $ex_{\mathcal{P}}(n, \{H\})$. Given a planar graph H, let $\mathcal{H} = \{H - e : e \in E(H)\}$. Let G be an $\mathcal{H}$-free plane subgraph of a plane triangulation $T \in \mathcal{T}_n(H)$ with $e(G) = ex_{\mathcal{P}}(n, \mathcal{H})$. We then obtain an H-free edge-coloring of T by coloring the edges of G with distinct colors and then coloring the edges in $E(T) \setminus E(G)$ with a new color. Hence, $1 + ex_{\mathcal{P}}(n, \mathcal{H}) \leq ar_{\mathcal{P}}(n, H)$ for any $n \geq |H|$. We obtain the following analogous result.

Proposition 1.1 *Given a planar graph H and a positive integer $n \geq |H|$,*

$$1 + ex_{\mathcal{P}}(n, \mathcal{H}) \leq ar_{\mathcal{P}}(n, H) \leq ex_{\mathcal{P}}(n, H),$$

where $\mathcal{H} = \{H - e : e \in E(H)\}$.

Colorings of plane graphs that avoid rainbow faces have also been studied, see, e.g., [5, 7, 15, 16]. Various results on anti-Ramsey numbers can be found in: [1, 2, 8, 9, 10, 11, 13, 14] to name a few. The study of planar anti-Ramsey numbers $ar_{\mathcal{P}}(n, H)$ was initiated by Horňák, Jendrol′, Schiermeyer and Soták [6] (under the name of rainbow numbers). We summarize their results in [6] as follows, where given two positive integers a and b, we use $a \textit{ mod } b$ to denote the remainder when a is divided by b. We use P_k and C_k to denote the path and cycle on k vertices, respectively.

Theorem 1.2 ([6]) *Let n, k be positive integers.*

(a) $ar_{\mathcal{P}}(n, C_3) = \lfloor (3n-6)/2 \rfloor$ *for* $n \geq 4$.
(b) $ar_{\mathcal{P}}(n, C_4) \leq 2(n-2)$ *for* $n \geq 4$, *and* $ar_{\mathcal{P}}(n, C_4) \geq (9(n-2)-4r)/5$ *for* $n \geq 42$ *and* $r = (n-2) \textit{ mod } 20$.
(c) $ar_{\mathcal{P}}(n, C_5) \leq 5(n-2)/2$ *for* $n \geq 5$, *and* $ar_{\mathcal{P}}(n, C_5) \geq (19(n-2)-10r)/9$ *for* $n \geq 20$ *and* $r = (n-2) \textit{ mod } 18$.
(d) $ar_{\mathcal{P}}(n, C_k) \geq (3n-6) \cdot \frac{k-3}{k-2} - \frac{2k-7}{k-2}$ *for* $6 \leq k \leq n$.

Finding exact values of $ar_{\mathcal{P}}(n, H)$ is far from trivial. As observed in [6], an induction argument in general cannot be applied to compute $ar_{\mathcal{P}}(n, H)$ because deleting a vertex from a plane triangulation may result in a graph that is no longer a plane triangulation.

Dowden [3] began the study of planar Turán numbers $ex_{\mathcal{P}}(n, H)$ (under the name of "extremal" planar graphs) and proved Theorem 1.3 below, where each bound is tight.

Theorem 1.3 ([3]) *Let n be a positive integer.*

(a) $ex_{\mathcal{P}}(n, C_3) = 2n-4$ *for* $n \geq 3$.
(b) $ex_{\mathcal{P}}(n, C_4) \leq 15(n-2)/7$ *for* $n \geq 4$.
(c) $ex_{\mathcal{P}}(n, C_5) \leq (12n-33)/5$ *for* $n \geq 11$.

By Proposition 1.1 and Theorem 1.3(c), we see that $ar_{\mathcal{P}}(n, C_5) \leq (12n-33)/5$ for $n \geq 11$. This improves the upper bound for $ar_{\mathcal{P}}(n, C_5)$ in Theorem 1.2(c) when $n \geq 11$. Notice that the upper bound in Proposition 1.1 in general is quite loose, for example, $ex_{\mathcal{P}}(n, C_3) - ar_{\mathcal{P}}(n, C_3) = \lceil n/2 \rceil - 1$ for all $n \geq 4$. In this paper we study planar anti-Ramsey numbers for paths and cycles. In Section 2, we establish lower bounds for $ar_{\mathcal{P}}(n, P_k)$ when $n \geq k \geq 8$. In Section 3, we first improve the existing lower bounds for $ar_{\mathcal{P}}(n, C_k)$ when $k \geq 5$ and $n \geq k^2 - k$, which improves Theorem 1.2(c,d). We then use the main ideas in [6] by studying lower and upper bounds for the planar anti-Ramsey numbers when host graphs are wheels to obtain upper bounds for $ar_{\mathcal{P}}(n, C_6)$ when $n \geq 8$ and $ar_{\mathcal{P}}(n, C_7)$ when $n \geq 13$, respectively.

We need to introduce more notation. For a graph G we use $V(G)$, $|G|$, $E(G)$, $e(G)$, $\delta(G)$ and $\alpha(G)$ to denote the vertex set, number of vertices, edge set, number of edges, minimum degree, and independence number of G, respectively. For a vertex $x \in V(G)$, we will use $N_G(x)$ to denote the set of vertices in G which are adjacent to x. We define $N_G[x] = N_G(x) \cup \{x\}$ and $d_G(x) = |N_G(x)|$. The subgraph of G induced by A, denoted $G[A]$, is the graph with vertex set A and edge set $\{xy \in E(G) : x, y \in A\}$. We denote by $B \setminus A$ the set $B - A$ and $G \setminus A$ the subgraph of G induced on $V(G) \setminus A$, respectively. If $A = \{a\}$, we simply write $B \setminus a$ and $G \setminus a$, respectively. Given two graphs G and H, the *union* of G and H, denoted $G \cup H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. Given two isomorphic graphs G and H, we may (with a slight but common abuse of notation) write $G = H$. Given a plane graph G and an integer $i \geq 3$, an *i-face* in G is a face of size i. Let $f_i(G)$ denote the number of i-faces in G and $n_i(G)$ denote the number of vertices of degree i in G. Given an edge-coloring c of G, let $c(G)$ denote the number of colors used under c. For any positive integer k, let $[k] := \{1, 2, \ldots, k\}$.

2 Rainbow Paths

In this section, we study planar anti-Ramsey numbers for paths. We begin with a construction of a plane triangulation T_H that will be needed in the proof of Theorem 2.3.

Lemma 2.1 *For any integers $p \geq 1$ and $n = 3p + 2$, there exist plane triangulations H on $p + 2$ vertices and T_H on n vertices such that H and T_H satisfy the following.*

(a) $H \subseteq T_H$ and H is hamiltonian;

(b) $V(T_H) \setminus V(H)$ is an independent set in T_H;

(c) The longest path in T_H has $2p + 5 - \max\{0, 3 - p\}$ vertices; and

(d) The longest path in T_H with both endpoints in $V(H)$ has $2p + 3$ vertices.

Proof. Let P be a path with vertices $v_1, v_2, \ldots, v_p$ in order. Let H be the plane triangulation obtained from P by adding two adjacent vertices x, y and joining each of x and y to all vertices on P with the outer face of H having vertices x, y, v_p on its boundary. Then $|H| = p + 2$ and H is hamiltonian. Let T_H be the plane triangulation obtained from H by adding a new vertex to each 3-face F of H and then joining it to all vertices on the boundary of F. For each $i \in \{1, 2, \ldots, p - 1\}$, let u_i and w_i be the new vertices added to the faces with vertices v_i, v_{i+1}, x and v_i, v_{i+1}, y on the boundary, respectively. Let w, z be the new vertices added to the outer face of H and the face of H with vertices x, y, v_1 on its boundary. The construction of T_H when $p = 5$ is depicted in Figure 1. Then $|T_H| = |H| + f_3(H) = 3p+2 = n$ and $V(T_H) \setminus V(H) = \{u_1, \ldots, u_{p-1}, w_1, \ldots, w_{p-1}, w, z\}$. Clearly, $V(T_H) \setminus V(H)$ is a maximal independent set of T_H with $|V(T_H) \setminus V(H)| = f_3(H) = 2p$ and $|V(T_H) \setminus V(H)| \geq |H| + 1 - \max\{0, 3 - p\}$.

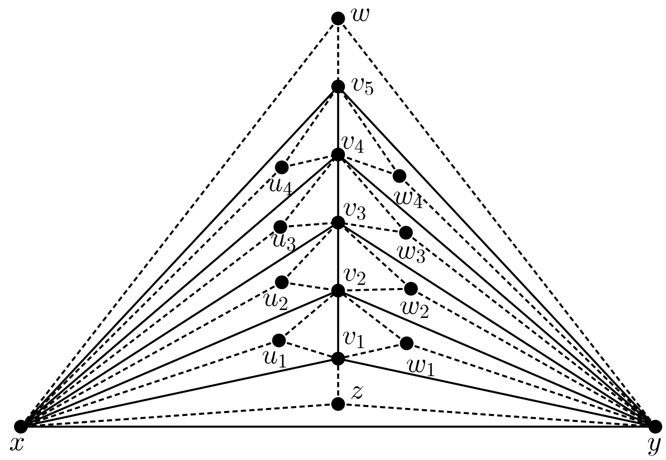


Figure 1: The construction of T_H when $p = 5$.

It can be easily checked that the longest path in T_H has $2p+5-\max\{0, 3-p\}$ vertices, and the longest path with both endpoints in $V(H)$ has $2p+3$ vertices. ■

Theorem 2.2 *For any $k \in \{8, 9\}$, let $\varepsilon = k \bmod 2$ and $n \geq k$ be an integer. Then $ar_{\mathcal{P}}(n, P_k) \geq (3n + 3\varepsilon - \varepsilon^* - 3)/2$, where $\varepsilon^* = (n + 1 + \varepsilon) \bmod 2$.*

Proof. Let $k \in \{8, 9\}, n, \varepsilon, \varepsilon^*$ be given as in the statement. Let t be a positive integer satisfying $2t - 3 - \varepsilon + \varepsilon^* = n$. Then $t \geq k - 3$ because $n \geq k$. Let H be a plane drawing of $K_2 + (\overline{K}_{t-3-\varepsilon} \cup K_{\varepsilon+1})$. Clearly, H has 3-faces and 4-faces only. Notice that $|H| = t$, $f_3(H) = 2 + 2\varepsilon$, $f_4(H) = t - 3 - \varepsilon$, $e(H) = 2t - 3 + \varepsilon$, and H is P_{k-2}-free but not P_{k-3}-free. Let $\mathcal{F}$ be a set which consists of all 4-faces of H and ε^* of the 3-faces of H. Let T^* be the plane triangulation obtained from H by adding a new vertex to each face $F \in \mathcal{F}$ and then joining it to all vertices on F. Then $|T^*| = |H| + |\mathcal{F}| = |H| + f_4(H) + \varepsilon^* = 2t - 3 - \varepsilon + \varepsilon^* = n$. Clearly, $T^* \in \mathcal{T}_n(P_k)$. Now let c be an edge-coloring of T^* defined as follows: edges in $E(H)$ are colored with distinct colors under c (that is, T^* contains a rainbow copy of H under c), and for each $F \in \mathcal{F}$, all the new edges added inside F are colored the same, but for distinct faces $F, F' \in \mathcal{F}$, new edges inside F are colored differently than the new edges inside F'. It can be easily checked that T^* has no rainbow P_k but contains a rainbow copy of P_{k-1} under c. Then $c(T^*) = e(H) + f_4(H) + \varepsilon^* = 3t - 6 + \varepsilon^* = (3n + 3\varepsilon - \varepsilon^* - 3)/2$, since $n = 2t - 3 - \varepsilon + \varepsilon^*$. Hence, $ar_{\mathcal{P}}(n, P_k) \geq c(T^*) \geq (3n + 3\varepsilon - \varepsilon^* - 3)/2$, as desired. This completes the proof of Theorem 2.2. ■

We next prove a lower bound for $ar_{\mathcal{P}}(n, P_k)$ when $k \geq 10$.

Theorem 2.3 *Let k and n be two integers such that $n \geq k \geq 10$. Let $\varepsilon = k$ mod 2. Then*

$$ar_{\mathcal{P}}(n, P_k) \geq \begin{cases} n + 2k - 12 & if \quad k \leq n < 3\lfloor k/2 \rfloor + \varepsilon - 5, \\ (3n + 9\lfloor k/2 \rfloor + 3\varepsilon - 43)/2 & if \quad 3\lfloor k/2 \rfloor + \varepsilon - 5 \leq n \leq 5\lfloor k/2 \rfloor + \varepsilon - 15, \\ 2n + k - 14 & if \quad n > 5\lfloor k/2 \rfloor + \varepsilon - 15. \end{cases}$$

Proof. Let k, n, ε be given as in the statement. Assume first that $k \leq n < 3\lfloor k/2 \rfloor + \varepsilon - 5$. Then $k \geq 12$. Let $p = k - 5$ and let P and H be defined in the proof of Lemma 2.1. By Lemma 2.1, $|H| = k - 3$, $f_3(H) = 2k - 10$, $e(H) = 3k - 15$ and H is hamiltonian. Since $n < 3\lfloor k/2 \rfloor + \varepsilon - 5$, we see that $n - k + 3 < f_3(H)$. Let $\mathcal{F}$ be a set which consists of $n - k + 3$ many 3-faces of H. Let T^* be the plane triangulation obtained from H by adding a new vertex to each face $F \in \mathcal{F}$ and then joining it to all vertices on the boundary of F. Clearly, $T^* \in \mathcal{T}_n(P_k)$. Now let c be an edge-coloring of T^* defined as follows: edges in $E(H)$ are colored with distinct colors under c (that is, T^* contains a rainbow copy of H under c), and for each $F \in \mathcal{F}$, all the new edges added inside F are colored the same, but for distinct faces $F, F' \in \mathcal{F}$, new edges inside F are colored differently than the new edges inside F'. It can be easily checked that T^* has no rainbow P_k but contains a rainbow copy of P_{k-1} under c. Then $c(T^*) = e(H) + |\mathcal{F}| = 3k - 15 + n - k + 3 = n + 2k - 12$. Hence, $ar_{\mathcal{P}}(n, P_k) \geq c(T^*) \geq n + 2k - 12$.

Next assume that $3\lfloor k/2 \rfloor + \varepsilon - 5 \leq n \leq 5\lfloor k/2 \rfloor + \varepsilon - 15$. Let $\varepsilon^* = (n + \lceil k/2 \rceil)$ mod 2. By the choice of ε^*, let t be a positive integer satisfying $2t + \varepsilon^* + 10 - 3\lfloor k/2 \rfloor - \varepsilon = n$. Since $n \geq 3\lfloor k/2 \rfloor + \varepsilon - 5$, it follows that $t - 3\lfloor k/2 \rfloor + 10 \geq 2 + \varepsilon$. Let $p = \lfloor k/2 \rfloor - 4$ and let P, H, T_H, $x, y, w, v_{\lfloor k/2 \rfloor - 4}$ be defined in the proof of Lemma 2.1. By Lemma 2.1, $|H| = \lfloor k/2 \rfloor - 2$, $f_3(H) = 2|H| - 4 = 2\lfloor k/2 \rfloor - 8$ and $|T_H| = |H| + f_3(H) = 3\lfloor k/2 \rfloor - 10 \geq k - 5 - \varepsilon$. Let F^* be the outer face of T_H and F_0 be the 3-face of T_H with vertices $x, w, v_{\lfloor k/2 \rfloor - 4}$ on its boundary. Let T be the plane graph on t vertices obtained from T_H by adding $t - 3\lfloor k/2 \rfloor + 10 \geq 2 + \varepsilon$ new vertices to the face F^* and then joining each of the new vertices to both x and y (and further adding exactly one edge among the new vertices added inside F^* when $\varepsilon = 1$). Then all 4-faces of T are inside the face F^* of T_H, $e(T) = e(T_H) + 2(t - 3\lfloor k/2 \rfloor + 10) + \varepsilon = 2t + 3\lfloor k/2 \rfloor - 16 + \varepsilon$ and $f_4(T) = t - 3\lfloor k/2 \rfloor + 10 - \varepsilon$. Let $\mathcal{F}$ be a set which consists of all 4-faces of T (and F_0 when $\varepsilon^* = 1$). Finally, let T^* be the plane triangulation obtained from T by adding a new vertex to each face $F \in \mathcal{F}$ and then joining it to all vertices on the boundary of F. Then $|T^*| = |T| + f_4(T) + \varepsilon^* = 2t - 3\lfloor k/2 \rfloor + 10 + \varepsilon^* - \varepsilon = n$. By Lemma 2.1, the longest (x, y)-path in T_H has $k - 5 - \varepsilon$ vertices. Clearly, the longest (x, y)-path in T^* with all its internal vertices inside the face F^* contains all the new vertices added to F^*.

Thus T^* contains P_k as a subgraph and so $T^* \in \mathcal{T}_n(P_k)$. Now let c be an edge-coloring of T^* defined as follows: edges in $E(T)$ are colored with distinct colors under c (that is, T^* contains a rainbow copy of T under c), and for each $F \in \mathcal{F}$, all the new edges added inside F are colored the same, but for distinct $F, F' \in \mathcal{F}$, new edges inside F are colored differently than the new edges inside F'. We see that T^* has no rainbow P_k but contains a rainbow P_{k-1} under c. Since $n = 2t + \varepsilon^* + 10 - 3\lfloor k/2 \rfloor - \varepsilon$, we see that

$$\begin{aligned} c(T^*) = e(T) + f_4(T) + \varepsilon^* &= (2t + 3\lfloor k/2 \rfloor - 16 + \varepsilon) + (t - 3\lfloor k/2 \rfloor + 10 - \varepsilon) + \varepsilon^* \\ &= (3n + 9\lfloor k/2 \rfloor + 3\varepsilon - 42 - \varepsilon^*)/2 \\ &\geq (3n + 9\lfloor k/2 \rfloor + 3\varepsilon - 43)/2, \end{aligned}$$

Hence, $ar_{\mathcal{P}}(n, P_k) \geq c(T^*) \geq (3n + 9\lfloor k/2 \rfloor + 3\varepsilon - 43)/2$, as desired.

Finally assume that $n \geq 5\lfloor k/2 \rfloor + \varepsilon - 14$. Let $n - k + 7 = 3m + r$, where m is a positive integer and $r \in \{0, 1, 2\}$. Since $k \geq 10$ and $n \geq 5\lfloor k/2 \rfloor + \varepsilon - 14$, we have $m \geq 3$ or $m = r = 2$. Let $t := k + 2m - 7 + \lfloor r/2 \rfloor$. Then $t \geq k - 2$ because $m \geq 3$ or $m = r = 2$, and $t + \lceil (t-k+7)/2 \rceil = n - \varepsilon'$, where $\varepsilon' = 1$ when $r = 1$ and $\varepsilon' = 0$ when $r \in \{0, 2\}$. Let $p = k-9$ and let P, H, $x, y, v_1, \ldots, v_{k-9}$ be defined in the proof of Lemma 2.1. Then $|H| = k - 7$ and the longest path between x an y in H has $k-7$ vertices. Let T' be the plane triangulation on t vertices obtained from H by: adding $t-k+7 \geq 5$ new vertices to the outer face of H, then adding a matching of size $\lfloor (t - k + 7)/2 \rfloor \geq 2$ among the new vertices, and finally joining each of the new vertices to both x and y. We see that T' is a connected P_{k-2}-free plane graph with only 3-faces and 4-faces. It can be easily checked that $f_4(T') = \lceil (t - k + 7)/2 \rceil$ and $e(T') = 2t + k - 13 + \lfloor (t - k + 7)/2 \rfloor$. Let F_0 be the 3-face of T' with vertices x, y, v_{k-9} when $k = 10$ and x, v_{k-10}, v_{k-9} when $k \geq 11$ on its boundary. Let $\mathcal{F}$ be a set which consists of all 4-faces of T' (and F_0 when $\varepsilon' = 1$). Let T^* be the plane triangulation obtained from T' by adding a new vertex to each $F \in \mathcal{F}$ and then joining it to all vertices on the boundary of F. Then $|T^*| = |T'| + |\mathcal{F}| = |T'| + f_4(T') + \varepsilon' = t + \lceil (t - k + 7)/2 \rceil + \varepsilon' = n$. Clearly, T^* contains P_k as a subgraph and so $T^* \in \mathcal{T}_n(P_k)$. Now let c be an edge-coloring of T^* defined as follows: edges in $E(T')$ are colored with distinct colors under c (that is, T^* contains a rainbow copy of T' under c), and for each $F \in \mathcal{F}$, all the new edges added inside F are colored the same, but for distinct faces $F, F' \in \mathcal{F}$, new edges inside F are colored differently than the new edges inside F'. We see that T^* has no rainbow P_k under c but contains a rainbow copy of P_{k-1}. Then

$$
\begin{aligned}
c(T^*) = e(T') + f_4(T') + \varepsilon' &= \left(2t + k - 13 + \left\lfloor \frac{t-k+7}{2} \right\rfloor\right) + \left(\left\lceil \frac{t-k+7}{2} \right\rceil + \varepsilon'\right) \\
&= 2n + k - 13 - \varepsilon' + \left\lfloor \frac{t-k+7}{2} \right\rfloor - \left\lceil \frac{t-k+7}{2} \right\rceil \\
&= 2n + k - 13 - \varepsilon' + \left\lfloor \frac{\lfloor r/2 \rfloor}{2} \right\rfloor - \left\lceil \frac{\lfloor r/2 \rfloor}{2} \right\rceil \\
&\geq 2n + k - 14,
\end{aligned}
$$

since $n = t + \lceil (t-k+7)/2 \rceil + \varepsilon'$ and $t = k + 2m - 7 + \lfloor r/2 \rfloor$. Hence, $ar_{\mathcal{P}}(n, P_k) \geq c(T^*) \geq 2n + k - 14$, as desired. This completes the proof of Theorem 2.3. ∎

Remark. In the proofs of Theorem 2.2 and Theorem 2.3, $T^* \in \mathcal{T}_n(P_k)$ has no rainbow P_k but does contain a rainbow copy of P_{k-1} under the coloring c we found.

3 Rainbow Cycles

In this section, we study planar anti-Ramsey numbers for cycles.

3.1 Improving the existing lower bound for $ar_{\mathcal{P}}(n, C_k)$

We first prove a lower bound for $ar_{\mathcal{P}}(n, C_5)$, which improves Theorem 1.2(c).

Theorem 3.1 *Let $n \geq 119$ be an integer and let $r = (n+7)$ mod 18. Then $ar_{\mathcal{P}}(n, C_5) \geq (39n - 123 - 21r)/18$.*

Proof. Let r, n be given as in the statement. Let $t \geq 6$ be a positive integer satisfying $18t + 11 + r = n$. This is possible because $n \geq 119$ and $r = (n+7)$ mod 18. Let H be a connected C_5-free plane graph with $15t + 9$ vertices and $(12|H| - 33)/5$ edges such that H has only 3-faces and 6-faces, and no two 6-faces share an edge in common. The existence of such a graph H is due to Dowden (see Theorem 4 in [3]). Notice that $f_6(H) = 3t + 2$ and $f_3(H) = 18t + 6$. Let $\mathcal{F}$ be a set which consists of all 6-faces and r of the 3-faces of H. Then $|\mathcal{F}| = f_6(H) + r$. Let T^* be the plane triangulation obtained from H by adding a new vertex to each face $F \in \mathcal{F}$ and then joining it to all vertices on the boundary of F. Then $|T^*| = |H| + |\mathcal{F}| = |H| + f_6(H) + r = (15t + 9) + (3t + 2) + r = 18t + 11 + r = n$ and so $T^* \in \mathcal{T}_n(C_5)$. Finally let c be an edge-coloring of T^* defined as follows: edges in $E(H)$ are colored with distinct colors under c (that is, T^* contains a rainbow copy of H under c), and for each $F \in \mathcal{F}$, all the new edges added inside F are colored the same, but for distinct

$F, F' \in \mathcal{F}$, new edges inside F are colored differently than the new edges inside F'. We see that T^* has no rainbow C_5 under c because H is C_5-free and no rainbow C_5 in T^* can contain any new edges added to H. Then

$$c(T^*) = e(H) + f_6(H) + r = (36t + 15) + (3t + 2) + r = (39n - 123 - 21r)/18,$$

since $n = 18t + 11 + r$. Therefore, $ar_{\mathcal{P}}(n, C_5) \geq c(T^*) \geq (39n - 123 - 21r)/18$, as desired. This completes the proof of Theorem 3.1. ■

Remark. By Proposition 1.1 and Theorem 1.3(c), $ar_{\mathcal{P}}(n, C_5) \leq ex_{\mathcal{P}}(n, C_5) \leq (12n - 33)/5$ for all $n \geq 11$. It then follows from Theorem 3.1 that $(39n - 123 - 21r)/18 \leq ar_{\mathcal{P}}(n, C_5) \leq ex_{\mathcal{P}}(n, C_5) \leq (12n - 33)/5$ for all $n \geq 119$, where $r = (n + 7)$ mod 18.

Theorem 3.2 below provides a new lower bound for $ar_{\mathcal{P}}(n, C_k)$ when $k \geq 5$, which improves Theorem 1.2(d).

Theorem 3.2 *For integers $k \geq 5$, $n \geq k^2 - k$, and $r = (n - 2)$ mod $(k^2 - k - 2)$,*

$$ar_{\mathcal{P}}(n, C_k) \geq \left(\frac{k-3}{k-2} + \frac{2}{3(k+1)(k-2)} \right) (3n - 6) - \frac{2k^2 - 5k - 5}{k^2 - k - 2} r.$$

Proof. Let n, k, r be given as in the statement. Let $t \geq 3$ be an integer satisfying $(k^2 - k - 2)(t - 2) + 2 + r = n$. This is possible because $r = (n - 2)$ mod $(k^2 - k - 2)$ and $n \geq k^2 - k$. Let T be a plane triangulation on t vertices. Then $f_3(T) = 2t - 4$. Let $k := 3m + q$, where $q \in \{0, 1, 2\}$ and $m \geq 1$ is an integer. Let T' be obtained from T as follows. For each face F in T: first subdivide each of the $q + 1$ of the edges of F m times; next, subdivide each of the remaining $2 - q$ edges of F $m - 1$ times; and finally, replace each edge from the subdivision of T by any plane triangulation on $k - 1$ vertices. Examples of constructions of T' when $k \in \{5, 6, 7\}$ are depicted in Figure 2 and Figure 3.

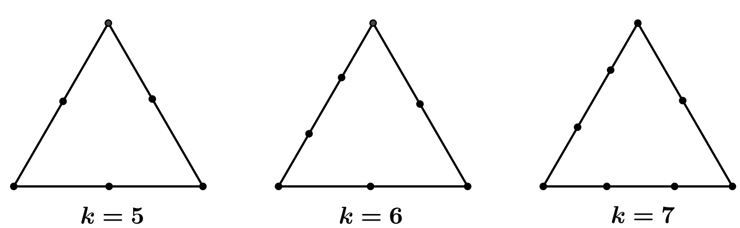
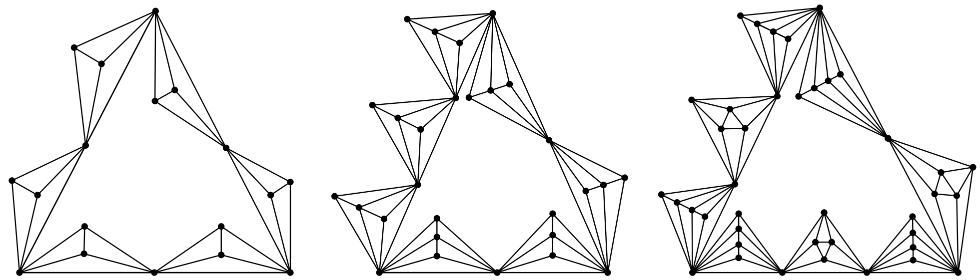


Figure 2: Subdividing one 3-face of T when $k \in \{5, 6, 7\}$.

It is worth noting that different edges of the subdivision of T may be replaced by different plane triangulations on $k - 1$ vertices. Such a subdivision of T is possible when $q \in \{0, 1, 2\}$



$k = 5$ **$k = 6$** **$k = 7$**

Figure 3: One possible construction of T' when $k \in \{5, 6, 7\}$.

because when $q = 2$, every edge of T is subdivided m times; and when $q \in \{0, 1\}$, the dual of T has a perfect matching, say M. Let M^* be the dual edges of M in T. Then every face F in T contains exactly one edge in M^* and $|M^*| = t - 2$. When $q = 0$, each edge in M^* is divided m times, and when $q = 1$, each edge in M^* is divided $m - 1$ times. Thus $(q + 1)(t - 2)$ many edges of T are each subdivided m times and $(2 - q)(t - 2)$ many edges of T are each subdivided $m - 1$ times. One can check that

$$\begin{aligned}
|T'| &= t + (q+1)(t-2)[(m+1)(k-3) + m] + (2-q)(t-2)[((m-1)+1)(k-3) + (m-1)] \\
&= t + (t-2)[(q+1)(mk - 2m + k - 3) + (2-q)(mk - 2m - 1)] \\
&= t + (t-2)[(q+1+2-q)(mk - 2m) + (q+1)(k-3) - (2-q)] \\
&= t + (k^2 - k - 5)(t-2)
\end{aligned}$$

and

$$\begin{aligned}
e(T') &= (q+1)(t-2)(m+1)[(3(k-1) - 6] + (2-q)(t-2)m[3(k-1) - 6] \\
&= (t-2)(3k-9)[(q+1+2-q)m + q + 1] \\
&= (t-2)(3k-9)(3m + q + 1) \\
&= (t-2)(3k-9)(k+1) = 3(k^2 - 2k - 3)(t-2).
\end{aligned}$$

By the construction of T', we see that T' is C_k-free (but contains C_{k+1} as a subgraph), T' has $f_3(T)$ many i-faces with $i > 3$ and at least

$$\begin{aligned}
&(q+1)(t-2)(m+1)[(2(k-1) - 5] + (2-q)(t-2)m[(2(k-1) - 5] \\
&= (t-2)(2k-7)[(q+1+2-q)m + q + 1] \\
&= (t-2)(2k^2 - 5k - 7) \\
&\geq k^2 - k - 2
\end{aligned}$$

many 3-faces because $t \geq 3$ and $k \geq 5$. Let $\mathcal{F}$ be a set which consists of all i-faces of T' with $i > 4$ and r of the 3-faces of T'. Let T^* be the plane triangulation obtained from T' by adding a new vertex to each face $F \in \mathcal{F}$ and then joining it to all vertices on the boundary of F. Then $|T^*| = |T'|+f_3(T)+r = [t+(k^2-k-5)(t-2)]+(2t-4)+r = (k^2-k-2)(t-2)+2+r = n$ and so $T^* \in \mathcal{T}_n(C_k)$. Now let c be an edge-coloring of T^* defined as follows: edges in $E(T')$ are colored with distinct colors under c (that is, T^* contains a rainbow copy of T' under c), and for each $F \in \mathcal{F}$, all the new edges added inside F are colored the same, but for distinct $F, F' \in \mathcal{F}$, new edges inside F are colored differently than the new edges inside F'. We see that T^* has no rainbow C_k (but contains a rainbow copy of C_{k+1}) under c because T' is C_k-free (but contains C_{k+1} as a subgraph) and no rainbow C_k in T^* can contain any new edges added to T'. Hence,

$$\begin{aligned}
c(T^*) = e(T') + f_3(T) + r &= 3(k^2-2k-3)(t-2) + 2(t-2) + r \\
&= (3k^2-6k-7)(t-2) + r \\
&= (3k^2-6k-7)\frac{n-r-2}{k^2-k-2} + r \\
&= \left(\frac{k-3}{k-2} + \frac{2}{3(k+1)(k-2)}\right)(3n-6) - \frac{2k^2-5k-5}{k^2-k-2}r,
\end{aligned}$$

since $n = (k^2-k-2)(t-2)+2+r$. Therefore, $ar_{\mathcal{P}}(n, C_k) \geq c(T^*) \geq \left(\frac{k-3}{k-2} + \frac{2}{3(k+1)(k-2)}\right)(3n-6) - \frac{2k^2-5k-5}{k^2-k-2}r$. This completes the proof of Theorem 3.2. ■

3.2 New upper bounds for $ar_{\mathcal{P}}(n, C_k)$ when $k \in \{6, 7\}$

Finally, we use the main ideas in [6] to establish upper bounds for $ar_{\mathcal{P}}(n, C_k)$ when $k \in \{6, 7\}$. We need to introduce more notation. Let C_q be a cycle with vertices $v_1, v_2, \ldots, v_q$ in order, where $q \geq 3$. Let W_q be a wheel obtained from C_q by adding a new vertex v, the *central vertex* of W_q, and joining v to all vertices of C_q. Vertices $v_1, v_2, \ldots, v_q$ are called *rim vertices* of W_q. A cycle $C \subseteq W_q$ is a *central k-cycle* if it contains the central vertex of W_q and $|C| = k$. For any plane triangulation T with at least four vertices and any $v \in V(T)$, the subgraph of T induced by $N_T[v]$ contains the wheel $W_{d_T(v)}$ with central vertex v as a subgraph. Let $c(v)$ be the set of all colors such that each is used to color the edges of $W_{d_T(v)}$ under any edge-coloring c of T. Lemma 3.3 below will be used in our proof.

Lemma 3.3 ([6]) *Let T be a plane triangulation and let $c : E(T) \to [m]$ be a surjection, where m is a positive integer. Then*

$$\sum_{v \in V(T)} |c(v)| \geq 4m.$$

To establish an upper bound for $ar_{\mathcal{P}}(n, C_k)$ when $k \in \{6, 7\}$, we use the main ideas in [6] by studying lower and upper bounds for the planar anti-Ramsey numbers when host graphs are wheels. For integers $k \geq 4$ and $q \geq k-1$, we define $ar_{\mathcal{P}}(W_q, C_k)$ to be the maximum number of colors in an edge-coloring of W_q that has no rainbow copy of C_k.

Theorem 3.4 *For integers $k \geq 5$ and $q \geq k-1$, $\lfloor \frac{2k-7}{k-3} q \rfloor \leq ar_{\mathcal{P}}(W_q, C_k) \leq \lfloor \frac{2k-5}{k-2} q \rfloor$.*

Proof. Let W_q be a wheel with rim vertices $v_1, v_2, \ldots, v_q$ and central vertex v. To obtain the desired lower bound, let $c : E(W_q) \to [\lfloor (2k-7)q/(k-3) \rfloor]$ be an edge-coloring of W_q defined as follows: for each $i \in [q]$, let $r := i \bmod (k-3)$ and $c(vv_i) := i$,

$$c(v_i v_{i+1}) = \begin{cases} (k-4) \cdot \frac{i-r}{k-3} + q + r - 1, & if\ r \in \{3, 4, \ldots, k-4\}, \\ (k-4) \cdot \frac{i-2}{k-3} + q + 1, & if\ r = 2, \\ (k-4) \cdot \frac{i}{k-3} + q, & if\ r = 0, \end{cases}$$

and

$$c(v_i v_{i+1}) = \begin{cases} (k-4) \cdot \frac{i-1}{k-3} + q + 1, & if\ i \neq q\ and\ r = 1, \\ (k-4) \cdot \frac{i-1}{k-3} + q, & if\ i = q\ and\ r = 1, \end{cases}$$

where all arithmetic on the index $i+1$ here and henceforth is done modulo q. It can be easily checked that c is a surjection and W_q has no rainbow C_k (but contains a rainbow copy of C_{k-1}) under the coloring c. Hence, $ar_{\mathcal{P}}(W_q, C_k) \geq \lfloor (2k-7)q/(k-3) \rfloor$.

Next we prove that $ar_{\mathcal{P}}(W_q, C_k) \leq (2k-5)q/(k-2)$. Let $c : E(W_q) \to [m]$ be any surjection such that W_q contains no rainbow C_k under the coloring c. It suffices to show that $m \leq (2k-5)q/(k-2)$. For any integer ℓ, let A_ℓ be the set of colors used ℓ times under the coloring c. For integers $\alpha \in [m]$ and $j \geq 1$, let: $\eta_j(\alpha)$ be the number of central k-cycles in W_q containing j edges colored α under c, $\eta(\alpha) := \sum_{j=2}^{k} \eta_j(\alpha)$, $\beta(\alpha) := |\{i \in [q] : c(vv_i) = \alpha\}|$ and $\beta'(\alpha) := |\{i \in [q] : c(v_i v_{i+1}) = \alpha\}|$. For any integer ℓ, it is easy to check that $\beta(\alpha) + \beta'(\alpha) = \ell$ for any $\alpha \in A_\ell$. Notice that for any integer $i \in [q]$, vv_i belongs to exactly two central k-cycles and $v_i v_{i+1}$ belongs to exactly $k-2$ central k-cycles in W_q. For any $\alpha \in A_\ell$, we see that

$$2\eta(\alpha) \leq 2\eta(\alpha) + \eta_1(\alpha) \leq \sum_{j \geq 1} j\eta_j(\alpha) = 2\beta(\alpha) + (k-2)\beta'(\alpha) \leq (k-2)\ell,$$

which implies that $\eta(\alpha) \leq (k-2)\ell/2$. Since each of the q central k-cycles of W_q contains a color α with $\eta(\alpha) \geq 1$, we have

$$q \leq \sum_{\ell \geq 2} \sum_{\alpha \in A_\ell} \eta(\alpha) \leq \sum_{\ell \geq 2} (k-2)\ell |A_\ell|/2,$$

which implies $2q/(k-2) \le \sum_{\ell\ge 2} \ell|A_\ell|$. This, together with $2q = e(W_q) = \sum_{\ell\ge 1} \ell|A_\ell|$, implies that $|A_1| \le (2k-6)q/(k-2)$. Then

$$m = |A_1| + \sum_{\ell\ge 2} |A_\ell| \le |A_1| + \sum_{\ell\ge 2} |A_\ell|/2 = |A_1|/2 + \sum_{\ell\ge 1} \ell|A_\ell|/2 \le (2k-5)q/(k-2),$$

as desired. ∎

Corollary 3.5 below follows from the fact that $\lfloor \frac{2k-7}{k-3} q\rfloor = 2q - \lfloor \frac{q}{k-3}\rfloor$, $\lfloor \frac{2k-5}{k-2} q\rfloor = 2q - \lfloor \frac{q}{k-2}\rfloor$ and $ar_{\mathcal{P}}(W_q, C_k) = 2q - \lfloor \frac{q}{k-3}\rfloor$ if $\lfloor \frac{q}{k-2}\rfloor = \lfloor \frac{q}{k-3}\rfloor$. One can see that $\lfloor \frac{q}{k-2}\rfloor = \lfloor \frac{q}{k-3}\rfloor$ when $q \in \{t(k-2), \ldots, t(k-2)+k-4-t\}$ for any integer $t \in [k-4]$.

Corollary 3.5 *Let $k \ge 5$ and $q \ge k-1$ be integers. If $q \in \{t(k-2), \ldots, t(k-2)+k-4-t\}$ for some integer $t \in [k-4]$, then $ar_{\mathcal{P}}(W_q, C_k) = 2q - \lfloor \frac{q}{k-3}\rfloor$.*

We are ready to determine the exact value for $ar_{\mathcal{P}}(W_q, C_6)$ when $q \ge 5$.

Theorem 3.6 *For integer $q \ge 5$, $ar_{\mathcal{P}}(W_q, C_6) = \lfloor 5q/3\rfloor$.*

Proof. By Theorem 3.4, $ar_{\mathcal{P}}(W_q, C_6) \ge \lfloor 5q/3\rfloor$. To prove that $ar_{\mathcal{P}}(W_q, C_6) \le \lfloor 5q/3\rfloor$, it suffices to show that for any surjection $c : E(W_q) \to [m]$ such that W_q contains no rainbow C_6 under the coloring c, we must have $m \le \lfloor 5q/3\rfloor$. We do that next.

Let A_ℓ be the set of colors used ℓ times under the coloring c. For $\alpha \in [m]$, let $\eta_j(\alpha)$ be the number of central 6-cycles in W_q containing j edges colored α under c, $\eta(\alpha) := \sum_{j=2}^{6} \eta_j(\alpha)$, $\beta(\alpha) := |\{i \in [q] : c(vv_i) = \alpha\}|$ and $\beta'(\alpha) := |\{i \in [q] : c(v_iv_{i+1}) = \alpha\}|$. Then $\beta(\alpha)+\beta'(\alpha) = \ell$ for all $\alpha \in A_\ell$. Notice that for any integer $i \in [q]$, vv_i belongs to exactly two central 6-cycles and v_iv_{i+1} belongs to exactly four central 6-cycles. For any $\alpha \in A_\ell$, we see that

$$2\eta(\alpha) \le 2\eta(\alpha) + \eta_1(\alpha) \le \sum_{j\ge 1} j\eta_j(\alpha) = 2\beta(\alpha) + 4\beta'(\alpha) \le 4\ell.$$

This implies that $\eta(\alpha) \le 2\ell$. Notice that for any $\alpha \in A_2$, two edges of W_q colored by α can prevent at most three central 6-cycles from being rainbow under the coloring c, and so $\eta(\alpha) = \eta_2(\alpha) \le 3$. Since each of the q central 6-cycles of W_q contains a color, say $\alpha \in [m]$, with $\eta(\alpha) \ge 1$, it follows that

$$q \le \sum_{\ell\ge 2} \sum_{\alpha\in A_\ell} \eta(\alpha) \le 3|A_2| + \sum_{\ell\ge 3} 2\ell|A_\ell|.$$

Thus $q/2 \leq 3|A_2|/2 + \sum_{\ell\geq 3} \ell|A_\ell|$. This, together with $2q = e(W_q) = \sum_{\ell\geq 1} \ell|A_\ell|$, implies that $2|A_1| + |A_2| \leq 3q$. Then

$$\begin{aligned} m &= |A_1| + |A_2| + \sum_{\ell\geq 3} |A_\ell| \leq |A_1| + |A_2| + \sum_{\ell\geq 3} \ell|A_\ell|/3 \\ &= (2|A_1| + |A_2|)/3 + \sum_{\ell\geq 1} \ell|A_\ell|/3 = (2|A_1| + |A_2|)/3 + 2q/3 \leq 5q/3, \end{aligned}$$

as desired. ■

Finally, we obtain new upper bounds for $ar_{\mathcal{P}}(n, C_6)$ when $n \geq 8$ and $ar_{\mathcal{P}}(n, C_7)$ when $n \geq 13$, respectively.

Theorem 3.7 *$ar_{\mathcal{P}}(n, C_6) \leq 17(n-2)/6$ for all $n \geq 8$, and $ar_{\mathcal{P}}(n, C_7) \leq (59n - 113)/20$ for all $n \geq 13$.*

Proof. We first prove that $ar_{\mathcal{P}}(n, C_6) \leq 17(n-2)/6$ for all integers $n \geq 8$. Let $n \geq 8$ be given and let T be any plane triangulation on n vertices such that T contains C_6 as a subgraph. Let $c : E(T) \to [m]$ be any surjection such that T contains no rainbow C_6 under the coloring c. It suffices to show that $m \leq 17(n-2)/6$. Since $e(T) = 3n-6$ and $n \geq 8$, T must have at least two vertices each with degree at least five. Thus, $n_4(T) \leq n - 2 - n_3(T)$ and $n_3(T) \geq 0$. For any $v \in V(T)$, we see that $|c(v)| \leq e(W_{d_T(v)}) = 2d_T(v)$. But for any $v \in V(T)$ with $d_T(v) \geq 5$, by Theorem 3.6, $|c(v)| \leq ar_{\mathcal{P}}(W_{d_T(v)}, C_6) = \lfloor 5d_T(v)/3 \rfloor$. By Lemma 3.3,

$$\begin{aligned} 4m &\leq \sum_{v\in V(T)} |c(v)| \leq 6n_3(T) + 8n_4(T) + \sum_{v\in V(T), d_T(v)\geq 5} \lfloor 5d_T(v)/3 \rfloor \\ &\leq n_3(T) + 4n_4(T)/3 + 5/3 \cdot \sum_{v\in V(T)} d_T(v) \\ &\leq 4(n-2)/3 - n_3(T)/3 + 5/3 \cdot 2(3n-6) \leq 34(n-2)/3, \end{aligned}$$

which implies that $m \leq 17(n-2)/6$, as desired.

It remains to prove that $ar_{\mathcal{P}}(n, C_7) \leq (59n - 113)/20$ for all $n \geq 13$. The proof is similar to the proof of $ar_{\mathcal{P}}(n, C_6) \leq 17(n-2)/6$. We include a proof here for completeness. Let $n \geq 13$ be given and let T be any plane triangulation on n vertices such that T contains C_7 as a subgraph. Let $c : E(T) \to [m]$ be any surjection such that T contains no rainbow C_7 under the coloring c. It suffices to show that $m \leq (59n - 113)/20$. Since $e(T) = 3n - 6$ and $n \geq 13$, T must have at least one vertex of degree six. Thus, $n_5(T) \leq n - 1 - n_3(T) - n_4(T)$ and $n_i(T) \geq 0$ ($i = 3, 4$). For any $v \in V(T)$, we see that $|c(v)| \leq e(W_{d_T(v)}) = 2d_T(v)$. But for any $v \in V(T)$ with $d_T(v) \geq 6$, by Theorem 3.4, $|c(v)| \leq ar_{\mathcal{P}}(W_{d_T(v)}, C_7) \leq \lfloor 9d_T(v)/5 \rfloor$. By Lemma 3.3,

$$
\begin{aligned}
4m &\leq \sum_{v \in V(T)} |c(v)| \leq 6n_3(T) + 8n_4(T) + 10n_5(T) + \sum_{v \in V(T),\ d_T(v) \geq 6} \lfloor 9d_T(v)/5 \rfloor \\
&\leq 3n_3(T)/5 + 4n_4(T)/5 + n_5(T) + 9/5 \cdot \sum_{v \in V(T)} d_T(v) \\
&\leq n - 1 - 2n_3(T)/5 - n_4(T)/5 + 9/5 \cdot 2(3n-6) \leq 59(n-2)/5 + 1,
\end{aligned}
$$

which implies that $m \leq (59n - 113)/20$, as desired.

This completes the proof of Theorem 3.7. ■

Remark. A better upper bound for $ar_{\mathcal{P}}(n, C_6)$ can be obtained using a result in [12] that $ex_{\mathcal{P}}(n, C_6) \leq 18(n-2)/7$ when $n \geq 6$. By Proposition 1.1 and Theorem 3.2, we see that $\frac{65(n-2)}{28} - \frac{37r}{28} \leq ar_{\mathcal{P}}(n, C_6) \leq ex_{\mathcal{P}}(n, C_6) \leq \frac{72(n-2)}{28}$ for all $n \geq 30$, where $r = (n-2) \bmod 28$.

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