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The extremal function for K_8^- minors

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Abstract

A graph H is a minor of a graph G if H can be obtained from a subgraph of G by contracting edges. Let K_8^- be the graph obtained from K_8 by deleting one edge. We prove a conjecture of Jakobsen that every simple graph on $n \geqslant 8$ vertices and at least $(11n - 35)/2$ edges either has a K_8^- minor, or is isomorphic to a graph obtained from disjoint copies of $K_{1,2,2,2,2}$ and/or K_7 by identifying cliques of size five.

Keywords: Graph; Minor; (H_1, H_2, k)-cockade

1. Introduction

All graphs considered in this paper are finite and simple. Let G be a graph and let x and y be adjacent vertices in G. We denote by G/xy the graph obtained from G by contracting the edge xy, i.e., by replacing x and y by one new vertex adjacent to every vertex that is adjacent to x or y in G. A graph is a *minor* of another if the first can be obtained from a subgraph of the second by contracting edges. We say that a graph G has an *H minor* (denoted by $G > H$) if G has a minor isomorphic to H.

One of the central problems of Graph Theory is the following conjecture due to Hadwiger [3].

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Conjecture 1.1. *For every integer $t \geqslant 1$, every graph with no K_{t+1} minor is t-colorable.*

Hadwiger's conjecture is trivially true for $t \leqslant 2$, and reasonably easy for $t = 3$, as shown by Dirac [2]. However, for $t \geqslant 4$, Hadwiger's conjecture implies the Four Color Theorem. (To see that, let H be a planar graph, and let G be obtained from H by adding $t-4$ vertices, each joined to every other vertex of the graph. Then G has no K_{t+1} minor, and hence is t-colorable by Hadwiger's conjecture, and hence H is 4-colorable). Wagner [14] proved that the case $t = 4$ of Hadwiger's conjecture is, in fact, equivalent to the Four Color Theorem, and the same was shown for $t = 5$ by Robertson et al. [10]. Hadwiger's conjecture remains open for $t \geqslant 6$. For $t = 6$, Kawarabayashi and Toft [8] proved that any 7-chromatic graph has either K_7 or $K_{4,4}$ as a minor. Jacobsen [4] proved that every 7-chromatic graph has a $K_7^{=}$ minor, where for integer $p>0$, K_p^- (resp. $K_p^{=}$) denotes the graph obtained from K_p by removing one edge (resp. two edges).

Mader [9] showed that for $p \leqslant 7$ every graph with $e(G) \geqslant (p-2)|G| - \binom{p-1}{2} + 1$ has a K_p minor. For $p = 6$, this result was instrumental in the proof of Hadwiger's conjecture for $t = 5$ mentioned above, and so it is reasonable to expect that further progress will be tied to a suitable generalization of Mader's result. Unfortunately, Mader's theorem does not extend for $p \geqslant 8$: $K_{2,2,2,2,2}$ is a counterexample for $p = 8$, and further counterexamples may be constructed by adding new vertices joined to all existing ones. On the other hand, Jørgensen [7] proved that every graph G with $e(G) \geqslant 6|G| - 20$ either has a K_8 minor or is a $(K_{2,2,2,2,2}, 5)$-cockade, where cockades are defined recursively as follows. Let H_1, H_2 be graphs and let k be an integer. Any graph isomorphic to H_1 or H_2 is an (H_1, H_2, k)-cockade. Now let G_1, G_2 be (H_1, H_2, k)-cockades and let G be obtained from the disjoint union of G_1 and G_2 by identifying a clique of size k in G_1 with a clique of the same size in G_2. Then the graph G is also an (H_1, H_2, k)-cockade, and every (H_1, H_2, k)-cockade can be constructed this way. In the case when $H_1 = H_2 = H$, it will be called an (H, k)-cockade. Thomas and the author [12] proved that every graph G with $e(G) \geqslant 7|G| - 27$ either has a K_9 minor or is a $(K_{1,2,2,2,2,2}, 6)$-cockade, or is isomorphic to $K_{2,2,2,3,3}$. More generally, Seymour and Thomas (see [12]) conjectured the following:

Conjecture 1.2. *For every $p \geqslant 1$ there exists a constant $N = N(p)$ such that every $(p-2)$-connected graph on $n \geqslant N$ vertices and at least $(p-2)n - \binom{p-1}{2} + 1$ edges has a K_p minor.*

In [1], Chen, Gould, Kawarabayashi, Pfender and Wei proved that every simple graph on n vertices and at least $9n - 46$ edges has a K_9^- minor, and used that to deduce that if, in addition, G is 6-connected, then it is 3-linked. The work of Chen, Gould, Kawarabayashi, Pfender and Wei suggested that there may be interest in the extremal problem for K_p^- minors.

Jakobsen [4,5] proved the following:

Theorem 1.3. *For $p = 5, 6, 7$, if G is a graph with $n \geqslant p$ vertices and at least $(p - \frac{5}{2})n - \frac{1}{2}(p-3)(p-1)$ edges, then $G > K_p^-$, or G is a $(K_{p-1}, p-3)$-cockade when $p \neq 7$, or $p = 7$ and G is a $(K_{2,2,2,2}, K_6, 4)$-cockade.*

In [5], Jakobsen also conjectured that Theorem 1.3 extends to $p = 8$ as follows:

Conjecture 1.4. *If G is a graph with $n \geqslant 8$ vertices and at least $\frac{11n-35}{2}$ edges, then $G > K_8^-$ or G is a $(K_{1,2,2,2,2}, K_7, 5)$-cockade.*

The purpose of this paper is to prove Conjecture 1.4, as follows.

Theorem 1.5. *If G is a graph with $n \geqslant 8$ vertices and at least $\frac{11n-35}{2}$ edges, then $G > K_8^-$ or G is a $(K_{1,2,2,2,2}, K_7, 5)$-cockade.*

Jakobsen [5] pointed out that the graph $K_{2,2,2,2,3}$ contains no K_9^- minor. In fact, there are many more small counterexamples to an analogue of Conjecture 1.4 for $p = 9$: $K_{1,1,2,2,2,2}$, $K_{1,2,2,3,3}$, $K_{3,3,3,3}$ and $K_{2,3,3,4}$. Thus an analogue of Conjecture 1.4 for $p = 9$ will have to include the conclusion that G is isomorphic to one of these graphs.

2. Preliminaries

We need to introduce more notation. For a graph G, we use $|G|$ and $e(G)$ to denote the order and size of G, respectively. The *complement* $\overline{G}$ of a graph G has the same vertex set as G, and distinct vertices u, v are adjacent in $\overline{G}$ just when they are not adjacent in G. The complement of a complete graph K_t will be denoted by $\overline{K_t}$. For any vertex v of a graph G, we use $N(v)$ or $N_G(v)$ to denote the subgraph of G spanned by the neighbors of v. The subgraph spanned by x and the neighbors of x is denoted by $N[v]$ or $N_G[v]$. For any subgraph H of G we denote by $N(H)$ the subgraph of G spanned by the vertices in $V(G)\backslash V(H)$ that are adjacent to a vertex in H.

For a graph G, $A, B \subset V(G)$ and two nonadjacent vertices $x, y \in V(G)$, we will use $e_G(A, B)$ to denote the number of edges between A and B in G and $G + xy$ to denote the graph obtained from G by adding an edge joining x to y. The *join* $G + H$ (resp. *union* $G \cup H$) of two vertex disjoint graphs G and H is the graph having vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy | x \in V(G), y \in V(H)\}$ (resp. $E(G) \cup E(H)$). The following results will be needed later. Theorem 2.1 is a result of Jørgensen [7], Theorem 2.2 was first proved by Jung [6]. For a complete characterization of the graphs with no pair of such paths, see [11,13].

Theorem 2.1. *Let G be a graph with $n \leqslant 11$ vertices and $\delta(G) \geqslant 6$. Then $G > K_6 \cup K_1$ or G is one of the graphs $K_{2,2,2,2}$, $K_{3,3,3}$ or the complement of the Petersen graph. In particular, $G > K_6^- \cup K_1$.*

Theorem 2.2. *Let G be a 4-connected graph and let x_1, x_2, y_1, y_2 be vertices in G. If G does not contain an $x_1 - y_1$ path and an $x_2 - y_2$ path that are disjoint, then G is planar and $e(G) \leqslant 3|G| - 7$.*

In the proof of Theorem 1.5, we shall consider graphs with n vertices and exactly $\lceil \frac{11n-35}{2} \rceil$ edges. Such graphs have vertices of degree at most 10. Since we want to consider contraction

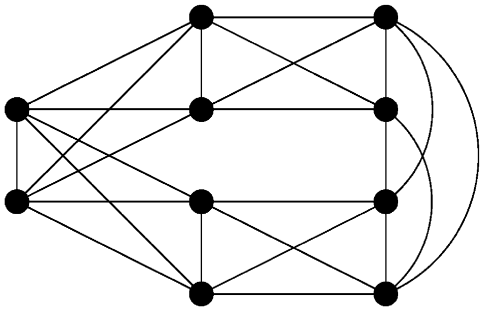

Fig. 1. Graph J.

in the graph spanned by the neighbors of a vertex of minimum degree, we need some results about contractions in graphs with at most 10 vertices.

Lemma 2.3. *Let G be a graph with 8 vertices and* $\delta(G)\geqslant 5$. *Then* $G>K_6^-\cup K_1$ *or G is isomorphic to* $\overline{C_8}$, $\overline{C_4}+\overline{C_4}$, $\overline{K_3}+C_5$, $\overline{K_2}+\overline{C_6}$, *or* $K_{2,3,3}$. *In particular, all these graphs are edge maximal subject to not having a* $K_6^-\cup K_1$ *minor. Moreover,* $\overline{C_8}>K_6$ *and* $\overline{C_4}+\overline{C_4}>K_6$.

Proof. It is not hard to verify that the graphs listed are edge maximal subject to not having a $K_6^-\cup K_1$ minor. Thus we may assume that every edge of G is incident with a vertex of degree five. Let $x\in V(G)$ be such that $d(x)=5$. If $e(G-x)\geqslant \frac{1}{2}(7|G-x|-15)=17$, by Theorem 1.3, $G-x>K_6^-$ or $G-x=K_3+(K_2\cup K_2)$. In the second case, x is adjacent to the four vertices of degree 4 in $K_3+(K_2\cup K_2)$. It is easy to check that $G>K_6^-\cup K_1$. Hence we may assume $e(G-x)\leqslant 16$, and so $20\leqslant e(G)\leqslant 21$. If $e(G)=20$, then G is 5-regular on 8 vertices. Thus $\overline{G}$ is 2-regular. It follows that $\overline{G}$ is isomorphic to C_8, $C_4\cup C_4$, or $C_3\cup C_5$, and so the lemma holds. If $e(G)=21$, then G has either one vertex of degree 7 and seven vertices of degree 5 or two vertices of degree 6 and six vertices of degree 5. In the first case, let y be the vertex of degree 7. Then $G-y$ is 4-regular on 7 vertices. Thus $\overline{G-y}=C_7$ or $C_3\cup C_4$. It is easy to check that $G-y>K_5^-\cup K_1$ and thus $G>K_6^-\cup K_1$. For the latter, let z, w be the two vertices of degree 6. Since G is edge minimal, we have $zw\notin E(G)$. It follows that $G-\{z,w\}$ is 3-regular on 6 vertices. Thus G is $\overline{K_2}+\overline{C_6}$ or $K_{2,3,3}$. The last assertion is easy to verify. □

Lemma 2.4. *Let G be a graph with* $9\leqslant n\leqslant 10$ *vertices and* $\delta(G)\geqslant 5$. *Then* $G>K_6^-\cup K_1$ *or G is isomorphic to J* (*given in* Fig. 1).

Proof. Lemma 2.4 can be checked by computers. However, a computer-free proof is given in the appendix. □

By Lemmas 2.3 and 2.4, it follows that

Corollary 2.5. *Let G be a graph with* $8 \leqslant |G| \leqslant 10$ *and* $\delta(G) \geqslant 5$. *Then* $G > K_6^- \cup K_1$ *or G is isomorphic to* $\overline{C_8}$, $\overline{C_4} + \overline{C_4}$, $\overline{K_3} + C_5$, $\overline{K_2} + \overline{C_6}$, $K_{2,3,3}$, *or J. In particular, all these graphs are edge maximal (subject to not having a* $K_6^- \cup K_1$ *minor) with maximum degree* $\leqslant |G| - 2$. *Moreover,* $\overline{C_8} > K_6$, $\overline{C_4} + \overline{C_4} > K_6$, *and* $J > K_6$.

Finally, we need some results about contractions in $(K_{1,2,2,2,2}, K_7, 5)$-cockades. Our proof of Conjecture 1.4 uses induction by deleting and contracting edges of G. We need to investigate graphs G such that the new graph $G - xy$ or G/xy is a $(K_{1,2,2,2,2}, K_7, 5)$-cockade, where $xy \in E(G)$. It turns out that contracting an edge of G in the proof of Conjecture 1.4 will not produce a $(K_{1,2,2,2,2}, K_7, 5)$-cockade. So we only consider the case when $G - xy$ is a $(K_{1,2,2,2,2}, K_7, 5)$-cockade. We do that next.

Lemma 2.6. *Let G be a* $(K_{1,2,2,2,2}, K_7, 5)$*-cockade and let x and y be nonadjacent vertices in G. Then* $G + xy$ *is contractible to* K_8^-.

Proof. This is obviously true if G is $K_{1,2,2,2,2}$. So we may assume that G is obtained from H_1 and H_2 by identifying on K_5, where both H_1 and H_2 are $(K_{1,2,2,2,2}, K_7, 5)$-cockades. If both $x, y \in V(H_i)$, then $H_i > K_8^-$ by induction. So we may assume that $x \in V(H_1) - V(H_2)$ and $y \in V(H_2) - V(H_1)$. If there exists $z \in V(H_1) \cap V(H_2)$ such that $yz \notin E(G)$, then by contracting $V(H_1) - V(H_1) \cap V(H_2)$ to z, the resulting graph will have a K_8^- minor by induction. So we may assume y is adjacent to all vertices in $V(H_1) \cap V(H_2)$. Similarly, we may assume that x is adjacent to all vertices in $V(H_1) \cap V(H_2)$. Hence there exists $w \in V(H_1)$ such that $H_1[\{w, x, V(H_1) \cap V(H_2)\}]$ is a K_7 subgraph in H_1. Clearly, $G[\{w, x, y, V(H_1) \cap V(H_2)\}] + xy > K_8^-$. □

It is easy to observe that

Lemma 2.7. *Let G be a* $(K_{1,2,2,2,2}, K_7, 5)$*-cockade. Then* $e(G) = \frac{11|G|-35}{2}$.

3. Proof of Theorem 1.5

In this section we prove Theorem 1.5 by induction on n. The only graphs G with 8 vertices and $e(G) \geqslant \frac{11\times 8-35}{2}$ are K_8^- and K_8. So we may assume that $n \geqslant 9$ and the assertion holds for smaller values of n.

Suppose G is a graph with n vertices and $e(G) \geqslant \frac{11n-35}{2}$ but G is not contractible to K_8^- and G is not a $(K_{1,2,2,2,2}, K_7, 5)$-cockade. By Lemma 2.6, we may assume that $e(G) = \lceil \frac{11n-35}{2} \rceil$.

If G has a vertex x with $d(x) \leqslant 5$, then $e(G-x) \geqslant \frac{11n-35}{2} - 5 > \frac{11|G-x|-35}{2}$. By the induction hypothesis and Lemma 2.7, $G - x > K_8^-$, a contradiction. Thus

(1) $\delta(G) \geqslant 6$.

(2) $\delta(N(x)) \geqslant 5$ for any $x \in V(G)$.

Proof. Suppose that there exists $y \in N(x)$ such that $d_{N(x)}(y) \leqslant 4$. Then $e(G/xy) \geqslant \frac{11(n-1)-34}{2} > \frac{11|G/xy|-35}{2}$. By the induction hypothesis and Lemma 2.7, $G - x > K_8^-$, a contradiction. □

Let S be a minimal separating set of vertices in G, and let G_1 and G_2 be proper subgraphs of G so that $G = G_1 \cup G_2$ and $G_1 \cap G_2 = G[S]$. For $i = 1, 2$, let d_i be the largest integer so that G_i contains disjoint set of vertices $V_1, V_2, \ldots, V_p$ so that $G_i[V_j]$ is connected and $|S \cap V_j| = 1$, $1 \leqslant j \leqslant p = |S|$, and so that the graph obtained from G_i by contracting $V_1, V_2, \ldots, V_p$ and deleting $V(G) - (\cup_j V_j)$ has $e(G[S]) + d_i$ edges. Let G_1' (resp. G_2') be obtained from G_1 (resp. G_2) by adding d_2 (resp. d_1) edges to $G[S]$. By (1), $|G_i| \geqslant 7$, $i = 1, 2$. Hence we may assume that $e(G_1) \leqslant \frac{11|G_1|-35}{2} - d_2$ (otherwise $e(G_1') > \frac{11|G_1'|-35}{2}$, in which case, $G_1' > K_8^-$ by induction). Similarly, we may assume that $e(G_2) \leqslant \frac{11|G_2|-35}{2} - d_1$. Consequently,

(3) $\frac{11n-35}{2} \leqslant e(G) = e(G_1) + e(G_2) - e(G[S]) \leqslant \frac{11n+11|S|-70}{2} - d_1 - d_2 - e(G[S])$, and so

(4) $11|S| \geqslant 35 + 2d_1 + 2d_2 + 2e(G[S])$.

(5) G is 5-connected.

Proof. It follows from (4) that $|S| \geqslant 4$. Note that $d_i \geqslant |S| - 1 - \delta(G[S])$, $i = 1, 2$, and $2e(G[S]) \geqslant |S|\delta(G[S])$. By (4), we have $7|S| \geqslant 31 + (|S| - 4)\delta(G[S])$, which implies that $|S| \geqslant 5$. □

(6) There is no minimal separating set S so that $G[S]$ is complete.

Proof. Suppose that $G[S]$ is complete. By (5), $|S| \geqslant 5$. If $|S| \geqslant 6$, by contracting $V(G_1) - S$ and $V(G_2) - S$ into two new vertices, we get $G > K_8^-$. So we may assume $|S| = 5$. Note that when $G[S] = K_5$, we get equality in (3). Thus $e(G_i) = \frac{11|G_i|-35}{2}$ for $i = 1, 2$ and $e(G) = \frac{11n-35}{2}$. It follows by induction that G is a $(K_{1,2,2,2,2}, K_7, 5)$-cockade, a contradiction. □

(7) There is no minimal separating set S with a vertex x so that $G[S - x]$ is complete.

Proof. Suppose that $G[S - x]$ is complete. By (5), $|S| \geqslant 5$. By (6), we may assume $\delta(G[S]) \leqslant |S| - 2$. Then $d_1 = d_2 = |S| - 1 - \delta(G[S])$ and $2e(G[S]) = (|S|-1)(|S|-2) + 2\delta(G[S])$. By (4), $11|S| \geqslant 35 + 4(|S| - 1 - \delta(G[S])) + (|S| - 1)(|S| - 2) + 2\delta(G[S]) = |S|^2 + |S| + 33 - 2\delta(G[S]) \geqslant |S|^2 + |S| + 33 - 2(|S| - 2)$. It follows that $|S|^2 - 12|S| + 37 \leqslant 0$, which is impossible. □

(8) $7 \leqslant \delta(G) \leqslant 10$.

Proof. Let $x \in V(G)$ be a vertex such that $d(x) = \delta(G)$. By (1), $d(x) \geqslant 6$. If $d(x) = 6$, by (2), $N(x) = K_6$. Now K_6 will be a minimal separating set, which contradicts (6). Thus $\delta(G) = d(x) \geqslant 7$. On the other hand, since $e(G) = \lceil \frac{11n-35}{2} \rceil$, we have $d(x) \leqslant 10$. □

(9) $\delta(G) \geqslant 8$.

Proof. Suppose that $d(x) \leqslant 7$. By (8), $d(x) = 7$. By (2), $\delta(N(x)) \geqslant 5$. Thus $N(x) = K_7 - M$, where M is a matching of $N(x)$. Let K be a component of $G - N[x]$. By (7), $N(K)$ contains two nonadjacent vertices, say a and b, in $N(x)$. Let P be an $a - b$ path with interior vertices in K. If $|M| \leqslant 2$, then by contracting all but one of the edges of the path P, $G > K_8^-$, a contradiction. So we may assume that $|M| = 3$, that is $N(x) = K_{1,2,2,2}$.

Let $V(N(x)) = \{y, z_1, z_2, z_3, w_1, w_2, w_3\}$ so that y is adjacent to all vertices in $N(x) - y$ and $z_i w_i \notin E(G)$. Suppose that $G - N[x]$ is disconnected. Let K and K' be two components of $G - N[x]$. Since $N(x) = K_{1,2,2,2}$, by (7), $N(K)$ and $N(K')$ contain two pairs of nonadjacent vertices of $N(x)$, respectively. We may assume that $z_1, w_1 \in N(K)$ and $z_2, w_2 \in N(K')$. Let P be a z_1-w_1 path in K and P' be a z_2-w_2 path in K'. Then by contracting all but one of the edges of P and P', respectively, we get a K_8^- minor of G, a contradiction. Hence

(9a) $G - N[x]$ is connected.

(9b) There is no vertex in $G - N[x]$ that is adjacent to a pair of nonadjacent vertices in $N(x)$.

Proof. Suppose that there exists $v \in V(G) - N[x]$ adjacent to, say z_1 and w_1. Let K be a component of $G - N[x] - v$. If $N(K)$ contains a pair of nonadjacent vertices of $\{z_2, z_3, w_2, w_3\}$, say, z_2 and w_2, then there is a z_2-w_2 path P in K. Now by contracting v to z_1 and all but one of the edges of the path P, we get a K_8^- minor of G, a contradiction. Thus by (7), we may assume $z_1, w_1 \in N(K)$. Let $K' = G - N[x] - K$. Clearly, K' is connected. If $N(K')$ contains a pair of nonadjacent vertices, other that z_1 and w_1 of $N(x)$, then G would have a K_8^- minor, a contradiction. Therefore, we may assume that $w_2, w_3 \in N(K) - N(K')$ and $z_2, z_3 \in N(K') - N(K)$. Since $w_2 z_3 \in E(G)$, w_2 and z_3 have at least one common neighbor in $G - N[x]$. It follows that $vw_2, vz_3 \in E(G)$ and thus $w_2 \in N(K')$, a contradiction. □

Let $v \in N(x)$ and $w \in V(G - N[x])$ be such that $v \neq y$ and $vw \in E(G)$. By (2) and (9b), v and w have at most three common neighbors in $N(x)$. Hence,

(9c) for any $v \in N(x) - y$, v has at least three neighbors in $G - N[x]$.

Suppose that w is a cut-vertex of $G - N[x]$. Let K be a component of $G - N[x] - w$ and let $K' = G - N[x] - K$. Then K' is connected. Since $N(x) = K_{1,2,2,2}$, by (7), $N(K)$ and $N(K')$ contain at least one pair of nonadjacent vertices of $N(x)$, respectively. If $N(K)$ and $N(K')$ contain distinct pairs of nonadjacent vertices of $N(x)$, then G would have a K_8^- minor by the existence of such two disjoint paths in K and K', respectively. So we may assume that $z_1, w_1 \in N(K) \cap N(K')$ and $N(K)$ and $N(K')$ contain no pair of nonadjacent vertices of $N(x)$ other than z_1, w_1. Thus we may assume that $z_2, z_3 \in N(K') - N(K)$ and $w_2, w_3 \in N(K) - N(K')$. Since $w_2 z_3 \in E(G)$, w_2 and z_3 have at least one common neighbor in $G - N[x]$. It follows that $ww_2, wz_3 \in E(G)$, and thus $w_2 \in N(K')$, a contradiction.Therefore

(9d) $G - N[x]$ is 2-connected.

Consider the graph $H = G - \{x, y, z_3, w_3\}$. We next show that H is 4-connected.

Let S be a minimal separating set of at most three vertices in H. By (9c) and (9d), $|S| \geqslant 2$ and $|S \cap N(x)| \leqslant 1$. If $|S \cap N(x)| = 1$, we may assume that $w_1 \in S$. Since $z_1z_2, z_1w_2 \in E(G)$, z_1, z_2, w_2 are in the same component of $H - S$. Denote this component by K. If $w_1 \notin S$, then also $w_1 \in K$, and in this case we assume that S and w_1 are chosen so that $|S \cap N(w_1)|$ is maximal. We next show that there exist z_2' and w_2' in $G - N[x] - S$ adjacent to z_2 and w_2, respectively. By (9b) and (9c), we may assume that w_2 has exactly three neighbors in $G - N[x]$, say a, b, c, and $S = \{a, b, c\}$. Clearly, $w_1 \notin S$. By the assumption that $|S \cap N(w_1)|$ is maximal, it follows that w_1 is adjacent to all vertices in S. Since $w_2z_1 \in E(G)$, by (2), z_1 and w_2 have at least one common neighbor in $G - N[x]$. Since w_2 has only three neighbors a, b, c in $G - N[x]$, we may assume $z_1a \in E(G)$. Now a is adjacent to both z_1 and w_1, which contradicts (9b). This proves that there exist $z_2', w_2' \in (V(G) - N[x] - S)$ such that $z_2z_2', w_2w_2' \in E(G)$.

Clearly, $z_2', w_2' \in K$. By (9d), $G - N[x]$ contains two independent z_2'-w_2' paths. One of these paths is contained in $G[K \cup S]$.

Since G is not contractible to $N[x] + z_2w_2 + z_3w_3$, there is no z_3-w_3 path in $G[K' \cup \{z_3, w_3\}]$, where $K' \neq K$ is another component of $H - S$. But this implies that K' is separated from x by S and two adjacent vertices in $N(x)$. We may assume that such two vertices are $\{y, w_3\}$. Since G is 5-connected, $|S| = 3$. Let $S = \{s_1, s_2, s_3\}$, where $s_1 = w_1$ if $w_1 \in S$, and $S' = S \cup \{y, w_3\}$. Then S' is a minimal separating set of G. Let $H_1 = G[K' \cup S'\}$ and $H_2 = G - K'$. Let d_1 and d_2 be defined as in the paragraph following (2). Clearly, $K \cup \{x, z_3\}$ is contained in H_2. By Menger's theorem, there exist three disjoint paths between $\{x, w_1, z_2\}$ and S in $G - \{y, w_3\}$. By contracting those paths, we get $d_2 + e_G(S') = e(K_5) = 10$. By (2), $d_1 \geqslant 1$. By (4), $55 = 11 \times 5 \geqslant 35 + 2(d_2 + e(S')) + 2d_1 = 35 + 20 + 2 = 57$, a contradiction. Thus H is 4-connected.

Since G is not contractible to K_8^-, it follows from Theorem 2.2 applied to the vertices z_1, z_2, w_1, w_2 that $e(H) \leqslant 3|H| - 7 = 3(n - 4) - 7$. Since the vertices z_3 and w_4 have no common neighbor in $G - N[x]$, they together have at most $|G| - |N[x]| = n - 8$ neighbors in $G - N[x]$. The vertices $\{z_3, w_3\}$ are incident with 8 edges of $N[x]$. Thus

$$\begin{aligned}\frac{11n - 35}{2} &\leqslant e(G) \leqslant d(x) + d(y) - 1 + e(H) + (n - 8) + 8\\ &\leqslant 7 + n - 2 + 3(n - 4) - 7 + (n - 8) + 8 = 5n - 14.\end{aligned}$$

It follows that $n \leqslant 7$, which contradicts the fact that $n \geqslant \delta(G) + 1 \geqslant 8$ by (8). □

(10) Let x be a vertex such that $8 \leqslant d(x) \leqslant 10$. Then there is no component K of $G - N[x]$ such that $N(K) = N(x)$.

Proof. Suppose such a component K exists. By (2), $\delta(N(x)) \geqslant 5$. By Corollary 2.5, $N(x) > K_6^- \cup K_1$ or $N(x) > K_6$ or $N(x) \in \{\overline{K_3} + C_5, K_{2,3,3}, \overline{K_2} + \overline{C_6}\}$. In the first case, there is a vertex $y \in N(x)$ such that $N(x) - y > K_6^-$. By contracting $V(K) \cup \{y\}$ to a single vertex we see that $G > K_8^-$, a contradiction. We will use this argument repeatedly later, and we shall refer to it as "contracting K onto a free vertex of $N(x)$". If $N(x) > K_6$, then we obtain the same conclusion by contracting K to a vertex. So we may assume that $N(x) \in \{\overline{K_3} + C_5, K_{2,3,3}, \overline{K_2} + \overline{C_6}\}$. We claim that $G - N[x]$ is connected. Suppose

$G - N[x]$ is disconnected. Let $K' \neq K$ be another component of $G - N[x]$. By (6), $N(K')$ is not complete. Let $a, b \in N(K')$ be such that $ab \notin E(G)$. Let P be an a-b path in K'. By Corollary 2.5, $N(x)$ is edge maximal, and so $N[x] \cup P > K_7^- \cup K_1$. By contracting K to a free vertex of $N(x) \cup P$, we get $G > K_8^-$, a contradiction. Thus $G - N[x]$ is connected, as claimed. We consider the following two cases.

Case 1: $G - N[x]$ is 2-connected.

Suppose $N(x) \in \{\overline{K_2} + \overline{C_6}, K_{2,3,3}\}$. By (2), there exist $x_1, x_2, y_1, y_2 \in N(x)$ such that $x_1x_2, y_1y_2 \in E(G)$, x_1 and x_2 (resp. y_1 and y_2) have at least two common neighbors in $G - N[x]$, and $x_1y_1, x_2y_2 \notin E(G)$ but $N[x] + x_1y_1 + x_2y_2 > K_8^-$. Let $u_1, u_2 \in V(K)$ be two distinct common neighbors of x_1 and x_2, and $w_1, w_2 \in V(K)$ be two distinct common neighbors of y_1 and y_2, respectively. By Menger's Theorem, K contains two disjoint paths from $\{u_1, u_2\}$ to $\{w_1, w_2\}$. Thus G has two disjoint paths with interiors in K, one with ends x_1, y_1, and the other with end x_2, y_2. Then $G > K_8^-$ by the existence of those two paths, a contradiction.

Suppose $N(x) = \overline{K_3} + C_5$. Let $V(\overline{K_3}) = \{a_1, a_2, a_3\}$ and let $\overline{C_5}$ have vertices y_1, y_2, y_3, y_4, y_5 in order. Let $w \in V(G - N[x])$. Then $G - N[x] - w$ is connected and each vertex of $N(x)$ is adjacent to at least one vertex of $G - N[x] - w$. If w is adjacent to two vertices of a_1, a_2, a_3, say a_1, a_2, then $G > N[x] + a_1a_2 + y_1y_2 + y_2y_3 > K_8^-$ by contracting wa_1 and $V(G - N[x] - w)$ onto y_2, respectively. Similarly, if w is adjacent to two nonadjacent vertices of $y_1, y_2, \ldots, y_5$, say y_1, y_2, then $G > N[x] + y_1y_2 + y_2y_3 + y_3y_4 > K_8^-$ by contracting wy_1 and $V(G - N[x] - w)$ onto y_3, respectively. So we may assume that any pair of nonadjacent vertices of $N(x)$ have no common neighbor in $G - N[x]$. By (2), there exist $w_1, w_2, w_3, w_4 \in V(G - N[x])$ such that w_i is a common neighbor of y_1 and a_i, $i = 1, 2, 3$, and w_4 a common neighbor of y_2 and y_5. Since any pair of nonadjacent vertices of $N(x)$ have no common neighbor in $G - N[x]$, we have $w_i \neq w_j$ for $i \neq j$. As $G - N[x]$ is 2-connected, there exist two disjoint paths, say P_1, P_2, between $\{w_1, w_4\}$ and $\{w_2, w_3\}$ in $G - N[x]$. We may assume that P_1 is a w_1-w_3 path. Now $G > N[x] + a_1a_3 + y_1y_2 + y_1y_5 > K_8^-$ by contracting a_1w_1, y_1w_2 and all but one of the edges of each of P_1, P_2, a contradiction.

Case 2: $G - N[x]$ is not 2-connected.

In this case, $G - N[x]$ is connected. Let w be a cut-vertex of $G - N[x]$ and let H_1 be a connected component of $G - N[x] - w$ with $N(H_1)$ minimal, and let $H_2 = G - N[x] - H_1$. Clearly, H_2 is also connected. If $N(H_1) \subseteq N(H_2)$ or $N(H_2) \subseteq N(H_1)$, say the latter. Then $N(H_1) = N(K) = N(x)$. By (6), there exists $e = ab \in E(\overline{N(H_2)})$. By Corollary 2.5, there exists $u \in N(x)$ such that $N(x) + e - u > K_6^-$. Then $G > K_8^-$ by contracting the a-b path in H_2 and contracting $V(H_1)$ to u. So we may assume that there exist $a \in N(H_1) - N(H_2)$ and $b \in N(H_2) - N(H_1)$. By (2), any two adjacent vertices in $N(x)$ have at least one common neighbor in $G - N[x]$. Thus $ab \notin E(G)$, $N_{N(x)}(a) \subseteq N(H_1)$ and $N_{N(x)}(b) \subseteq N(H_2)$. Suppose $N(x) \in \{\overline{K_2} + \overline{C_6}, K_{2,3,3}\}$. Since $ab \notin E(G)$, there exist $x_1, y_1 \in N_{N(x)}(a)$ and $x_2, y_2 \in N_{N(x)}(b)$ such that $x_1y_1, x_2y_2 \notin E(G)$ but $N[x] + x_1y_1 + x_2y_2 > K_8^-$. Then $G > K_8^-$ by the existence of x_i-y_i path in H_i, $i = 1, 2$, a contradiction. Suppose $N(x) = \overline{K_3} + C_5$. Let $V(\overline{K_3}) = \{a_1, a_2, a_3\}$ and let $\overline{C_5}$ have vertices y_1, y_2, y_3, y_4, y_5 in order. If $a, b \in \{a_1, a_2, a_3\}$, then $y_i \in (N_{N(x)}(a) \cap N_{N(x)}(b))$ for all $i = 1, 2, \ldots, 5$. Thus $G > K_8^-$ by contracting $V(H_1)$ to y_1 and $V(H_2)$ to y_2, respectively. So we may assume that $a, b \in \{y_1, \ldots, y_5\}$, say $a = y_1$ and $b = y_2$. Clearly, $a_1, a_2, a_3, y_3, y_4 \in N(H_1)$ and $a_1, a_2, a_3, y_4, y_5 \in N(H_2)$. By (2), y_3 and y_5 have at least one common neighbor, say y, in $G - N[x]$. We may assume that $y \in V(H_1)$. Then $y_5 \in N(H_1)$ and so $G > K_8^-$ by contracting $V(H_1)$ to y_4 and $V(H_2)$ to a_1, respectively, a contradiction. $\square$

(11) Let x be a vertex such that $8 \leqslant d(x) \leqslant 10$. Then there is no component K of $G - N[x]$ such that $N(K') \subseteq N(K)$ for every component K' of $G - N[x]$.

Proof. Suppose such a component K exists. Among all vertices x with $8 \leqslant d(x) \leqslant 10$ for which such a component exists, choose x to be of minimal degree. By (10), $N(K) \neq N(x)$. Let $y \in N(x) - N(K)$ be of smallest degree. Then $N(y) \subseteq N[x]$. Note that $d(y) \leqslant d(x) \leqslant d(y) + 2$. Suppose $d(x) = d(y)$. Then each vertex of $N(x)$ is either adjacent to all vertices in $N[x]$ or contained in $N(K)$, and $d_{N(x)}(y) = |N(x)| - 1$. By Corollary 2.5, $N(x) > K_6^- \cup K_1$. By contracting $N(K)$ to a free vertex of $N(x)$, we obtain $G > K_8^-$, a contradiction. Next, suppose $d(x) = d(y) + 1$. Let $\{z\} = N(x) - N[y]$. Then $z \notin N(K)$, for otherwise we would have chosen y for x. By the choice of y, $d(z) = d(x) - 1$. Thus $\{z\}$ is a component of $G - N[y]$ such that $N(\{z\}) = N(y)$, which contradicts (10). Finally, suppose $d(x) = d(y) + 2$. Then $d(x) = 10$. Let $\{z, w\} = N(x) - N[y]$. Clearly, z and w are not both in $N(K)$, otherwise we would have chosen y for x. So we may assume that $z \notin N(K)$. If $zw \notin E(G)$, then $\{z\}$ is a component of $G - N[y]$ such that z is adjacent to all the vertices in $N(y)$, which contradicts (10). So we may assume $zw \in E(G)$, and thus $w \notin N(K)$ (otherwise we would have chosen y for x, because $K \cup \{z, w\}$ is a component in $G - N[y]$ satisfying (11)). By the choice of y, $d(z), d(w) \geqslant d(y)$. Now $e(N(x)) \geqslant (d(y)-1)+(d(z)-2)+(d(w)-2)+1+\frac{4|N(x)\cap N(y)|}{2} \geqslant 3d(y)-4+2(d(y)-1) = 5d(y) - 6 = 5(d(x) - 2) - 6 = 5d(x) - 16 > \frac{9|N(x)|}{2} - 12$. By Theorem 1.3, $N(x) > K_7^-$ and so $G > N[x] > K_8^-$, a contradiction. $\square$

It follows from (11) that

(12) $G - N[x]$ is disconnected.

(13) Let x be a vertex such that $8 \leqslant d(x) \leqslant 10$. Then there is no component K of $G - N[x]$ with one vertex w so that $d_G(y) \geqslant 11$ for every vertex $y \neq w$ in K and $d_G(w) \geqslant d_G(x)$.

Proof. Assume that such a component K exists. Let $G_1 = G - K$ and $G_2 = G[K \cup N(K)]$. Let d_1 be defined as in the paragraph following (2). Let G_2' be a graph with $V(G_2') = V(G_2)$ and $e(G_2') = e(G_2) + d_1$ edges obtained by contracting edges in G_1. By (9), $|G_2'| \geqslant 9$. If $e(G_2') > \frac{11|G_2'|-35}{2}$, then $G > G_2' > K_8^-$ by induction, a contradiction. Thus $e(G_2) = e(G_2') - d_1 \leqslant \frac{11|G_2|-35}{2} - d_1 = \frac{11|N(K)|+11|K|-35}{2} - d_1$. On the other hand, for any $u \in N(K)$, there exists $w \in K$ such that $uw \in E(G)$. By (2), $d_{G_2}(u) \geqslant 6$. Thus $e(G_2) \geqslant \frac{1}{2}(6 \times |N(K)| + 11(|K| - 1) + d_G(w)) \geqslant \frac{6|N(K)|+11|K|-11+d(x)}{2}$. It follows that

(13a) $5|N(K)| \geqslant 24 + d(x) + 2d_1$ and so $|N(K)| \geqslant 7$ by (9).

Let $t = e_G(N(K), K)$ and $d = \delta(N(K))$. Then $e(G_2) = e(G[K]) + t + e(N(K)) \geqslant \frac{11(|K|-1)+d_G(w)-t}{2} + t + \frac{|N(K)|\times d}{2} \geqslant \frac{11|K|-11+d(x)+t+|N(K)|\times d}{2}$. It follows that

(13b) $\frac{-t+d(x)}{2} \geqslant d_1 + d(x) + 12 + \frac{d|N(K)|-11|N(K)|}{2} \geqslant (|N(K)| - 1 - d) + d(x) + 12 + \frac{d|N(K)|-11|N(K)|}{2} = 11 + d(x) + \frac{d(|N(K)|-2)}{2} - \frac{9|N(K)|}{2}$.

Note that $t \geqslant \sum_{v\in K} d_G(v) - 2e(G[K]) \geqslant 11(|K|-1) + d(w) - |K|(|K|-1) \geqslant -|K|^2 + 12|K| + d(x) - 11$. If $t \leqslant d(x) + s$, then

(13c) $|K|^2 - 12|K| + 11 + s \geqslant 0$.

By (10), $N(K) \neq N(x)$. This, together with (13a), implies that $7 \leqslant |N(K)| \leqslant 9$. Thus $|K| \geqslant (\Delta(G_2) + 1) - |N(K)| \geqslant (11 + 1) - 9 = 3$. We next show that $t \leqslant d(x) + s$, where $s = 14$.

By (2), $d \geqslant 5 - (|N(x)| - |N(K)|)$. If $|N(K)| = 7$, by (6) and (13a), we have $d_1 \geqslant 1$ and $d(x) + 2d_1 \leqslant 11$. Thus $d(x) \leqslant 9$ and $d \geqslant 5 - (9 - 7) = 3$. By (13b), $\frac{-t+d(x)}{2} \geqslant 1 + d(x) + 12 + \frac{3|N(K)|-11|N(K)|}{2} \geqslant -7$. If $|N(K)| = 8$, then $d(x) \leqslant 10$ and $d \geqslant 5 - (10 - 8) = 3$. By (13b), $\frac{-t+d(x)}{2} \geqslant 11 + d(x) + \frac{d(|N(K)|-2)}{2} - \frac{9|N(K)|}{2} \geqslant 11 + d(x) + \frac{3\times(8-2)}{2} - \frac{9\times 8}{2} \geqslant -7$. If $|N(K)| = 9$, then $d(x) = 10$ and $d \geqslant 5 - (10 - 9) = 4$. By (13b), $\frac{-t+d(x)}{2} \geqslant 11 + d(x) + \frac{d(|N(K)|-2)}{2} - \frac{9|N(K)|}{2} \geqslant 11 + d(x) + \frac{4\times(9-2)}{2} - \frac{9\times 9}{2} > -7$. In all cases, we have $t \leqslant d(x) + 14$ and $s = 14$.

Since $s = 14$ and $|K| \geqslant 3$, by 13(c), $|K| > 8$. Note that $e(G[K]) \geqslant \frac{11(|K|-1)+d(w)-t}{2} \geqslant \frac{11(|K|-1)}{2} + \frac{-t+d(x)}{2} \geqslant \frac{11|K|-25}{2}$. It follows that $G[K] > K_8^-$ by induction, a contradiction. □

By (9), G has a vertex of degree 8, 9 or 10. Among the vertices of degree 8, 9 or 10 for which the order of the largest component of $G - N[x]$ is maximum, choose x so that its degree is minimum. Let K be a largest component of $G - N[x]$.

By (12), there is another component K' of $G - N[x]$. By (13), there is a vertex x' in K' of degree $d_G(x') \leqslant 10$. By the maximality of the order of K, $N(K) \subseteq N(x') \cap N(x)$. Thus $N(K) \subseteq N(K')$ and K is also a component of $G - N[x']$. By the choice of x, $d(x') \geqslant d(x)$. By (13), there exists another vertex $y' \neq x'$ in K' of degree $d(x) \leqslant d(y') \leqslant 10$. Clearly, y' is adjacent to every vertex in $N(K)$. By (11), there is a third component K'' of $G - N[x]$. By symmetry, K'' has two vertices x'', y'' of degree at most 10 in G and $N(K) \subseteq N(x'') \cap N(y'')$. Let $G_1 = G - K$, $G_2 = G[N(K) \cup K]$ and let d_1 and d_2 be as in the paragraph following (2).

Since $\delta(N(x)) \geqslant 5$, $\delta(N(K)) \geqslant 5 - (10 - |N(K)|) = |N(K)| - 5$. Therefore there is a subgraph T of $N(K)$ with $|N(K)| - 5$ vertices and at least $|N(K)| - 6$ edges. Contract the vertices in $N(K) - T$ with different vertices in $\{x, x', y', x'', y''\}$, which are adjacent to every vertex in $N(K)$. It is easy to see that

$$d_1 + e(N(K)) \geqslant e(K_5) + 5(|N(K)| - 5) + (|N(K)| - 6) = 6|N(K)| - 21. \qquad (*)$$

By (4), $d_1 + e(N(K)) \leqslant \frac{11|N(K)|-35-2d_2}{2}$. It follows that $d_2 = 1$ and $|N(K)| = 5$. However, when $|N(K)| = 5$, by (*), $d_1 + e(N(K)) \geqslant e(K_5) + 5(|N(K)| - 5) = 5|N(K)| - 15 = 10$. By (4) again, $55 = 11|N(K)| \geqslant 35 + 2(d_1 + e(N(K)) + 2d_2 \geqslant 35 + 20 + 2 = 57$, which is impossible. This completes the proof of Theorem 1.5. □

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Appendix. Proof of Lemma 2.4

Here we give a computer-free proof of Lemma 2.4. We first prove two lemmas.

Lemma A.1. *Let G be a graph on 8 vertices. Let $u, w \in V(G)$ be such that $d(u) \geqslant 4$, $d(w) = 7$, and $d(v) \geqslant 5$ for every $v \neq u, w$. Then $G > K_6^- \cup K_1$.*

Proof. Suppose $d(u) \geqslant 5$. Then $\delta(G) \geqslant 5$ and $\Delta(G) = 7$. By Lemma 2.3, $G > K_6^- \cup K_1$. So we may assume that $d(u) = 4$. Then $e(G) \geqslant \lceil \frac{4+7+5\times 6}{2} \rceil = 21$. Note that $e(G - u) = e(G) - 4 \geqslant 17$ and $G - u$ has at most three vertices of degree 4. By Theorem 1.3, we have $G - u > K_6^-$. □

Lemma A.2. *Let G be a graph on 9 vertices. Let $uw \in E(G)$ be such that $d(u) = 4$, $d(w) \geqslant 7$ and $d(v) \geqslant 5$ for every $v \neq u, w$. Then $G > K_6^- \cup K_1$.*

Proof. Suppose G is not contractible to $K_6^- \cup K_1$. We may assume that G is edge minimal. We claim that $d(w) = 7$. Suppose $d(w) = 8$. Since the number of odd vertices of any graph is even, there exists another vertex, say $v \in V(G)$, such that $d(v) \geqslant 6$. Clearly, $vw \in E(G)$ and $d_{G-vw}(w) \geqslant 7$, $d_{G-vw}(u) = 4$, $d_{G-vw}(v) \geqslant 5$ for any $v \neq u, w$, which contradicts the fact that G is edge minimal. Hence $d(w) = 7$, as claimed.

We first show that G is 4-connected. Let S be a minimal separating set of G with $|S| \leqslant 3$. Since $|G| = 9$ and $d(v) \geqslant 5$ for any $v \neq u, w$, we have $|S| = 3$. Let H_1 and H_2 be the two connected components of $G - S$. Then $|H_1| = |H_2| = 3$. We may assume that $H_1 = K_3$ and each vertex of H_1 is adjacent to all vertices of S. Note that there exists a vertex, say $a \in V(H_2)$, adjacent to all vertices in S. Let $b \in S$. Now $G/ab - V(H_2 - a) > K_6^-$. This proves that G is 4-connected.

Since $uw \in E(G)$, let $V(N(u)) = \{w, a, b, c\}$ and $A = V(G) - V(N[u]) = \{d, e, f, g\}$. We next prove the following claim.

Claim. *For any $v \in \{a, b, c\}$, if $vw \in E(G)$, then $d_{N(u)}(v) \geqslant 2$.*

Proof. Suppose otherwise. We may assume that $aw \in E(G)$ and $ab, ac \notin E(G)$. Let w' be the new vertex in G/ua. Then $d_{G/ua}(w) = 6$, $d_{G/ua}(w') \geqslant 6$ and $ww' \in E(G)$. Note that $\delta(G/ua - ww') \geqslant 5$. By Lemma A.2, $G/ua > K_6^- \cup K_1$. □

Suppose that w is adjacent to all vertices of A. Since $d_G(w)=7$, we may assume that $cw \notin E(G)$. If $ca \notin E(G)$ or $d_G(a) \geqslant 6$, then $\Delta(G/uc) = 7$, $d_{G/uc}(b) \geqslant 4$ and $d_{G/uc}(v) \geqslant 5$ for any $v \in V(G/uc - b)$. By Lemma A.1, $G/uc > K_6^- \cup K_1$. Hence $ca \in E(G)$ and $d_G(a) = 5$. Similarly, $cb \in E(G)$ and $d_G(b) = 5$. Note that $e_G(v, \{a, b, c\}) \geqslant 1$ for any $v \in A$. If

$G[A] = K_4$ or K_4^-, then $G/ac/bc - u > K_6^-$. So we may assume that $e(G[A]) \leqslant 4$. Thus $e_G(A, N(u)) \geqslant 20 - 2e(G[A]) \geqslant 12$ and $e_G(\{a, b, c\}, A) = e_G(A, N(u)) - e_G(w, A) \geqslant 12 - 4 = 8$. Note that $d_G(a) = d_G(b) = 5$. It follows that $ab \notin E(G)$ and c is adjacent to all vertices of A. Hence $d_G(c) = 7$, $d_G(v) = 5$ for any $v \in A$, and $e(G[A]) = 4$. So $G[A] = C_4$ or $K_1 + (K_2 \cup K_1)$. In the first case, we may assume that $G[A]$ has vertices d, e, f, g in order and $ad \in E(G)$. Then by symmetry, either $af \in E(G)$ or $ae \in E(G)$. If $af \in E(G)$, then $be, bg \in E(G)$ and so $G/ad/be - u = K_6^-$. If $ae \in E(G)$, then $bf, bg \in E(G)$ and so $G/uw/de - a = K_6^-$. In the second case, we may assume that $ed, ef, eg, fg \in E(G)$. Then d is adjacent to all vertices of $N(u)$. Note that either $af, bg \in E(G)$ or $ag, bf \in E(G)$. In either case, $G/da/db - u > K_6^-$. This proves the case when w is adjacent to all vertices of A.

Suppose w is adjacent to all vertices of $N(u)$. Then $d_G(w, A) = 3$. By Claim, $\delta(G[\{a, b, c\}]) \geqslant 1$. We may assume that $ab, bc \in E(G)$. Note that $e_G(v, \{a, b, c\}) \geqslant 1$ for any $v \in A$. If $G[A] = K_4$, then $G/ab/bc - u > K_6^-$. So we may assume that $e(G[A]) \leqslant 5$. It follows that $e_G(A, N(u)) \geqslant 20 - 2e(G[A]) \geqslant 10$ and so $e_G(\{a, b, c\}, A) = e_G(A, N(u)) - e_G(w, A) \geqslant 10 - 3 = 7$. Thus $ca \notin E(G)$ (otherwise, since G is edge minimal, at most one of a, b, c could be of degree >5, and so $e(\{a, b, c\}, A) \leqslant 4 + 1 + 1 = 6$, a contradiction). If a is adjacent to all vertices of A, then $\Delta(G/uc) = 7$, $d_{G/uc}(b) = 4$ and $d_{G/uc}(v) \geqslant 5$ for any $v \in V(G/uc - b)$. By Lemma A.1, $G/uc > K_6^- \cup K_1$. Hence a, similarly c, is adjacent to at most three vertices of A. Thus $e_G(N(u), A) \leqslant 3 + 3 + 1 + 3 = 10 \leqslant e_G(A, N(u))$. It follows that $G[A] = K_4^-$, a (resp. c) is adjacent to exactly three vertices of A and b is adjacent to exactly one vertex of A, all vertices of A are of degree five. Since $G[A] = K_4^-$, we may assume that $de \notin E(G)$. Note that $e_G(b, A) = 1$, we may assume that $be \notin E(G)$. Then $ew, ea, ec \in E(G)$. Observe that $e_G(d, N(u)) = 3$ and if $v \in N(u)$ is not adjacent to d, then $vf \in E(G)$ or $vg \in E(G)$, say the later. Clearly, $G/ae/dg - f > K_6^-$. $\square$

Proof of Lemma 2.4. We may assume that G is minor minimal subject to $\delta(G) \geqslant 5$ and $|G| \geqslant 9$. If $\delta(G) \geqslant 6$, by Theorem 2.1, $G > K_6^- \cup K_1$. So we may assume that $\delta(G) = 5$. We first prove two claims.

Claim 1. *Every edge of G is in at least two triangles.*

Proof. Suppose $e = uv \in E(G)$ is in at most one triangle in G. Let w be the new vertex in G/e. Then $d_G(w) \geqslant 7$, and $d_G(y) \geqslant 4$, where y is the common neighbor of u and v in G. Clearly, $wy \in E(G/e)$ and $d_{G/e}(v) \geqslant 5$ for any $v \neq w, y$. Since G is minor minimal, by Lemmas A.1 and A.2, $G > G/e > K_6^- \cup K_1$. $\square$

Claim 2. *There is no edge of G with both ends of degree at least six in G.*

Proof. Suppose $e = uv \in E(G)$ is such that $d(u), d(v) \geqslant 6$. Then $\delta(G - e) \geqslant 5$ and $|G| \geqslant 9$, which contradicts the fact that G is minor minimal. $\square$

We next show that G is 4-connected. Let S be a minimal separating set of G with $|S| \leqslant 3$. Let H_1 be a component of $G - S$ with minimal order and $H_2 = G - S - H_1$. If $|S| \leqslant 2$, then, since $\delta(G) \geqslant 5$, $|H_1|, |H_2| \geqslant 4$, and hence $|S| = 2$, H_1 and H_2 are isomorphic to K_4, because $|G| \leqslant 10$. But then, clearly, $G > K_6^- \cup K_1$. Suppose $|S| = 3$. Then $H_1 = K_3$ and $3 \leqslant |H_2| \leqslant 4$. Note that every vertex of H_1 is adjacent to every vertex of S. If there is a vertex $b \in V(H_2)$ such that b is adjacent to all vertices in S, then $G/ab - V(H_2 - b) > K_6^-$, where $a \in S$. Otherwise $H_2 = K_4$. By the minimality of $|S|$, G has a matching from S into H_2. By contracting this matching, it follows that $G > K_6^- \cup K_1$. This shows that G is 4-connected.

Since $\delta(G) = 5$, let $x \in V(G)$ be such that $d(x) = 5$. We may assume that $V(N(x)) = \{a, b, c, d, e\}$ and $A = V(G) - V(N[x]) = \{y_1, y_2, \ldots, y_{|G|-6}\}$.

Claim 3. *$N(x)$ contains no subgraph isomorphic to $K_{2,3}$.*

Proof. Suppose that $N(x)$ has a subgraph H isomorphic to $K_{2,3}$. We may assume that $d_H(a) = d_H(e) = 3$ and $d_H(b) = d_H(c) = d_H(d) = 2$. Suppose that there exists a vertex of A, say y_1, such that $y_1b, y_1c, y_1d \in E(G)$. If $G[\{b, c, d\}] \neq \overline{K_3}$, say $bc \in E(G)$, then $G/y_1d - y_2 > K_6^-$. So we may assume that $G[\{b, c, d\}] = \overline{K_3}$. If two of b, c, d, say b, c, have a common neighbor, say y_2, of $A - y_1$ in G, then $G/by_2/dy_1 - y_3 > K_6^-$. It follows that any two vertices of b, c, d have no common neighbors in A, thus there is a matching M from $\{b, c, d\}$ into $A - y_1 = \{y_2, y_3, y_4\}$, and $V(M) \cap A$ is not a stable set in G. We may assume that $y_2y_3 \in E(G)$ and $by_2, cy_3 \in M$. Now $G/by_2/y_2y_3/dy_1 - y_4 > K_6^-$. This proves that there is no vertex of A adjacent to all b, c, d in G. Next, suppose that $G[\{b, c, d\}]$ induces at least two edges, say $bc, cd \in E(G)$. We may assume that $bd, ae \notin E(G)$, otherwise $N[x] > K_6^-$. Among a, b, d, e, by Claim 2, we may assume that $d_G(e) = 5$. Let $ey_1 \in E(G)$. If $cy_1 \notin E(G)$, by Claim 1, $\delta(N(e)) \geqslant 2$. Thus $by_1, dy_1 \in E(G)$ and so $G/by_1 - y_2 > K_6^-$. It follows that $cy_1 \in E(G)$. Then $d_G(c) \geqslant 6$. By Claim 2, $d_G(a) = d_G(b) = d_G(d) = 5$. By Claim 1 and the symmetry of b and d, we may assume that $by_1 \in E(G)$. Then $dy_1 \notin E(G)$, otherwise y_1 is adjacent to all b, c, d in G. Similarly, let $dy_2 \in E(G)$. Then $cy_2 \in E(G)$ and $by_2, ey_2 \notin E(G)$. Thus $ay_2 \in E(G)$. Now y_3 is only adjacent to c, y_1, y_2, y_4, which contradicts the fact that $d_G(y_3) \geqslant 5$. This proves that $G[\{b, c, d\}]$ contains at most one edge. We may assume that $bc, bd \notin E(G)$.

Suppose that $d_G(a), d_G(e) \geqslant 6$. Then $\delta(G/xb) \geqslant 5$. Since G is minor minimal, we have $|G| = 9$. Let w be the new vertex in G/xb. Then $d_{G/xb}(w) \geqslant 6$. If $d_{G/xb}(a) \geqslant 6$ or $d_{G/xb}(e) \geqslant 6$, say the latter, then $\delta(G/xb - ew) \geqslant 5$. By Lemma 2.3, $G/xb > K_6^- \cup K_1$. It follows that $d_G(a) = d_G(e) = 6$. Since $|G| = 9$ and the number of odd vertices of a graph is even, there exists a vertex of A, say y_1, such that $d_G(y_1) \geqslant 6$. Then $d_{G/xb}(y_1) \geqslant 6$ and $wy_1 \in E(G/xb)$. Now $\delta(G/xb - wy_1) \geqslant 5$. By Lemma 2.3, $G/xb > K_6^- \cup K_1$. Consequently, $d_G(a) = 5$ or $d_G(e) = 5$. We may assume that $d_G(a) = 5$. If $ae \in E(G)$, then, since G is 4-connected, e has at least one neighbor in A. It follows that $d_G(e) \geqslant 6$ and so $d_G(b) = d_G(c) = d_G(d) = 5$. Now x and b have exactly two common neighbors a and e in G. If $d_G(e) \geqslant 8$, then in G/xb, $\Delta(G/xb) \geqslant 7$, $d_{G/xb}(a) = 4$ and $d_{G/xb}(v) \geqslant 5$ for any $v \in V(G/xb - a)$. By Lemmas A.1 and A.2, $G/xb > K_6^- \cup K_1$. So we may assume that e is adjacent to at most two vertices of A in G. Then $e_G(N(x), A) \leqslant 8$. It follows that $e_G(N(x), A) = 8$, $|A| = 4$, $G[A] = K_4$, and $G[\{b, c, d\}] = \overline{K_3}$. We may assume that $by_1, cy_4 \in E(G)$. Then $G/by_1/y_1y_2/y_2y_3 - y_4 = K_6^-$. Hence $ae \notin E(G)$. Let $ay_1 \in E(G)$. Then $cd \in E(G)$, otherwise, by Claim 1, $\delta(N(a)) \geqslant 2$, but then y_1 is adjacent to all b, c, d in G. Again, by Claim 1, $y_1b \in E(G)$. By symmetry of c and d, we may assume that $cy_1 \in E(G)$ and so $dy_1 \notin E(G)$ (otherwise y_1 is adjacent to all b, c, d). Let $dy_2 \in E(G)$. Then $ay_2 \notin E(G)$ and y_2 is adjacent to at most one of b and c in G. It follows that either $y_2y_1 \in E(G)$ (in this case $G/by_1/y_1y_2 - y_3 > K_6^-$) or $y_2y_3, y_2y_4 \in E(G)$ and y_1 is adjacent to at least one of y_3, y_4, say y_3 (in this case $G/by_1/y_1y_3/y_3y_2 - y_4 > K_6^-$). □

Claim 4. *$N(x)$ contains no subgraph isomorphic to $K_1 + (K_2 \cup K_2)$.*

Proof. Suppose that $N(x)$ has a subgraph H isomorphic to $K_1+(K_2\cup K_2)$. We may assume that $d_H(c) = 4$, and $ab, de \in E(H)$. By Claim 3, there exists at most one edge between $\{a, b\}$ and $\{d, e\}$ in G. Suppose such an edge exists. By symmetry, we may assume that $ad \in E(G)$. By Claim 2, we may assume that $d_G(a) = 5$. Let $ay_1 \in E(G)$. By Claim 1, $\delta(N(a)) \geqslant 2$. By Claim 3, we may assume that $cy_1 \in E(G)$. It follows that $d_G(c) \geqslant 6$ and by Claim 2, $d_G(b) = d_G(d) = d_G(e) = 5$. If $e_G(c, A) \geqslant 3$, then $d_{G/xe}(c) \geqslant 7$, $d_{G/xe}(d) = 4$ and $d_{G/xe}(v) \geqslant 5$ for any $v \neq e$. By Lemmas A.1 and A.2, $G > G/xe > K_6^- \cup K_1$. Hence $e_G(c, A) \leqslant 2$. By counting the number of edges between $N(x)$ and A in G, it follows that $e_G(A, N(x)) = 8$ and $G[A] = K_4$. Let $by_i, ey_j \in E(G)$, where y_i, y_j, y_1 could be the same. Clearly, $G/ey_j/y_jy_i/y_iy_1 - (A - \{y_1, y_i, y_j\}) = K_6^-$. This shows that there exists no edge between $\{a, b\}$ and $\{d, e\}$ in G. By Claim 2, we may assume that $d_G(b) = d_G(e) = 5$. Let $by_1, by_2 \in E(G)$.

Suppose that $d_G(c) = 5$. Then by Claim 1, $y_1y_2, ay_1, ay_2 \in E(G)$. Let y_i, y_j be the two neighbors of e in A. By Claim 1, $y_iy_j, dy_i, dy_j \in E(G)$. If $y_i = y_1$ and $y_j = y_2$, then $G/ey_1/dy_2 - y_3 > K_6^-$. If $y_i = y_1$ and $y_j \neq y_2$, we may assume that $y_j = y_3$. Then $G/ey_1/ay_3 - y_2 > K_6^-$ if $ay_3 \in E(G)$ or $G/ey_1/dy_2 - y_3 > K_6^-$ if $dy_2 \in E(G)$. It follows that $G[A] = K_4$. Now $G/ey_1/ay_2/y_2y_3 - y_4 > K_6^-$. Hence, by symmetry, we may assume that $y_i, y_j \neq y_1, y_2$ and so $ey_3, ey_4 \in E(G)$. Clearly, $G > K_6^- \cup K_1$ or G is isomorphic to J. This proves that $d_G(c) \geqslant 6$. By Claim 2, $d_G(a) = d_G(b) = d_G(d) = d_G(e) = 5$. If $d_G(c) \geqslant 8$, then $d_{G/xa}(c) \geqslant 7$, $d_{G/xa}(b) = 4$ and $d_{G/xa}(v) \geqslant 5$ for any $v \neq c, b$. By Lemmas A.1 and A.2, $G/xa > K_6^- \cup K_1$. It follows that $6 \leqslant d_G(c) \leqslant 7$. Since $by_1, by_2 \in E(G)$, by the symmetry of a, b, d, e, we may assume that $cy_1 \notin E(G)$. By Claim 1, $y_1y_2, ay_1 \in E(G)$.

Suppose $ey_1 \in E(G)$. By Claim 1, $dy_1 \in E(G)$. If $dy_2 \in E(G)$ or $ey_2 \in E(G)$, say the latter, then $G/ay_1/by_2 - y_3 > K_6^-$. So we may assume that $dy_2, ey_2 \notin E(G)$. Let $ey_3 \in E(G)$. By Claim 1, $y_1y_3 \in E(G)$. By symmetry of a, b, d, e, $ay_3, by_3 \notin E(G)$. If $|A| = 3$, then by Claim 1, $cy_2, cy_3, ay_2, dy_3, y_2y_3 \in E(G)$ and so $G/xd/y_2y_3 - e = K_6^-$. If $|A| = 4$, since y_4 is adjacent to at least two vertices other than b, e of H, we may assume that $ay_4 \in E(G)$. Then $G/ay_4/by_1 - \{y_2, y_3\} = K_6^-$ if $dy_4 \in E(G)$, otherwise $y_3y_4 \in E(G)$ and $G/by_1/ey_3/y_3y_4 - y_2 = K_6^-$. This proves that $ey_1 \notin E(G)$ and similarly, $dy_1 \notin E(G)$. Thus $y_1y_i \in E(G), i = 2, 3, 4$, and $d_G(y_1) = 5$. We claim that $G[A] = K_4$. If $ay_2 \in E(G)$, by Claim 1, $\delta(N(y_1)) \geqslant 2$ and so $G[A] = K_4$. If $ay_2 \notin E(G)$, we may assume that $ay_3 \in E(G)$. By Claim 1, $\delta(N(b)) \geqslant 2$ and so $cy_2, cy_3 \in E(G)$. Since $d_G(c) \leqslant 7$, we have $cy_4 \notin E(G)$ and so y_4 is adjacent to d, e, y_1, y_2, y_3. Then either $G[A] = K_4$ or $y_2y_3 \notin E(G)$ (in this case, we may assume that $ey_3 \in E(G)$. Then $G/by_1/y_1y_4/ey_3 - y_2 = K_6^-$). Hence $G[A] = K_4$, as claimed. Since $e_G(N(x), A) \geqslant 9$, there exists a vertex $y_i \in A$ such that $d_G(y_i) \geqslant 6$. Note that $d_G(y_1) = 5$, we have $y_i \neq y_1$. By Claim 2, $cy_i \notin E(G)$ and so $e_G(y_i, \{a, b, d, e\}) \geqslant 3$. Since $y_1e, y_1d \notin E(G)$, let $ey_j, dy_k \in E(G)$, where $y_j, y_k \neq y_i$. If $y_i \neq y_2$, then $G/ay_i/by_1/y_1y_j > K_6^- \cup K_1$. So we may assume that $y_i = y_2$. If $ay_2 \notin E(G)$, then $G/by_2/ay_1/y_1y_j > K_6^- \cup K_1$. If $ay_2 \in E(G)$, we may assume that $ey_2 \in E(G)$. Then $G/ey_2/by_1/y_1y_k > K_6^- \cup K_1$. □

By Claim 1, $\delta(N(x)) \geqslant 2$. Hence, by Claims 3 and 4, $N(x)$ is isomorphic to either C_5 or C_5 with exactly one chord.

Suppose that $N(x)$ is isomorphic to C_5 and $N(x)$ has vertices a, b, c, d and e in order. By Claim 2, $N(x)$ contains at most two vertices of degree $\geqslant 6$. Suppose that $N(x)$ contains exactly two vertices of degree $\geqslant 6$, say b and d. Then $\delta(G/xc) \geqslant 5$. Since G is minor minimal, we have $|G| = 9$, $d_G(b) = d_G(d) = 6$, and by Claim 2, $d_G(v) = 5$ for any $v \in V(G - \{b, d\})$, which contradicts the fact the number of odd vertices of G is even. This implies that $N(x)$ contains at most one vertex of degree greater than five (we may assume $d_G(e) \geqslant 6$ if such a vertex exists). Thus $d_G(a) = d_G(b) = d_G(c) = d_G(d) = 5$. Let $cy_1, cy_2 \in E(G)$. By Claims 3 and 4, $N(c)$ contains no subgraph isomorphic to $K_{2,3}$ and $K_1 + (K_2 \cup K_2)$. Thus by Claim 1, $y_1y_2 \in E(G)$. We may assume that $by_1, dy_2 \in E(G)$. Then by_2, dy_1 cannot be both in $E(G)$, otherwise$N(c) > K_{2,3}$.

Suppose $by_2, dy_1 \notin E(G)$. Since $d_G(b) = 5$, let $by_3 \in E(G)$. By Claim 1, $ay_3, y_3y_1 \in E(G)$. We claim that $dy_3 \notin E(G)$. Suppose $dy_3 \in E(G)$. By Claim 1, $y_3y_2, ey_3 \in E(G)$. Thus $d_G(y_3) \geqslant 6$ and so $d_G(e) = d_G(y_1) = d_G(y_2) = 5$. If $|A| = 3$, by Claim 1, $ay_1, ey_2 \in E(G)$. Clearly, $G/xb/y_1y_2 - c > K_6^-$. If $|A| = 4$, then y_4 is adjacent to a, e, y_1, y_2, y_3, and so $G/by_3/ay_4/y_4y_2 - y_1 = K_6^-$. This proves that $dy_3 \notin E(G)$. Since $d_G(d) = 5$, let $dy_4 \in E(G)$. Then by Claim 1, $ey_4, y_2y_4 \in E(G)$. If $ay_4 \notin E(G)$, then $d_G(y_4) = 5$ and $y_4y_1, y_4y_3 \in E(G)$. By Claim 1, $\delta(N(y_4)) \geqslant 2$ and so $ey_3 \in E(G)$. Note that a is adjacent to exactly one vertex of $\{y_1, y_2\}$. Now $G/ay_1/by_3/cd - y_2 = K_6^-$ if $ay_1 \in E(G)$ or $G/xc/xe/ay_2 - d = K_6^-$ if $ay_2 \in E(G)$. This proves that $ay_4 \in E(G)$. By Claim 1, $\delta(N(a)) \geqslant 2$ and so $y_3y_4 \in E(G)$. Clearly, $y_1y_4 \notin E(G)$ (otherwise $d_G(y_4) \geqslant 6$ and so by Claim 2, e is adjacent to exactly one of y_2 and y_3, say y_2. Then $d_G(y_3) = 4$, which is a contradiction). It follows that $ey_1 \in E(G)$ and $d_G(y_1) = 5$. By Claim 1, $\delta(N(y_1)) \geqslant 2$, we have $ey_2, ey_3 \in E(G)$. Now $G/xa/xc/y_3y_4 - b = K_6^-$.

Suppose $by_2 \notin E(G)$ but $dy_1 \in E(G)$. Since $d_G(b) = 5$, let $by_3 \in E(G)$. By Claim 1, $ay_3, y_3y_1 \in E(G)$. Suppose $|A| = 4$. Then y_4 is adjacent to a, e, y_1, y_2, y_3. Then $d_G(y_1) \geqslant 6$. By Claim 2, $d_G(y_2) = d_G(y_3) = 5$. By Claim 1, $\delta(N(y_4)) \geqslant 2$, we have $ey_2, ey_3 \in E(G)$. Now $G/ab/cy_2/dy_1 - y_3 = K_6^-$. So we may assume that $|A| = 3$. Since $cy_3, dy_3 \notin E(G)$, it follows that $d_G(y_3) = 5$ and $y_3e, y_3y_2 \in E(G)$. By Claim 1, $ey_2 \in E(G)$. Note that a is adjacent to exactly one vertex of y_1, y_2. Now $G/xa/y_2y_3/ - e = K_6^-$ if $ay_1 \in E(G)$ or $G/xa/y_1y_3 - b = K_6^-$ if $ay_2 \in E(G)$.

Finally, assume that $dy_1 \notin E(G)$ but $by_2 \in E(G)$. Since $d_G(d) = 5$, let $dy_3 \in E(G)$. By Claim 1, $ey_3, y_3y_2 \in E(G)$. Suppose $|A| = 4$. Then y_4 is adjacent to a, e, y_1, y_2, y_3. Thus $d_G(y_2) \geqslant 6$. By Claim 1, $\delta(N(y_4)) \geqslant 2$ and so $ay_1 \in E(G)$. Since $d_G(y_2) \geqslant 6$, by Claim 2, y_3 is only adjacent to d, e, y_2, y_4, which contradicts the fact that $d_G(y_3) \geqslant 5$. So we may assume that $|A| = 3$. Since $cy_3, by_3 \notin E(G)$, it follows that $d_G(y_3) = 5$ and $y_3a, y_3y_1 \in E(G)$. Suppose $ay_1 \in E(G)$. By Claim 2, e is adjacent to exactly one vertex of y_1, y_2. Thus $G/xe/y_2y_3/ - d = K_6^-$ if $ey_1 \in E(G)$ or $G/xe/y_1y_3 - a = K_6^-$ if $ey_2 \in E(G)$. Suppose $ay_1 \notin E(G)$. Then $ey_1, ay_2 \in E(G)$. Now $G/ay_2/ey_1 - y_3 = K_6^-$. This completes the proof that $N(x)$ is isomorphic to C_5.

It remains to consider the case when $N(x)$ is isomorphic to C_5 with exactly one chord. We may assume that $E(N(x)) = \{ab, bc, cd, de, ea, be\}$. By Claim 2, one of b and e, say e, is of degree five in G. Let $ey_1 \in E(G)$. By Claim 1, $\delta(N(e)) \geqslant 2$ and so $dy_1 \in E(G)$. Suppose $ay_1, by_1 \in E(G)$. We claim that $d_G(a) \geqslant 6$. Suppose $d_G(a) = 5$. Let $ay_2 \in E(G)$. By Claim 1, $\delta(N(a)) \geqslant 2$ and so $by_2,\ y_2y_1 \in E(G)$. It follows that $N(a) > K_1 + (K_2 \cup K_2)$, which contradicts Claim 4. Hence $d_G(a) \geqslant 6$, as claimed. By Claims 1 and 2, $d_G(b) = 5$ and $cy_1 \in E(G)$. But now $N(b) > K_{2,3}$, which contradicts Claim 3. This proves that at most one of by_1, ay_1 are in $E(G)$.

Suppose $by_1 \in E(G)$ but $ay_1 \notin E(G)$. If $d_G(b) = 5$, since $\delta(N(b)) \geqslant 2$, $cy_1 \in E(G)$. By Claim 1, $\delta(N(a)) \geqslant 2$. Hence $ay_i \in E(G)$, $i = 2, 3, 4$, and $G[\{y_2, y_3, y_4\}] = K_3$. Since there is no edge between $\{b, e\}$ and $\{y_2, y_3, y_4\}$ in G, $e_G(\{y_2, y_3, y_4\}, \{c, d, y_1\}) \geqslant 6$. However, by Claim 2, $e_G(\{c, d, y_1\}, \{y_2, y_3, y_4\}) \leqslant 5$, which is a contradiction. So we may assume that $d_G(b) \geqslant 6$. By Claim 2, $d_G(a) = 5$. Let $ay_2, ay_3 \in E(G)$. By Claim 1, $\delta(N(a)) \geqslant 2$ and so $y_2y_3, by_2, by_3 \in E(G)$. Then $N(a) > K_1 + (K_2 \cup K_2)$, which contradicts Claim 4.

Finally, suppose $ay_1 \in E(G)$ but $by_1 \notin E(G)$. We claim that $d_G(d) \geqslant 6$. Suppose $d_G(d) = 5$. Let $dy_2 \in E(G)$. We may assume that $N(d) \neq C_5$. By Claim 1, $\delta(N(d)) \geqslant 2$ and so $y_1y_2, cy_1, cy_2 \in E(G)$. It follows that $G/ay_1/by_2 - y_3 > K_6^-$ if $by_2 \in E(G)$ or $G/ay_2/ey_1 - y_3 > K_6^-$ if $ay_2 \in E(G)$. So we may assume that $ay_2, by_2 \notin E(G)$. Then $y_2y_3, y_2y_4 \in E(G)$. Since $by_1, by_2 \notin E(G)$, we may assume that $by_3 \in E(G)$. Now $G/ay_1/by_3/y_3y_2 - y_4 = K_6^-$. This proves that $d_G(d) \geqslant 6$. By Claim 2, $d_G(c) = 5$ and so $d_G(b) \geqslant 6$ (otherwise, by symmetry of b and e, $d_G(c) \geqslant 6$). Now $\delta(G/xc) \geqslant 5$. Since G is minor minimal, we have $|G| = 9$. Let w be the new vertex in G/xc. Then $d_{G/xc}(w) \geqslant 6$. If $d_{G/xc}(b) \geqslant 6$ or $d_{G/xc}(d) \geqslant 6$, say the latter, then $\delta(G/xb - dw) \geqslant 5$. By Lemma 2.3, $G/xc > K_6^- \cup K_1$. It follows that $d_G(b) = d_G(d) = 6$. Since $|G| = 9$ and the number of odd vertices of G is even, there exists a vertex, say y_1, of A such that $d_G(y_1) \geqslant 6$. Note that $d_{G/xc}(y_1) \geqslant 6$ and $y_1c \in E(G)$. Now $y_1w \in E(G/xc)$ and $\delta(G/xc - y_1w) \geqslant 5$. By Lemma 2.3, $G/xc > K_6^- \cup K_1$. □

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