The Smallest Degree Sum that Yields Potentially P_k-Graphical Sequences*

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Abstract: A simple graph G is said to have property P_k if it contains a complete subgraph of order $k+1$, and a sequence π is potentially P_k-graphical if it has a realization having property P_k. Let $\sigma(k,n)$ denote the smallest degree sum such that every n-term graphical sequence π without zero terms and with degree sum $\sigma(\pi) \geq \sigma(k,n)$ is potentially P_k-graphical. Erdös, Jacobson, and Lehel [Graph Theory, 1991, 439–449] conjectured that $\sigma(k,n) = (k-1)(2n-k)+2$. In this article, we prove that the conjecture is true for $k=4$ and $n \geq 10$. © 1998 John Wiley & Sons, Inc. J Graph Theory 29: 63–72, 1998

1. INTRODUCTION

Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphical sequence. For each $m = 1, 2, \ldots, n$, denote $\sigma_m(\pi) = d_1 + d_2 + \cdots + d_m$ and $\sigma_n(\pi) = \sigma(\pi)$. $\sigma(\pi)$ is called the degree sum of π. A simple graph G is said to have property P_k if it has a complete subgraph of order $k+1$ and a graphical sequence $\pi = (d_1, d_2, \ldots, d_n)$ with $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$

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is said to be potentially P_k-graphical if it has a realization having property P_k. In [2], Erdös, Jacobson, and Lehel raised the following problem: Determine the smallest degree sum $\sigma(k, n)$ such that every graphical sequence $\pi = (d_1, d_2, \ldots, d_n)$ with $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$ and $\sigma(\pi) \geq \sigma(k, n)$ is potentially P_k-graphical. Using an example, they pointed out that $\sigma(k, n) \geq (k-1)(2n-k)+2$ and conjectured that $\sigma(k, n) = (k-1)(2n-k)+2$. They also proved that the conjecture was true for $k = 2$. The following is one of their results.

Theorem 1.1. *Let $n \geq 6$ and $\pi = (d_1, d_2, \ldots, d_n)$ be a graphical sequence, where $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$. If $\sigma(\pi) \geq 2n$, then π is potentially P_2-graphical. In other words, $\sigma(2, n) = 2n$ for $n \geq 6$.*

Recently, J. S. Li and Z. X. Song [6] and Gould, Jacobson, and Lehel [3] proved, respectively, the following.

Theorem 1.2. *If $n \geq 8$, then $\sigma(3, n) = 4n - 4$.*

Moreover, [6] also determined the value of $\sigma(k, n)$ for $n = 2k + 1$.

Theorem 1.3. $\sigma(k, 2k+1) = 2k(2k-1)$.

In this article, we will determine the value $\sigma(4, n)$. The following theorems are repeatedly used in our proofs.

For a nonincreasing sequence $\pi = (d_1, d_2, \ldots, d_n)$ of nonnegative integers, $d_i \leq n-1, i = 1, 2, \ldots, n$, define an n-by-n matrix $\bar{A} = (a_{ij})$ as follows: If $d_i \geq i$, then

$$a_{ij} = \begin{cases} 1 & \text{if } 1 \leq j \leq d_i + 1 \text{ and } j \neq i \\ 0 & \text{otherwise,} \end{cases}$$

and if $d_i < i$, then

$$a_{ij} = \begin{cases} 1 & \text{if } 1 \leq j \leq d_i \\ 0 & \text{otherwise.} \end{cases}$$

The matrix $\bar{A}$ is called the off-diagonal matrix of π. The column sum vector of $\bar{A}$ is denoted by $\bar{\pi}$, and called the corrected conjugate vector of π. Clearly, the row sum vector of $\bar{A}$ is π.

Theorem 1.4 (Berge [1]). *Let $\pi = (d_1, d_2, \ldots, d_n)$ be a nonincreasing sequence of nonnegative integers, where $d_i \leq n-1, i = 1, 2, \ldots, n$ and $\sigma(\pi)$ is even. Then π is graphical if and only if $\sigma_i(\pi) \leq \sigma_i(\bar{\pi})$ for $i = 1, 2, \ldots, n-1$, and $\sigma(\pi) = \sigma(\bar{\pi})$.*

For a nonincreasing sequence $\pi = (d_1, d_2, \ldots, d_n)$ of nonnegative integers, let

$$\pi' = \begin{cases} (d_1 - 1, \ldots, d_{k-1} - 1, d_{k+1} - 1 \cdots, d_{d_k+1} - 1, d_{d_k+2}, \ldots, d_n) \\ \quad \text{if } d_k \geq k \\ (d_1 - 1, \ldots, d_{d_k} - 1, d_{d_k+1}, \ldots, d_{k-1}, d_{k+1}, \ldots, d_n) \\ \quad \text{if } d_k < k. \end{cases}$$

Then π' is called the residual sequence after laying off d_k from π.

Theorem 1.5 (Kleitman and D. L. Wang [5]). *A nonincreasing sequence π of nonnegative integers is graphical if and only if π' is graphical.*

In [7], A. R. Rao introduced the following concept: Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let $\pi = (d_1, d_2, \ldots, d_n)$ be the degree sequence of G, where d_i is the degree of v_i. Then G is said to have property A_k if the subgraph induced by $V = \{v_1, v_2, \ldots, v_{k+1}\}$ is complete. A graphical sequence π is said to be potentially A_k-graphical if it has a realization having property A_k. A. R. Rao proved the following.

Theorem 1.6. *A graphical sequence π is potentially P_k-graphical if and only if π is potentially A_k-graphical.*

A. R. Rao [8] also gave a criterion for a sequence π being potentially A_k-graphical.

Theorem 1.7. *Let $\pi = (d_1, d_2, \ldots, d_n)$ be a sequence of nonnegative integers in which $d_1 \geq d_2 \geq \cdots \geq d_{k+1}$ and $d_{k+2} \geq d_{k+3} \geq \cdots \geq d_n$. Then π is potentially A_k-graphical if and only if the following conditions hold*:

(1) $d_{k+1} \geq k$,

(2) $\sigma(\pi)$ *is even*,

(3) *for any s and $t, 0 \leq s \leq k+1$ and $0 \leq t \leq n-k-1$,*

$$\sum_{i=1}^{s} d_i + \sum_{j=1}^{t} d_{k+1+j} \leq (s+t)(s+t-1) + \sum_{i=s+1}^{k+1} \min\{s+t, d_i - k + s\} + \sum_{j=k+2+t}^{n} \min\{s+t, d_j\}.$$

Although the proof of Theorem 1.7 is unpublished until now, Kézdy and Lehel [4] gave a proof of Theorem 1.7.

2. MAIN RESULTS

Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphical sequence, where $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$ and $n \geq 2k+2$. For given s and $t, 0 \leq s \leq k+1$ and $0 \leq t \leq n-k-1$, let

$$L(s,t) = \sum_{i=1}^{s} d_i + \sum_{j=1}^{t} d_{k+1+j}$$

and

$$R(s,t) = (s+t)(s+t-1) + \sum_{i=s+1}^{k+1} \min\{s+t, d_i - k + s\} + \sum_{j=k+2+t}^{n} \min\{s+t, d_j\}.$$

We first prove the following.

Theorem 2.1. *Let $n \geq 2k+2$ and let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphical sequence. If*

$$n-2 \geq d_1 \geq \cdots \geq d_k = d_{k+1} = \cdots = d_{d_1+2} \geq d_{d_1+3} \geq \cdots \geq d_n \geq k,$$

then π is potentially P_k graphical.

Proof. By Theorem 1.7, we need to verify only that

$$R(s,t) \geq L(s,t) \tag{1}$$

for any s and $t, 0 \leq s \leq k+1$ and $0 \leq t \leq n-k-1$. We consider the following cases.

Case 1. $d_k \leq s+t-1$. If $s \geq k$, then $d_i - k + s \geq (d_k - k) + s \geq s \geq k$ for $1 \leq i \leq k+1$. Hence, $\min\{s+t, d_i - k + s\} \geq s \geq k$ for $s+1 \leq i \leq k+1$. Observe that $s+t > d_k \geq \cdots \geq d_n \geq k$. So we have

$$\begin{aligned} R(s,t) &\geq (s+t)(s+t-1) + (k+1-s)k + (n-k-1-t)k \\ &= (k-1)(n-2) + (s-k+1+t)(s+t-1) + (n+k-1-s-t) \\ &\geq (k-1)(n-2) + (s-k+1)(s+t-1) + t(s+t-1) \\ &\geq (k-1)d_1 + (s-k+1)d_k + td_k \geq L(s,t). \end{aligned}$$

If $s \leq k-1$, then $d_i - k + s \geq (d_k - k) + s \geq s$ for $1 \leq i \leq k+1$. Moreover, $s+t > d_k \geq \cdots \geq k > s$. Hence,

$$\begin{aligned} R(s,t) &\geq (s+t)(s+t-1) + (k+1-s)s + (n-k-1-t)s \\ &= s(n-1) + t(s+t-1) > sd_1 + td_k \geq L(s,t). \end{aligned}$$

Thus, (1) holds for $d_k \leq s+t-1$.

Case 2. $d_k \geq s+t$. Clearly,

$$d_k = d_{k+1} = \cdots = d_{t+k+2} = \cdots = d_{d_1+2} \geq s+t.$$

If $d_k \geq t+k$, then $d_i - k + s \geq d_k - k + s \geq s+t$ for $s+1 \leq i \leq k+1$. Hence,

$$\begin{aligned} R(s,t) &\geq (s+t)(s+t-1) + \sum_{i=s+1}^{k+1} \min\{s+t, d_i - k + s\} \\ &\quad + \sum_{j=k+2+t}^{d_1+2} \min\{s+t, d_j\} \\ &= (s+t)(s+t-1) + (k+1-s)(s+t) + (d_1+1-k-t)(s+t) \\ &= (s+t)(d_1+1) \geq (s+t)d_1 \geq L(s,t). \end{aligned}$$

Now assume that $s+t \leq d_k \leq t+k-1$. Then $s \leq k-1$ and $d_k - k + s \leq (t+k-1) + (-k+s) \leq s+t-1$. Moreover $d_k - k + s \leq d_i - k + s$ for $1 \leq i \leq k+1$. Hence, $\min\{s+t, d_i-k+s\} \geq d_k-k+s$ for $1 \leq i \leq k+1$. Denote

$d_k = t+m$, where $0 \le s \le m \le k-1$. Then $s+t = d_k-(m-s) \ge k-(m-s)$. In addition, $d_n \ge k \ge k-(m-s)$. Therefore,

$$\begin{aligned} R(s,t) &\ge (s+t)(s+t-1)+(k+1-s)(d_k-k+s) \\ &\quad + (k-m+s)(n-t-k-1) \\ &= s(n-2)+td_k+(k-m)(n-2k-2)+s(k+1-m) \\ &\ge s(n-2)+td_k \ge sd_1+td_k \ge L(s,t). \end{aligned}$$

This shows that (1) holds for $d_k \ge s+t$. ∎

We now will determine the values of $\sigma(4,n)$ for $n \ge 10$. The main tools are induction, the laying off technique due to Kleitman and D. L. Wang, and A. R. Rao's Theorem 1.6 concerning equivalence on potentially P_k-graphical sequences and potentially A_k-graphical sequences. It is easy to see that, if the residual sequence $\pi' = (d_1', d_2', \dots, d_{n-1}')$ obtained laying off d_1 from a graphical sequence $\pi = (d_1, d_2, \dots, d_n)$ is potentially A_{k-1}-graphical and $d_2-1, \dots, d_{k+1}-1$ are the k largest numbers in π', then π is potentially P_k-graphical. We begin with the value $\sigma(4,10)$.

Theorem 2.2. $\sigma(4,10) = 50$.

Proof. It is enough to prove that, if $\pi = (d_1, d_2, \dots, d_{10})$ is a graphical sequence without zero terms and the degree sum $\sigma(\pi)$ is at least 50, then π is potentially P_4-graphical.

First, $d_1 \ge 5$, since $50 \le \sigma(\pi) \le 10d_1$. Next, $d_5 \ge 4$, otherwise by the off-diagonal matrix $\bar{A}$ of π, we have $\overline{d_i} \le 9$ for $1 \le i \le 3$ and $\overline{d_4} \le 3$. Hence, $\sigma_4(\bar{\pi}) \le 30$. On the other hand, we have $50 \le \sigma(\pi) \le \sigma_4(\pi) + 6d_5 \le \sigma_4(\pi)+18$. Hence, $\sigma_4(\pi) \ge 32 > 30 \ge \sigma_4(\bar{\pi})$. By Theorem 1.4, π is not graphical, contradiction. Finally, $d_8 \ge 2$, otherwise by the off-diagonal matrix $\bar{A}$ of π, we have $\overline{d_1} = 9$ and $\overline{d_i} \le 6$ for $2 \le i \le 7$. Hence, $\sigma_7(\bar{\pi}) \le 9 + 6 \times 6 = 45$. On the other hand, we have $50 \le \sigma(\pi) \le \sigma_7(\pi)+3$. Hence, $\sigma_7(\pi) \ge 47 > 45 \ge \sigma_7(\bar{\pi})$. By Theorem 1.4, π is not graphical, contradiction. In other words, π contains at most two 1's.

The residual sequence obtained by laying off d_1 from π is denoted by $\pi' = (d_2-1, \dots, d_{d_1+1}-1, d_{d_1+2}, \dots, d_{10})$. In realizing π', we delete the zero terms; this leaves 9, 8, or 7 vertices. By Theorems 1.2 and 1.3, we have $\sigma(3,9) = 32, \sigma(3,8) = 28$, and $\sigma(3,7) = 30$. Since $\sigma(\pi') = \sigma(\pi) - 2d_1 \ge 50 - 18 \ge 32 = \sigma(3,9) > \sigma(3,7) > \sigma(3,8), \pi'$ is potentially P_3-graphical. By Theorem 1.6, π' is potentially A_3-graphical. If π' has zero terms, then $d_8 \ge 2$ implies that $d_1 \ge 8$ and $d_{d_1+2} \le 1$. Since $d_5 \ge 4, d_2-1, \dots, d_5-1$ are the four largest numbers in π'. Thus, π is potentially A_4-graphical.

Now suppose that π' has no zero terms. If there exists an integer $t, 5 \le t \le d_1+1$ such that $d_t > d_{t+1}$, then $d_2-1, \dots, d_5-1$ are the four largest numbers in π'. Since π' is potentially A_3-graphical, π is potentially A_4-graphical. So we may assume that

$$d_1 \ge \cdots \ge d_4 \ge d_5 = \cdots = d_{d_1+2} \ge d_{d_1+3} \ge \cdots \ge d_{10}.$$

If $d_4 > d_5$, then by laying off $d_{d_1+1} = l$ from π, we obtain a sequence $\pi'' = (d_1 - 1, \ldots, d_l - 1, d_{l+1}, \ldots, d_{d_1}, d_{d_1+2}, \ldots, d_{10})$ without zero terms. Clearly, $\sigma(\pi'') = 50 - 2d_{d_1+1} \geq 32 = \sigma(3, 9)$. Hence π'' is potentially A_3-graphical. Since $d_1 - 1, d_2 - 1, d_3 - 1$ and $d_4 - 1$ are the four largest numbers in π'', π is potentially A_4-graphical. So we may further assume that

$$d_1 \geq \cdots \geq d_4 = d_5 = \cdots = d_{d_1+2} \geq d_{d_1+3} \geq \cdots \geq d_{10}.$$

If $d_1 = 9$, then

$$d_1 \geq \cdots \geq d_4 = d_5 = \cdots = d_{10} \geq 4.$$

Since $\pi' = (d_2 - 1, d_3 - 1, \ldots, d_{10} - 1)$ is potentially A_3-graphical, π is potentially p_4-graphical. So we assume that $d_1 \leq 8$. If $d_{10} \geq 4$, then by Theorem 2.1, π is potentially P_4-graphical. Now assume $d_{10} \leq 3$. Since $d_{d_1+2} = d_5 \geq 4$, we have $d_1 + 2 \leq 9$, i.e., $d_1 \leq 7$. Observe that $50 \leq \sigma(\pi) \leq 3d_1 + 6d_4 + d_{10}$. Hence, $6 \leq d_1 \leq 7$ and $5 \leq d_4 \leq 7$. We distinguish three cases according to the number of zeros in π' to prove that π is one of the following sequences:

$\pi_1 = (7^9, 3^1)$, $\pi_2 = (7^3, 6^6, 3^1)$, $\pi_3 = (7^1, 6^8, 3^1)$,
$\pi_4 = (7^3, 5^6, 3^1)$, $\pi_5 = (7^2, 5^7, 3^1)$, $\pi_6 = (7^1, 6^2, 5^6, 3^1)$,
$\pi_7 = (7^1, 5^8, 3^1)$, $\pi_8 = (6^8, 5^1, 3^1)$, $\pi_9 = (6^8, 3^2)$,
$\pi_{10} = (6^3, 5^5, 4^1, 3^1)$, $\pi_{11} = (6^2, 5^7, 3^1)$, $\pi_{12} = (7^2, 6^7, 2^1)$,
$\pi_{13} = (7^2, 6^1, 5^6, 2^1)$, $\pi_{14} = (7^1, 6^1, 5^7, 2^1)$, $\pi_{15} = (6^9, 2^1)$,
$\pi_{16} = (6^8, 4^1, 2^1)$, $\pi_{17} = (6^8, 2^2)$, $\pi_{18} = (6^3, 5^6, 2^1)$,
$\pi_{19} = (7^9, 1^1)$, $\pi_{20} = (7^3, 6^6, 1^1)$, $\pi_{21} = (7^1, 6^8, 1^1)$,
$\pi_{22} = (7^1, 6^2, 5^6, 1^1)$, $\pi_{23} = (7^3, 5^6, 1^1)$, $\pi_{24} = (6^8, 1^2)$,
$\pi_{25} = (6^8, 3^1, 1^1)$, $\pi_{26} = (6^8, 5^1, 1^1)$,

Case 1. $d_{10} = 3$. If $d_1 = 7$, then $d_4 = \cdots = d_9$. If $d_4 = 7$, then $\pi = \pi_1$. If $d_4 = 6$, then $\pi = (7^1, d_2, d_3, 6^6, 3^1)$, where $12 \leq d_2 + d_3 \leq 14$. Note that $\sigma(\pi)$ is even. So $\pi = \pi_2$ or π_3. If $d_4 = 5$, then $\pi = (7^1, d_2, d_3, 5^6, 3^1)$, where $10 \leq d_2 + d_3 \leq 14$. Hence, π is one of π_4, π_5, π_6, and π_7. Therefore, we assume $d_1 = 6$. If $d_4 = 6$, then $\pi = (6^8, d_9, 3^1)$. Hence, $\pi = \pi_8$ or π_9, because $\sigma(\pi)$ is even. If $d_4 = 5$, then $\pi = (6^1, d_2, d_3, 5^5, d_9, 3^1)$. Since $50 \leq \sigma(\pi) \leq 18 + 25 + d_9 + 3$, we have $4 \leq d_9 \leq 5$. Thus, $\pi = \pi_{10}$ or π_{11}.
Case 2. $d_{10} = 2$. If $d_1 = 7$, then $d_4 = \cdots = d_9$. If $d_4 = 7$, then $\pi = (7^9, 2)$ is not graphical. If $d_4 = 6$, then $12 \leq d_2 + d_3 \leq 14$. Since $\sigma(\pi)$ is even, we have $\pi = \pi_{12}$. If $d_4 = 5$, then $\pi = \pi_{13}$ or π_{14}. If $d_1 = 6$, then $d_4 = \cdots = d_8$. If $d_4 = 6$, then $\pi = (6^8, d_9, 2^1)$ and π is one of π_{15}, π_{16}, and π_{17}, since $\sigma(\pi)$ is even. If $d_4 = 5$, then $50 \leq \sigma(\pi) \leq 18 + 25 + d_9 + 2$. Hence, $d_9 = 5$ and $\pi = \pi_{18}$.
Case 3. $d_{10} = 1$. If $d_1 = 7$, then $d_4 = \cdots = d_9$. If $d_4 = 7$, then $\pi = \pi_{19}$. If $d_4 = 6$, then $\pi = (7^1, d_2, d_3, 6^6, 1^1)$, where $12 \leq d_2 + d_3 \leq 14$. Since $\sigma(\pi)$ is even, $\pi = \pi_{20}$ or π_{21}. If $d_4 = 5$, then $\pi = (7^1, d_2, d_3, 5^6, 1^1)$, where $12 \leq d_2 + d_3 \leq 14$ and $d_2 + d_3$ is even. Hence, $\pi = \pi_{22}$ or π_{23}. If $d_1 = 6$, then $d_4 = \cdots = d_8$. Note that $50 \leq \sigma(\pi) \leq 18 + 6d_4 + 1$. Hence, $d_4 = 6$, and $\pi(6^8, d_9, 1^1)$, where d_9 is odd. Thus, $\pi = \pi_{24}, \pi_{25}$, or π_{26}.

We now will check that all sequences $\pi_1, \pi_2, \ldots, \pi_{26}$ are potentially A_k-graphical. First $\sigma(4, 9) = 56$ by Theorem 1.3. Hence, π_1, π_{19}, and π_{20} are potentially A_4-graphical. Next for every remained sequence $\pi = (d_1, d_2, \ldots, d_{10})$, we may construct its realization G having property A_4 in the following way: Place a chique on the set S consisted of the 5 vertices of largest degree. The number $r = \sum_{i=1}^{5}(d_i - 4)$ is the number of edges that must go from S to $V(G) - S$ as a bipartite subgraph. Hence, $V(G) - S$ induces a subgraph with $((\sum_{i=6}^{10} d_i) - r)/2$ edges. After forming such a subgraph, it is generally easy to insert the r edges between the sets S and $V(G) - S$ to obtain the realization G.

Figure 1 in the Appendix shows that all these remained sequences are potentially A_4-graphical. ∎

Theorem 2.3. *If* $n \geq 10$, *then* $\sigma(4, n) = 6n - 10$.

Proof. We only need to prove that if $\pi = (d_1, d_2, \ldots, d_n)$ is a graphical sequence without zero terms and the degree sum $\sigma(\pi)$ is at least $6n - 10$, then π is potentially P_4-graphical. We use induction on n. By Theorem 2.2, the theorem holds for $n = 10$. Now suppose that $n > 10$. If $d_n \leq 3$, then $\pi'' = (d_1 - 1, \ldots, d_{d_n} - 1, d_{d_n+1}, \ldots, d_{n-1})$ is graphical and has no zero terms. Clearly $\sigma(\pi'') = \sigma(\pi) - 2d_n \geq 6(n-1) - 10$. By induction hypothesis, π'' is potentially P_4-graphical, and, therefore, π is potentially P_4-graphical. Hence, we may assume that $d_n \geq 4$. If $d_1 = n - 1$, then $\pi' = (d_1 - 1, d_2 - 1, \ldots, d_n - 1)$ is graphical and has no zero terms. Clearly, $\sigma(\pi') = \sigma(\pi) - 2(n-1) \geq 4(n-1) - 4 = \sigma(3, n-1)$, where $n - 1 \geq 9$. By Theorem 1.2, π' is potentially P_3-graphical, so π is potentially P_4-graphical. Hence, we may further assume that $d_1 \leq n - 2$. If there exists an integer $t, 5 \leq t \leq d_1 + 1$ such that $d_t > d_{t+1}$, then $d_2 - 1, \ldots, d_5 - 1$ are the four largest numbers in π'. Since π' is potentially A_3-graphical, π is potentially A_4-graphical. Consequently, we may assume that

$$n - 2 \geq d_1 \geq \cdots \geq d_4 \geq d_5 = \cdots = d_{d_1+2} \geq d_{d_1+3} \geq \cdots \geq d_n \geq 4.$$

If $d_4 > d_5$, then by laying off $d_{d_1+2} = l$ from π, the sequence $\pi''' = (d_1 - 1, \ldots, d_l - 1, d_{l+1}, \ldots, d_{d_l+1}, d_{d_1+3}, \ldots, d_n)$ has no zero terms and $\sigma(\pi''') = \sigma(\pi) - 2l \geq 4(n-1) - 4$. By Theorem 1.2, π''' is potentially A_3-graphical. Since $d_1 - 1, d_2 - 1, d_3 - 1$, and $d_4 - 1$ are the four largest numbers in π''', π is potentially A_4-graphical. Hence, we may assume that

$$n - 2 \geq d_1 \geq \cdots \geq d_4 = d_5 = \cdots = d_{d_1+2} \geq d_{d_1+3} \geq \cdots \geq d_n \geq 4.$$

By Theorem 2.1, π is potentially P_4-graphical. ∎

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Appendix

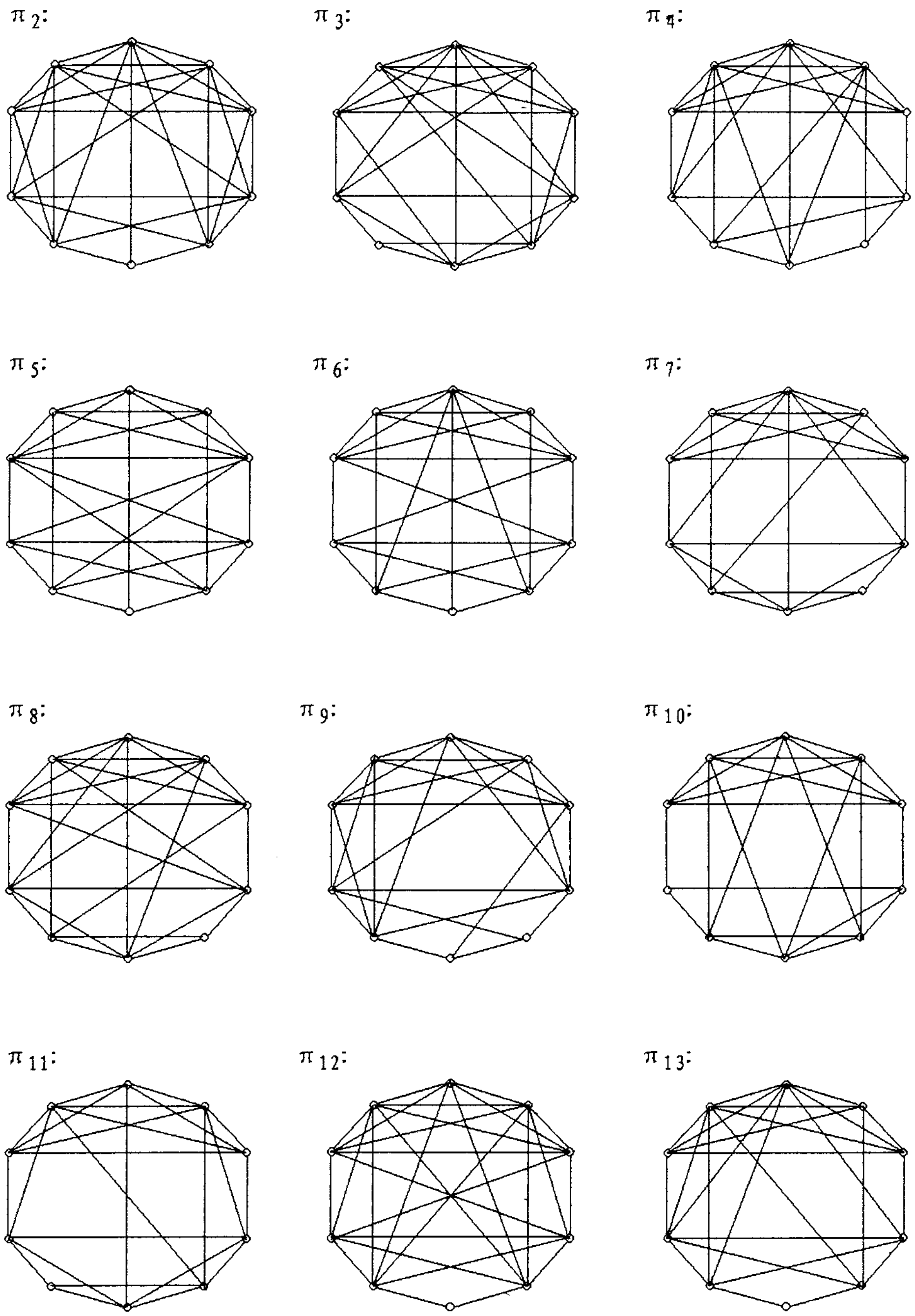

FIGURE 1. Potentially A_4-graphical sequences.

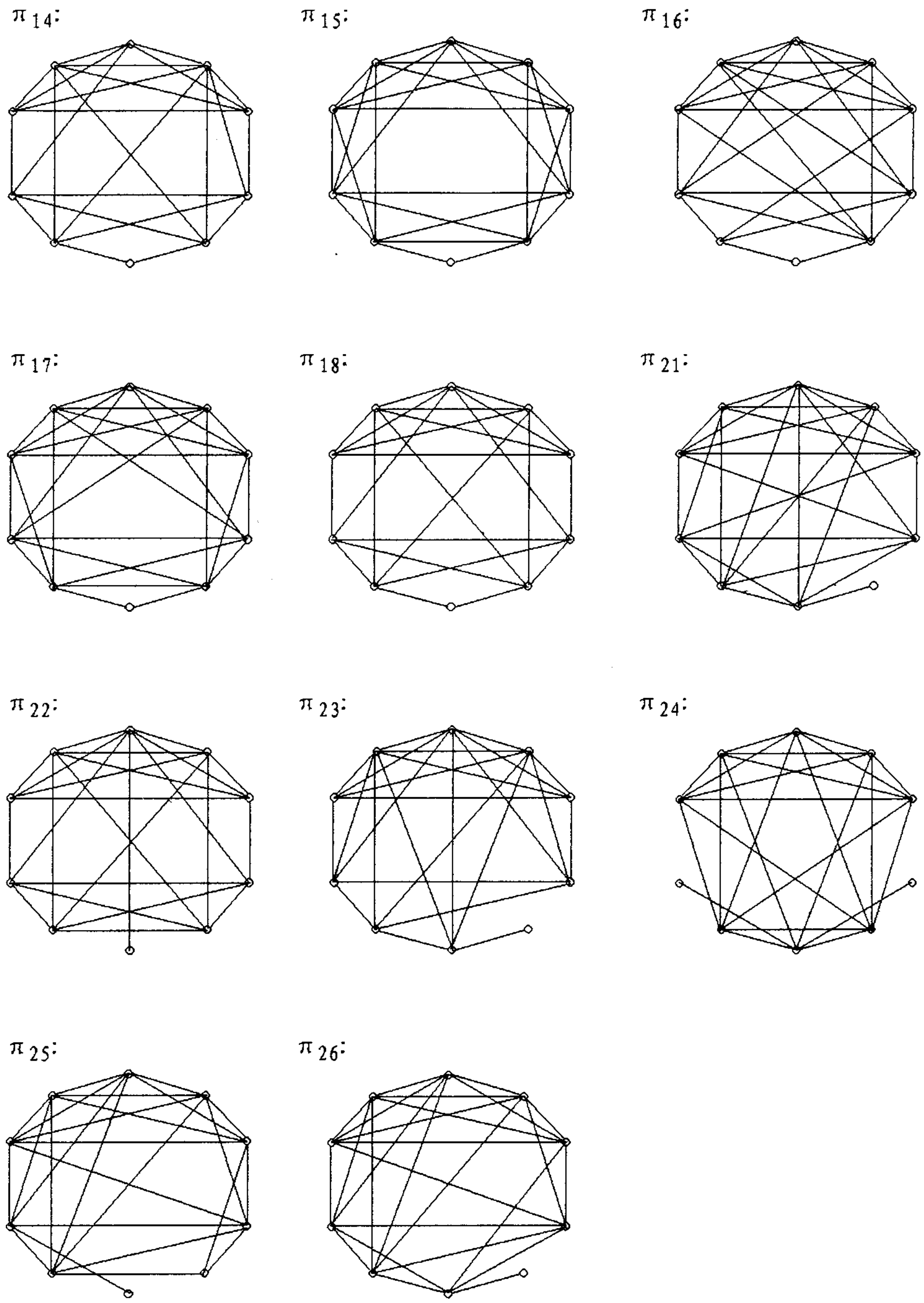

FIGURE 1. (continued)

References

[1] C. Berge, *Graphs and Hypergraphs*, North-Holland, Amsterdam, 1973.

[2] P. Erdös, M. S. Jacobson, and J. Lehel, Graphs realizing the same degree sequences and their respective clique numbers, in *Graph Theory, Combinatorics & Applications* (Y. Alaui, G. Chartrand, O.R. Oellermann and A.J. Schwenk eds.) John Wiley & Sons, Inc., New York (1991), 439–449.

[3] R. J. Gould, M. S. Jacobson, and J. Lehel, Potentially G-graphical sequences, a lecture presented at Kalamazoo Meeting in 1996.

[4] A. E. Kézdy and J. Lehel, Degree sequences of graphs with prescribed clique size, a lecture presented at Kalamazoo Meeting in 1996.

[5] D. J. Kleitman and D. L. Wang, Algorithm for constructing graphs and digraphs with given valences and factors, *Discrete Math.* **6** (1973), 79–88.

[6] Li Jiong-Sheng and Song Zi-Xia, An extremal problem on the potentially P_k-graphical sequences, *The International Symposium on Combiratorics and Applications*, June 28–30, 1996, Tainjin, W. Y. C. Chen, D. Z. Du, D. F. Hsu and H. P. Yap, Eds., Nankai University, 1996, pp. 269–276.

[7] A. R. Rao, The clique number of a graph with a given degree sequence, in *Proc. Symposium on Graph Theory*, A. R. Rao, Ed., MacMillan and Co. India Ltd., I. S. I. Lecture Notes Series, **4** (1979), 251–267.

[8] A. R. Rao, An Erdös–Gallai type result on the clique number of a realization of a degree sequence, unpublished.