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# The extremal function for $K_9$ minors

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**Abstract**

We prove that every (simple) graph on $n \geqslant 9$ vertices and at least $7n - 27$ edges either has a $K_9$ minor, or is isomorphic to $K_{2,2,2,3,3}$, or is isomorphic to a graph obtained from disjoint copies of $K_{1,2,2,2,2,2}$ by identifying cliques of size six. The proof of one of our lemmas is computer-assisted.


*Keywords:* Graph; Minor; Complete graph minor; Extremal function

## 1. Introduction

All graphs in this paper are finite and simple. Our work is motivated by the following theorem of Mader [18].

**Theorem 1.1.** *For every integer* $p = 1, 2, \dots, 7$, *a graph on* $n \geqslant p$ *vertices and at least* $(p - 2)n - \binom{p-1}{2} + 1$ *edges has a* $K_p$ *minor.*

For $p \leqslant 5$, this was first proved by Dirac [5]. Some years later but independently of Mader, Györi [7] proved Theorem 1.1 for $p \leqslant 6$.

Mader pointed out that Theorem 1.1 does not hold for $p = 8$: the graph $K_{2,2,2,2,2}$ is a counterexample. However, one can construct further counterexamples by repeatedly identifying cliques of size five. So for graphs $H_1$, $H_2$ and an integer $k$, let us define an $(H_1, H_2, k)$-*cockade* recursively as follows. Any graph isomorphic to $H_1$ or $H_2$ is an $(H_1, H_2, k)$-cockade. Now let $G_1$, $G_2$ be $(H_1, H_2, k)$-cockades and let $G$ be obtained from the disjoint union of $G_1$ and $G_2$ by identifying a clique of size $k$ in $G_1$ with a clique of the same size in $G_2$. Then the graph $G$

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is also an $(H_1, H_2, k)$-cockade, and every $(H_1, H_2, k)$-cockade can be constructed this way. If $H_1 = H_2 = H$, then $G$ is called an $(H, k)$-cockade. Jørgensen [13] generalized Theorem 1.1 as follows.

**Theorem 1.2.** *Every graph on $n \geqslant 8$ vertices and at least $6n - 20$ edges either has a $K_8$ minor, or is a $(K_{2,2,2,2,2}, 5)$-cockade.*

To see that Theorem 1.2 implies Theorem 1.1, let $G$ and $p$ be as in Theorem 1.1, and apply Theorem 1.2 to the graph obtained from $G$ by adding $8 - p$ vertices, each adjacent to every other vertex of the graph.

Our main result is the following next step. Note that every $(K_{2,2,2,3,3}, 6)$-cockade is isomorphic to $K_{2,2,2,3,3}$.

**Theorem 1.3.** *Every graph on $n \geqslant 9$ vertices and at least $7n - 27$ edges either has a $K_9$ minor, or is a $(K_{1,2,2,2,2,2}, 6)$-cockade, or is isomorphic to $K_{2,2,2,3,3}$.*

Our motivation was threefold. First, the bound in Theorem 1.1 is related to Hadwiger's conjecture [8], the following.

**Conjecture 1.4.** *For every integer $t \geqslant 1$, every graph with no $K_{t+1}$ minor is t-colorable.*

Hadwiger's conjecture is trivially true for $t \leqslant 2$, and reasonably easy for $t = 3$, as shown by Dirac [4]. However, for $t \geqslant 4$, Hadwiger's conjecture implies the Four Color Theorem. (To see that, let $H$ be a planar graph, and let $G$ be obtained from $H$ by adding $t - 4$ vertices, each joined to every other vertex of the graph. Then $G$ has no $K_{t+1}$ minor, and hence is $t$-colorable by Hadwiger's conjecture, and hence $H$ is 4-colorable.) Wagner [26] proved that the case $t = 4$ of Hadwiger's conjecture is, in fact, equivalent to the Four Color Theorem, and the same was shown for $t = 5$ by Robertson et al. [19]. Their proof made use of Theorem 1.1 for $p = 6$. Hadwiger's conjecture remains open for $t \geqslant 6$.

In [3] Chen et al. proved that every graph on $n$ vertices and at least $9n - 45$ edges has a $K_9$ minor, and used that to deduce that if, in addition, $G$ is 6-connected, then it is 3-linked. It turns out [22] that the latter conclusion can be obtained from a weaker bound on the number of edges by a more direct argument, but the work of Chen et al. suggested that there may be interest in the extremal problem for $K_9$ minors.

Theorem 1.1 is such a nice result that it raises the question of whether it can be generalized to all values of $p$. But there are more depressing news than Mader's example above: for large $p$ a graph must have at least $\Omega(p\sqrt{\log p}n)$ edges in order to guarantee a $K_p$ minor, because, as noted by several people (Kostochka [15,16], and Fernandez de la Vega [6] based on Bollobás et al. [2]), a random graph with no $K_p$ minor may have average degree of order $p\sqrt{\log p}$. Kostochka [15,16] and Thomason [23] proved that this is indeed the correct order of magnitude, and in a remarkable recent result [24] Thomason was able to determine the constant of proportionality. Thus it may seem that an effort to generalize Theorem 1.1 will be in vain, but there is still the following possibility. The random graph examples provide only finitely many counterexamples for any given value of $p$. Of course, more counterexamples can be obtained by taking disjoint unions or even gluing counterexamples along small cutsets, but we know of no construction of highly connected infinite families of counterexamples. More specifically, Seymour and the second author conjecture the following.

**Conjecture 1.5.** *For every $p \geqslant 1$ there exists a constant $N = N(p)$ such that every $(p-2)$-connected graph on $n \geqslant N$ vertices and at least $(p-2)n - \binom{p-1}{2} + 1$ edges has a $K_p$ minor.*

It was this conjecture that was the third motivating factor for our research. Thus our main result implies that Conjecture 1.5 holds for $p \leqslant 9$. Recently, Böhme et al. [1] showed that Conjecture 1.5 holds for $31(p+1)/2$-connected graphs.

The extremal functions for $K_p^-$ minors have also been studied, where $K_p^-$ denotes the graph obtained from $K_p$ by removing one edge. Jakobsen [10,11] proved that, for $p \leqslant 7$, every graph on $n \geqslant p$ vertices and at least $(p - \frac{5}{2})n - \frac{1}{2}(p-3)(p-1)$ edges has a $K_p^-$ minor, or $G$ is a $(K_{p-1}, p-3)$-cockade, or $p = 7$ and $G$ is a $(K_{2,2,2,2}, K_6, 4)$-cockade. Recently, the first author [21] proved a conjecture of Jakobsen [11] that every graph on $n \geqslant 8$ vertices and at least $\frac{1}{2}(11n - 35)$ edges has a $K_8^-$ minor or is a $(K_{1,2,2,2,2}, K_7, 5)$-cockade. The extremal functions for the graphs obtained from $K_p$ by deleting two edges were determined in [9,10] when $p = 7$ or 8. In related work Jørgensen [14] proved that every 4-connected graph on $n \geqslant 8$ vertices and at least $4n - 7$ edges has a $K_{4,4}$ minor.

We need to introduce more notation. If $G$ is a graph and $K$ is a subgraph of $G$, then by $N(K)$ we denote the set of vertices of $V(G) - V(K)$ that are adjacent to a vertex of $K$. If $V(K) = \{x\}$, then we use $N(x)$ to denote $N(K)$. By abusing notation we will also denote by $N(x)$ the graph induced by the set $N(x)$. We define $N[x] = N(x) \cup \{x\}$, and similarly will use the same symbol for the graph induced by that set. If $x, y$ are adjacent vertices of a graph $G$, then we denote by $G/xy$ the graph obtained from $G$ by contracting the edge $xy$ and deleting all resulting parallel edges. If $u, v$ are distinct nonadjacent vertices of a graph $G$, then by $G + uv$ we denote the graph obtained from $G$ by adding an edge with ends $u$ and $v$. If $u, v$ are adjacent or equal, then we define $G + uv$ to be $G$. We write $G > H$ if a graph isomorphic to $H$ can be obtained from a subgraph of $G$ by contracting edges. In those circumstances we say that $G$ has an *$H$ minor*. For a graph $G$ we use $|G|$, $e(G)$, $\delta(G)$ to denote the number of vertices, number of edges and minimum degree of $G$, respectively. The degree of a vertex $v$ in a graph is denoted by $d_G(v)$ or simply $d(v)$.

## 2. Outline of proof

Suppose for a contradiction that $G$ is a counterexample to Theorem 1.3 with minimum number of vertices, say $n$. Since deletion or contraction of edges does not produce smaller counterexamples, it follows easily that $G$ has minimum degree at least eight, and with some effort it can be shown that every edge of $G$ is in at least seven triangles. It also follows by a straightforward counting argument that $G$ is 6-connected. Also $e(G) = 7n - 27$, and hence $G$ has a vertex $x$ of degree at least eight and at most thirteen. Fix such a vertex, and let $K$ be a component of $G - N[x]$. Assume for a moment that every vertex of $N(x)$ has a neighbor in $K$. If there exists a vertex $y \in N(x)$ such that $N(x) - y > K_7$, then by contracting the connected set $V(K) \cup \{y\}$ to a single vertex, we see that $G > K_9$. Thus $G - y \not> K_7$ for every vertex $y \in N(x)$. On the other hand, $N(x)$ has minimum degree at least seven and at most 13 vertices. Those properties are fairly restrictive: there are only 14 such graphs, and so they can be found explicitly. It turns out that they all have two properties in common (conditions (A) and (B) stated prior to Lemma 3.7) that allow us to find a $K_9$ minor in $G$ in a different way. This is how we deal with the case when there is a component $K$ of $G - N[x]$ satisfying $N(x) = N(K)$. In fact, the argument extends to the situation when there exists a component $K$ of $G - N[x]$ such that $N(K') \cap M \subseteq N(K)$ for every component $K'$ of $G - N[x]$, where $M$ is the set of all vertices of $N(x)$ that are not adjacent to every other vertex of $N(x)$.

Thus we may assume that for no vertex $x$ of degree at most 13 such a component exists. In the next step we prove a lemma inspired by Claim (15) of [13], namely that if $x \in V(G)$ has degree at most 13, then there is no component $K$ of $G - N[x]$ such that $d_G(v) \geqslant 14$ for all vertices $v \in V(K)$. This follows by counting edges, for if such a component exists, then we exhibit a proper minor of $G$ with $n' < n$ vertices and more than $7n' - 27$ edges. That minor of $G$ has a $K_9$ minor by the minimality of $G$, and hence $G$ has a $K_9$ minor, a contradiction. Finally, in the last step, we select a vertex $x \in V(G)$ of degree at most 13 to minimize the size of a component $K$ of $G - N[x]$. It follows easily that $K$ does not have a vertex whose degree in $G$ is at most 13.

## 3. Preliminaries

The following result of Jørgensen [13] follows from the proof of Lemma 3.7 below, but we state it separately for convenience.

**Theorem 3.1.** *Let $G$ be a graph with $n \leqslant 11$ vertices and $\delta(G) \geqslant 6$ such that for every vertex $x$ in $G$, $G - x$ is not contractible to $K_6$. Then $G$ is one of the graphs $K_{2,2,2,2}$, $K_{3,3,3}$ or the complement of the Petersen graph.*

The next theorem was first proved by Jung [12]. Seymour [20] and Thomassen [25] gave a complete characterization of all (not necessarily 4-connected) graphs that satisfy the hypothesis of the theorem.

**Theorem 3.2.** *Let $G$ be a 4-connected graph and let $x_1, x_2, y_1, y_2$ be distinct vertices in $G$. If $G$ does not contain an $x_1 - y_1$ path and an $x_2 - y_2$ path that are disjoint, then $G$ is planar and $e(G) \leqslant 3|G| - 7$.*

As noted in Section 2, our proof uses induction by deleting and contracting edges of $G$. Thus we need to investigate graphs $G$ such that the new graph $G - xy$ or $G/xy$ is a $(K_{1,2,2,2,2,2}, 6)$-cockade or is isomorphic to $K_{2,2,2,3,3}$. We do that next.

**Lemma 3.3.** *Let $G$ be $K_{2,2,2,3,3}$ or a $(K_{1,2,2,2,2,2}, 6)$-cockade and let $x$ and $y$ be nonadjacent vertices in $G$. Then $G + xy$ is contractible to $K_9$.*

**Proof.** This is easily checked if $G = K_{2,2,2,3,3}$ or $G = K_{1,2,2,2,2,2}$. So we may assume that $G$ is obtained from $H_1$ and $H_2$ by identifying cliques of size 6, where $H_1$ and $H_2$ are $(K_{1,2,2,2,2,2}, 6)$-cockades. If $x$ and $y$ are both in $H_1$ or $H_2$, then $H_1 + xy > K_9$ or $H_2 + xy > K_9$ by induction. So we may assume that $x \in V(H_1) - V(H_2)$ and $y \in V(H_2) - V(H_1)$. Note that no $(K_{1,2,2,2,2,2}, 6)$-cockade contains $K_7$ as a subgraph. Therefore there exists $z \in V(H_1) \cap V(H_2)$ such that $zy \notin V(G)$. Now by contracting $V(H_1) - V(H_2)$ to the vertex $z$ in $G + xy$, the resulting graph is $H_2 + zy$. By induction, $H_2 + zy > K_9$. □

**Lemma 3.4.** *Let $G$ be a graph. Let $x, y \in V(G)$ be such that $xy \in E(G)$ with exactly six common neighbors. If $G/xy$ is isomorphic to $K_{2,2,2,3,3}$, then $G > K_9$.*

**Proof.** Let $w$ be the new vertex in $G/xy$. Since $x$ and $y$ have exactly six common neighbors, there exist distinct vertices $w_1, w_2, w_3, w_4 \in V(G/xy) - w$ such that $w_1w_2, w_3, w_4 \notin E(G/xy)$, and $w_1, w_2, w_3$ are common neighbors of $x$ and $y$ in $G$. Moreover, $w_4$ is adjacent to $x$ or $y$, say to $y$, in $G$. By contracting the edges $xw_2$ and $yw_4$ we see that $G$ has a $K_9$ minor, as desired. □

**Lemma 3.5.** *Let $G$ be a graph and let $x, y$ be adjacent vertices of $G$ with exactly six common neighbors. If $G/xy$ is isomorphic to $K_{1,2,2,2,2,2}$, then $G$ has a $K_9$ minor, unless $G$ is isomorphic to $K_{2,2,2,3,3}$ and $x, y$ have degree nine in $G$.*

**Proof.** Let $w$ be the new vertex of $G/xy$, and let $z, x_1, y_1, \dots, x_5, y_5$ be the vertices of $G/xy$ numbered so that $x_i$ is not adjacent to $y_i$. Assume first that $w \neq z$, say $w = x_1$. Since $x$ and $y$ have six common neighbors, we may assume that $x_2, y_2, x_3$ are common neighbors of $x$ and $y$. Moreover, $y_3$ is adjacent to $x$ or $y$, say to $y$. By contracting the edges $xy_2$, $yy_3$ and $y_4y_5$ we see that $G$ has a $K_9$ minor, as desired.

Thus we may assume that $w = z$. Since $x, y$ have six common neighbors, their degree is at least seven. Assume for a moment that $d_G(x) = 7$. Since $x, y$ have six common neighbors in $G$, we deduce that $y$ is adjacent to all other vertices of $G$ and there exists an index $i$ such that $x_i, y_i$ are common neighbors of $x, y$. We may assume that $i = 1$. By contracting the edges $xx_1, x_2x_3$ and $x_4x_5$, we obtain a $K_9$ minor of $G$. Hence we may assume that $d(x), d(y) \geqslant 8$. We may also assume that $G$ is not isomorphic to $K_{2,2,2,3,3}$ with $x, y$ of degree nine, and so it follows that one of $x, y$ is adjacent to $x_i$ or $y_i$ for every $i = 1, 2, 3, 4, 5$. Thus we may assume (by swapping $x_i$ and $y_i$) that $x$ is adjacent to all of $X$, where $X = \{x_1, \dots, x_5\}$. Moreover, we may assume that if $y$ is also adjacent to every vertex of $X$, then $d(x) \leqslant d(y)$. Let $Y = \{y_1, \dots, y_5\}$. Since $y$ has degree at least eight, there is some $i$ such that $y$ is adjacent to $x_i$ and $y_i$. We claim that $y$ is adjacent to at least three vertices of $Y$. For if not, then $x$ is adjacent to at least three vertices of $Y$ (the non-neighbors of $y$) and, since $d(y) \geqslant 8$, $y$ is adjacent to all vertices of $X$. But then $d(x) > d(y)$, a contradiction. Thus $y$ is adjacent to at least three vertices of $Y$.

Thus there exist distinct indices $i, j, k$ such that $y$ is adjacent to $x_i, y_i, y_j, y_k$. Choose such indices so that, if possible, $x$ is not adjacent to $y_i$. We may assume that $i = 1$, $j = 2$ and $k = 3$. We claim that $x$ is adjacent to at least two vertices of $Y - \{y_1\}$. For if not, then $y$ has at least four neighbors in $Y$, and hence $x, y$ have at least four common neighbors in $X$, and so the indices $i, j, k$ above can be chosen so that $x$ is not adjacent to $y_i$. Thus $x$ is not adjacent to $y_1$, and hence $x$ has at most one neighbor in $Y$, implying that $d(x) = 7$, a contradiction. Thus $x$ has at least two neighbors in $Y - \{y_1\}$, and so we may assume that $x$ has a neighbor in $\{y_2, y_4\}$ and a neighbor in $\{y_3, y_5\}$. By contracting the edges $yy_1$, $y_2y_4$ and $y_3y_5$ we see that $G$ has a $K_9$ minor, as required. □

**Lemma 3.6.** *Let $G$ be a graph with $\delta(G) \geqslant 7$. Let $x, y \in V(G)$ be such that $xy \in E(G)$ with exactly six common neighbors. If $G/xy$ is a $(K_{1,2,2,2,2,2}, 6)$-cockade, then either $G > K_9$, or $G$ is isomorphic to $K_{2,2,2,3,3}$ and $x, y$ have degree nine in $G$.*

**Proof.** We proceed by induction on $|G|$. By Lemma 3.5 we may assume that $G/xy = H_1 \cup H_2$, where $H_1 \cap H_2$ is a complete graph on six vertices and both $H_1$ and $H_2$ are $(K_{1,2,2,2,2,2}, 6)$-cockades. Let $w$ be the new vertex of $G/xy$. For $i = 1, 2$ let $H_i^* = G[(V(H_i) - \{w\}) \cup \{x, y\}]$. If $w \in V(H_1) - V(H_2)$, then $H_1^* \neq K_{2,2,2,3,3}$ (because the latter graph has no $K_6$ subgraph) and the result follows by induction applied to $H_1^*$. From the symmetry we may assume that $w \in V(H_1) \cap V(H_2)$. Let $S = V(H_1) \cap V(H_2) - \{w\}$; thus $V(H_1^*) \cap V(H_2^*) = S \cup \{x, y\}$. Let $Z$

denote the set of six common neighbors of $x$ and $y$ in $G$. If $Z \subseteq V(H_1^*)$, then by induction applied to $H_1^*$ we may assume that $H_1^*$ is isomorphic to $K_{2,2,2,3,3}$ and $x$, $y$ have degree nine in $H_1^*$. Since $H_1^*$ has no $K_6$ subgraph one of $x$, $y$, say $x$, is not adjacent to some $s \in S$ and $x$ has at least one neighbor in $V(H_2) - V(H_1)$. By using a path with ends $x$ and $s$ and interior in $H_2^* - V(H_1^*)$ we deduce that $G > H_1^* + sx > K_9$ by Lemma 3.3, as desired.

Thus we may assume that $Z - V(H_1^*) \neq \emptyset \neq Z - V(H_2^*)$. Since $H_2$ is a $(K_{1,2,2,2,2,2}, 6)$-cockade, it is 6-connected. Let $k = |Z - V(H_1)|$. Since $|Z \cap V(H_2)| \leqslant 5$ we have $|S - Z| = 5 - |Z \cap S| \geqslant k$. Thus there exist $k$ disjoint paths $P_1, P_2, \ldots, P_k$ in $H_2 - (Z \cap S) - w$ between $Z \cap V(H_2 - S)$ and $S - Z$. Consequently $H_1^*$ has a supergraph $H_1'$ on the same vertex set such that $H_1' < G$ and $x, y$ have exactly six common neighbors in $H_1'$. By induction $H_1'$ is isomorphic to $K_{2,2,2,3,3}$ and $x$, $y$ have degree nine in $H_1'$. By symmetry the same holds for the analogous graph $H_2'$. It follows that in $H_1'$ the vertex $x$ has a unique non-neighbor in $S$, say $x'$. Then $x' \notin V(P_1 \cup \cdots \cup P_k)$. From the symmetry between $H_1$ and $H_2$ we may assume that $k \leqslant 3$. (In fact, $|Z - V(H_1)| = |Z - V(H_2)| = 3$.) It follows that the $k$ disjoint paths $P_1, \ldots, P_k$ can each be chosen of length one, and that there exists a common neighbor of $x$ and $x'$ in $V(H_2^*)$, say $u$, that does not belong to any of the paths. Thus by contracting the edge $ux'$ and all the edges of the paths $P_1, \ldots, P_k$ we deduce that $G > H_1' + xx' > K_9$ by Lemma 3.3, as desired. □

As pointed out in Section 2, we need to examine graphs $G$ such that $|V(G)| \leqslant 13$, $\delta(G) \geqslant 7$ and $G \not> K_7 \cup K_1$. (Here $K_7 \cup K_1$ stands for a disjoint union of $K_7$ and $K_1$.) The next lemma shows that those graphs $G$ satisfy the following properties:

(A) either $G$ is isomorphic to $K_{1,2,2,2,2,2}$, or $G$ has four distinct vertices $a_1, b_1, a_2, b_2$ such that $a_1a_2, b_1b_2 \notin E(G)$ and for $i = 1, 2$ the vertex $a_i$ is adjacent to $b_i$, the vertices $a_i, b_i$ have at most four common neighbors, and $G + a_1a_2 + b_1b_2 > K_8$,

(B) for any two sets $A, B \subseteq V(G)$ of cardinality at least five such that neither is complete and $A \cup B$ includes all vertices of $G$ of degree at most $|G| - 2$, either
  - (B1) there exist $a \in A$ and $b \in B$ such that $G' > K_8$, where $G'$ is obtained from $G$ by adding all edges $aa'$ and $bb'$ for $a' \in A - \{a\}$ and $b' \in B - \{b\}$, or
  - (B2) there exist $a \in A - B$ and $b \in B - A$ such that $ab \in E(G)$ and the vertices $a$ and $b$ have at most five common neighbors in G, or
  - (B3) one of $A$ and $B$ contains the other and $G + ab > K_7 \cup K_1$ for all distinct nonadjacent vertices $a, b \in A \cap B$.

**Lemma 3.7.** *Let $n$ be an integer satisfying $9 \leqslant n \leqslant 13$ and let $G$ be a graph on $n$ vertices with $\delta(G) \geqslant 7$. Then either $G > K_7 \cup K_1$, or $G$ satisfies* (A) *and* (B).

**Proof.** By a computer search we have determined that the graphs $G$ with $9 \leqslant n \leqslant 13$ vertices, $\delta(G) \geqslant 7$ and $G \not> K_7 \cup K_1$ are the following ones: $K_{1,2,2,2,2,2}$, $K_{1,3,3,3,3}$, $K_{3,3} + P_4$, $K_{3,3} + \overline{C}_4$, $K_{2,2,3,3,3}$, $K_{2,3} + C_5$, $C_5 + C_5$, $\overline{K}_3 + \overline{C}_7$, $K_{3,4,4}$, $\overline{K}_3 + \overline{V}_8$, $K_1 + \overline{P}$, $\overline{P'}$, $\overline{J_1}$ and $K_1 + J_2$. Here $H + G$ stands for the graph obtained from $G \cup H$ by adding all edges with one end in $V(H)$ and the other in $V(G)$, $\overline{G}$ denotes the complement of a graph, $P_4$ denotes the path of four vertices, $V_8$ denotes the graph obtained from $C_8$ by joining all four pairs of diagonally opposite vertices, $P$ is the Petersen graph, $P'$ denotes the graph obtained from $P$ by subdividing one edge, and the graphs $J_1$ and $J_2$ are depicted in Fig. 1. It is straightforward to check that those graphs satisfy (A) and (B). The details of this and of the computer search can be obtained from the authors' websites. □

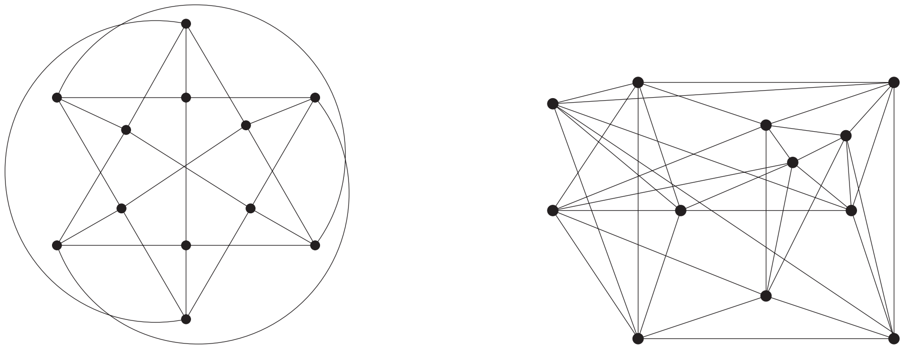

Fig. 1. Graph $J_1$ and graph $J_2$.

## 4. Proof of Theorem 1.3

In this section, we are going to prove Theorem 1.3 by induction on $n$. The only graph $G$ with 9 vertices and $e(G) \geqslant 7 \times 9 - 27 = 36$ is $K_9$. Thus we may assume that $n \geqslant 10$ and that the assertion holds for smaller values of $n$. Throughout this section we assume that $G$ is a graph with $n$ vertices and $e(G) \geqslant 7n - 27$ but $G$ is not contractible to $K_9$ and $G$ is not $K_{2,2,2,3,3}$ or a $(K_{1,2,2,2,2,2}, 6)$-cockade. By Lemma 3.3, we may assume that $e(G) = 7n - 27$.

Suppose that $G$ has a vertex $x$ of degree at most 6. Then $e(G - x) \geqslant 7(n-1) - 26$, and hence $G > G - x > K_9$ by induction, a contradiction. Suppose now that $G$ has two adjacent vertices $x$, $y$ with at most five common neighbors. Then $e(G/xy) \geqslant 7(n-1) - 26$. By induction, $G > K_9$, a contradiction. Thus $\delta(G) \geqslant 7$ and $\delta(N(x)) \geqslant 6$. If $G$ has a vertex $x$ of degree 7, then $N(x) = K_7$ and $e(G - x) \geqslant 7(n-1) - 27$. Note that neither a $(K_{1,2,2,2,2,2}, 6)$-cockade nor $K_{2,2,2,3,3}$ contain $K_7$ as a subgraph. Thus, by induction, $G - x > K_9$, a contradiction. Hence

(1) $\delta(G) \geqslant 8$ and $\delta(N(x)) \geqslant 6$ for any $x \in V(G)$.

Let $S$ be a separating set of vertices in $G$, and let $G_1$ and $G_2$ be proper subgraphs of $G$ so that $G = G_1 \cup G_2$ and $G_1 \cap G_2 = G[S]$. Let $m_i = 7|G_i| - 27 - e(G_i)$, $i = 1, 2$. Then $7n - 27 = e(G) = e(G_1) + e(G_2) - e(G[S]) = 7n + 7|S| - 54 - m_1 - m_2 - e(G[S])$, and so

(2) $7|S| = 27 + m_1 + m_2 + e(G[S])$.

For $i = 1, 2$, let $d_i$ be the maximum number of edges that can be added to $G_{3-i}$ by contracting edges of $G$ with at least one end in $G_i$. More precisely, let $d_i$ be the largest integer so that $G_i$ contains disjoint set of vertices $V_1, V_2, \ldots, V_p$ so that $G_i[V_j]$ is connected, $|S \cap V_j| = 1$ for $1 \leqslant j \leqslant p = |S|$, and so that the graph obtained from $G_i$ by contracting $V_1, V_2, \ldots, V_p$ and deleting $V(G) - (\bigcup_j V_j)$ has $e(G[S]) + d_i$ edges. By (1), $\delta(G) \geqslant 8$. Thus $|G_i| \geqslant 9$, $i = 1, 2$. By induction, $d_1 \leqslant m_2$ and $d_2 \leqslant m_1$. By (2),

(3) $7|S| \geqslant 27 + d_1 + d_2 + e(G[S])$.

In particular, $|S| \geqslant 4$. If $S$ is a minimal separating set, then let $v \in S$ be a vertex of minimum degree in $G[S]$. By choosing $V_1 = V(G_i) - (S - \{v\})$ and the rest of the sets $V_j$ to be singletons, we see that $d_i \geqslant |S| - 1 - \delta(G[S])$ for $i = 1, 2$. Thus

(4) if $S$ is a minimal separating set, then

$$5|S| \geqslant 25 + e(G[S]) - 2\delta(G[S])) \geqslant 25 + \frac{1}{2}(|S| - 4)\delta(G[S]).$$

**Lemma 4.1.** *$G$ is 6-connected.*

**Proof.** Suppose $G$ is not 6-connected. Let $S$ be a minimal separating set of $G$, and let $G_1, G_2, d_1, d_2$ be as above. By (4) $G$ is 5-connected and $G[S] = \overline{K_5}$. We next show that $d_1 \geqslant 5$. Let $x$ and $y$ be distinct vertices in $G_1 \backslash S$. By Menger's theorem, there exist five $x$-$S$ paths $P_1, P_2, \ldots, P_5$ in $G_1$ which have only the vertex $x$ in common. If all these paths have length 1, then, since there are at least four internally disjoint $y$-$S$ paths in $G_1 \backslash \{x\}$, by contracting these paths we deduce that $d_1 \geqslant 7$. We may now assume that $P_1$ has length at least 2. Let $V(P_1) \cap S = \{z\}$. As $\{x, z\}$ is not a separating set in $G$, there is a path $P$ from a vertex on $P_1 \backslash \{x, z\}$ to a vertex on some $P_i \backslash \{x\}$, $i \neq 1$, so that only the end vertices of $P$ belong to $\bigcup_{j=1}^{5} P_j$. By contracting a suitable subset of the edges of $P \cup P_1 \cup \cdots \cup P_5$ we deduce that $d_1 \geqslant 5$, as claimed.

By symmetry, $d_2 \geqslant 5$ and so $d_1 + d_2 \geqslant 10$. However, by (3), $d_1 + d_2 \leqslant 8$, which is a contradiction. □

**Lemma 4.2.** *There is no separating set $S$ with a vertex $x$ so that $G[S - x]$ is complete.*

**Proof.** Suppose that $G[S - x]$ is complete and let $G_1, G_2$ be as above. We may assume that $S$ is a minimal separating set. By Lemma 4.1, $|S| \geqslant 6$. If $|S| \geqslant 8$, by contracting $V(G_1) - S$ to $x$ and $V(G_2) - S$ to a new vertex, we get a $K_9$ minor, a contradiction. So we may assume that $|S| = 6$ or $|S| = 7$.

If $|S| = 6$, by (4), $5|S| \geqslant 25 + e(G[S]) - 2\delta(G[S]) \geqslant 25 + 10 + \delta(G[S]) - 2\delta(G[S])$, which implies that $G[S] = K_6$. By induction, we may assume $e(G_i) \leqslant 7|G_i| - 27$, $i = 1, 2$. Since $7n - 12 = 7n - 27 + 15 = e(G) + 15 = e(G_1) + e(G_2) \leqslant 7|G_1| - 27 + 7|G_2| - 27 = 7n - 12$, it follows that $e(G_i) = 7|G_i| - 27$, $i = 1, 2$. Since $K_{2,2,2,3,3}$ does not contain $K_6$ as a subgraph, by induction, $G_i > K_9$ or $G_i$ is a $(K_{1,2,2,2,2,2}, 6)$-cockade. Thus $G > K_9$ or $G$ is a $(K_{1,2,2,2,2,2}, 6)$-cockade, a contradiction.

If $|S| = 7$, by (4), $5|S| \geqslant 25 + e(G[S]) - 2\delta(G[S]) \geqslant 25 + 15 + \delta(G[S]) - 2\delta(G[S])$, which implies that $G[S]$ is isomorphic to $K_7$ or $K_7$ with an edge deleted. Let $e(G[S]) = 21 - t$, where $t = 0$ or 1. Suppose $e(G_1) \geqslant 7|G_1| - 27 - t$. Let $G'_1$ be obtained from $G$ by contracting $V(G_2) - S$ to $x$. Then $e(G'_1) = e(G_1) + t \geqslant 7|G'_1| - 27$. Since $G'_1$ contains a $K_7$ subgraph, it is not $K_{2,2,2,3,3}$ or a $(K_{1,2,2,2,2,2}, 6)$-cockade, and hence by induction, $G > G'_1 > K_9$. Thus $e(G_1) \leqslant 7|G_1| - 28 - t$. Similarly, we have $e(G_2) \leqslant 7|G_2| - 28 - t$. But now $e(G) = e(G_1) + e(G_2) - e(G[S]) \leqslant 7(n + 7) - 28 - t - 28 - t - 21 + t = 7n - 28 - t$, which is a contradiction. □

**Lemma 4.3.** *$\delta(N(x)) \geqslant 7$ for any $x \in V(G)$.*

**Proof.** Suppose $\delta(N(x)) \leqslant 6$. By (1) there exists a vertex $y \in N(x)$ such that $x$ and $y$ have exactly six common neighbors. Then $e(G/xy) = 7(n - 1) - 27$. Since $G \not> K_9$, the minimality of $|G|$ implies that $G/xy$ is isomorphic to $K_{2,2,2,3,3}$ or is a $(K_{1,2,2,2,2,2}, 6)$-cockade. In either case, by Lemma 3.4 or Lemma 3.6, $G > K_9$ or $G = K_{2,2,2,3,3}$, a contradiction. □

**Lemma 4.4.** $\delta(G) \geqslant 9$.

**Proof.** Let $x \in V(G)$ be such that $d(x) = \delta(G) \leqslant 8$. By Lemma 4.3, $N(x) = K_8$ and so $G > N[x] = K_9$, a contradiction. □

**Lemma 4.5.** *If $G - N[x]$ is 2-connected or has at most two vertices, then $N(x) \neq K_{1,2,2,2,2}$.*

**Proof.** Suppose for a contradiction that $N(x) = K_{1,2,2,2,2}$. Let $V(N(x)) = \{y, z_1, z_2, z_3, z_4, w_1, w_2, w_3, w_4\}$ so that $y$ is adjacent to all vertices in $N(x) - y$ and $z_i w_i \notin E(G)$.

We next show that $z_i$ and $w_i$ have no common neighbor in $G - N[x]$ for $i = 1, 2, 3, 4$. To this end suppose that there exists a vertex $v \in V(G - N[x])$ adjacent to, say $z_1$ and $w_1$. Let $K = G - N[x] - v$. Then $K$ is not null by Lemma 4.4, because $G$ is not isomorphic to $K_{1,2,2,2,2,2}$. Since $G - N[x]$ has no cut vertex, $K$ is connected. If $z_i, w_i \in N(K)$ for some $i \in \{2, 3, 4\}$, then let $P$ be a path with ends $z_i$ and $w_i$ and interior in $K$. By contracting the edge $z_1 v$ and all but one of the edges of $P$ we see that $G > N[x] + z_1 v_1 + z_i w_i > K_9$, a contradiction. Thus we may assume that $w_2, w_3, w_4 \notin N(K)$. Let $i \in \{2, 3, 4\}$. It follows from Lemma 4.3 applied to $w_i$ that $v$ is adjacent to $w_i$. By Lemma 4.3 the edge $v w_i$ is in at least seven triangles, and hence $z_2, z_3, z_4$ are all adjacent to $v$. By Lemma 4.2 the set $N(K) - \{v\}$ is not complete, and hence $z_1, w_1 \in N(K)$. By contracting the edge $v w_2$ and all but one edge of a $z_1$-$w_1$ path with interior in $K$ we deduce that $G > N[x] + z_1 w_1 + z_2 w_2 > K_9$, a contradiction. This proves that the vertices $z_i$ and $w_i$ have no common neighbor in $G - N[x]$.

Let $u \in V(G) - N[x]$ be a neighbor of $z_1$. By Lemma 4.3 the vertices $u$ and $z_1$ have at least seven common neighbors, and so by the result of the previous paragraph $z_1$ has at least four neighbors in $G - N[x]$. By symmetry the same holds for all $z_i$ and $w_i$.

Let $H = G - \{x, y, z_3, w_3, z_4, w_4\}$. We next show that $H$ is 4-connected. Suppose for a contradiction that $S$ is a minimal separating set of at most three vertices in $H$. Since $G - N[x]$ has no cut vertex, $|S| \geqslant 2$ and $|S \cap N(x)| \leqslant 1$. If $|S \cap N(x)| = 1$, we may assume that $w_1 \in S$. Since $z_1 z_2, z_1 w_2 \in E(G)$, $z_1, z_2, w_2$ are in the same component of $H - S$. Denote this component by $K$. If $w_1 \notin S$, then also $w_1 \in K$. Since $z_2, w_2$ have at least four neighbors in $G - N[x]$, there exist $z_2'$ and $w_2'$ in $G - N[x] - S$ adjacent to $z_2$ and $w_2$, respectively. Clearly, $z_2'$ and $w_2'$ belong to $K$. As $G - N[x]$ has no cut vertex, $G - N[x]$ contains two independent $z_2'$-$w_2'$ paths. One of these paths is contained in $G[K \cup S]$.

Since $G$ is not contractible to $N[x] + z_2 w_2 + z_i w_i > K_9$ for $i = 3, 4$, there is no $z_i$-$w_i$ path in $G[K' \cup \{z_i, w_i\}]$, where $K' \neq K$ is another component of $H - S$. But this implies that $K'$ is separated from $x$ by $S$ and three pairwise adjacent vertices. We may assume that such three vertices are $y, w_3, w_4$. Since $G$ is 6-connected, $|S| = 3$. Let $S = \{s_1, s_2, s_3\}$, where $s_1 = w_1$ if $w_1 \in S$, and let $S' = S \cup \{y, w_3, w_4\}$. Then $S'$ is a minimal separating set of $G$. Let $H_1 = G[K' \cup S']$ and $H_2 = G - K'$. Let $d_1$ and $d_2$ be defined as in the paragraph prior to (3). Clearly, $K \cup \{x, z_3, z_4\}$ is contained in $H_2$. By Menger's theorem, there exist three disjoint paths between $\{x, w_1, w_2\}$ and $S$ in $G - \{y, w_3, w_4\}$. Now by contracting those paths, we get $d_2 + e(G[S']) = e(K_6) = 15$. By Lemma 4.2, $d_1 \geqslant 1$. By (3), $42 = 7|S'| \geqslant 27 + 1 + 15 = 43$, a contradiction. Thus $H$ is 4-connected.

Since $G$ is not contractible to $K_9$, it follows from Theorem 3.2 applied to the vertices $z_1, z_2, w_1, w_2$ that $e(H) \leqslant 3|H| - 7 = 3(n - 6) - 7$. Since for $i \in \{3, 4\}$ the vertices $z_i$ and $w_i$ have no common neighbor in $G - N[x]$, they together have at most $|G| - |N[x]| = n - 10$ neighbors in $G - N[x]$. The vertices $\{z_3, w_3, z_4, w_4\}$ are incident with 20 edges of $N(x) - y$. Thus

$$\begin{aligned} 7n - 27 = e(G) &\leqslant d(x) + d(y) - 1 + e(H) + 2(n - 10) + 20 \\ &\leqslant 9 + n - 2 + 3(n - 6) - 7 + 2(n - 10) + 20 = 6n - 18. \end{aligned}$$

It follows that $n \leqslant 9$, a contradiction. □

**Lemma 4.6.** *Let $x \in V(G)$ be such that $9 \leqslant d(x) \leqslant 13$. Then there is no component $K$ of $G - N[x]$ such that $N(K') \cap M \subseteq N(K)$ for every component $K'$ of $G - N[x]$, where $M$ is the set of vertices of $N(x)$ not adjacent to all other vertices of $N(x)$.*

**Proof.** Assume such a component $K$ exists. Among all vertices $x$ with $9 \leqslant d(x) \leqslant 13$ for which such a component exists, choose $x$ to be of minimal degree. We first prove that $M \subseteq N(K)$. Suppose for a contradiction that $M - N(K) \neq \emptyset$, and let $y \in M - N(K)$ be such that $d(y)$ is minimum. Clearly, $d(y) < d(x)$. Let $J$ be the component of $G - N[y]$ containing $K$. Since $d(y) < d(x)$ the choice of $x$ implies that $N(x) - N[y] \not\subseteq V(J)$, and hence some component $H$ of $N(x) - N[y]$ is disjoint from $N(K)$. We have $d_G(z) \geqslant d_G(y)$ for all $z \in V(H)$ by the choice of $y$. Let $t = |V(H)|$. Then $t \geqslant 2$, for otherwise the vertex $y$ and component $H$ contradict the choice of $x$. On the other hand $t \leqslant d(x) - d(y) \leqslant 13 - 9 = 4$. From Lemma 4.3 applied to $y$ we deduce that $N(y) \cap N(x)$ has minimum degree at least six. Let $L$ be the subgraph of $G$ induced by $(N[y] \cap N(x)) \cup V(H)$. Then the edge-set of $L$ consists of edges of $N(x) \cap N(y)$, edges incident with $y$, and edges incident with $V(H)$. Thus

$$\begin{aligned} e(L) &\geqslant 3(d(y) - 1) + d(y) - 1 + t(d(y) - 1) - \frac{1}{2}t(t - 1) \\ &\geqslant 6(d(y) + t) + (t - 2)d(y) - 4 - 7t - \frac{1}{2}t(t - 1) \geqslant 6|V(L)| - 20, \end{aligned}$$

because $d(y) \geqslant 9$ and $2 \leqslant t \leqslant 4$. Since $11 \leqslant |V(L)| \leqslant 13$ the graph $L$ is not a $(K_{2,2,2,2,2}, 5)$-cockade, and hence $N(x) > L > K_8$ by Theorem 1.2. Thus $G > K_9$, a contradiction. This proves that $M \subseteq N(K)$.

If $N(x) > K_7 \cup K_1$, then $N(x)$ has a vertex $y$ such that $N(x) - y > K_7$. If $y \notin M$, then $N(x) > K_8$. Otherwise, by contracting the connected set $V(K) \cup \{y\}$ we can contract $K_8$ onto $N(x)$. Thus in either case $G > K_9$, a contradiction. Thus by Lemma 3.7, we may assume that $N(x)$ satisfies properties (A) and (B).

If $G - N[x]$ is 2-connected or has at most two vertices, then by Lemma 4.5, we may assume that $N(x) \neq K_{1,2,2,2,2,2}$. Then by property (A) and Lemma 4.3 the set $N(x)$ has four distinct vertices $a_1, b_1, a_2, b_2$ such that $a_1a_2, b_1b_2 \notin E(G)$, $N(x) + a_1a_2 + b_1b_2 > K_8$ and for $i = 1, 2$ the vertex $a_i$ is adjacent to $b_i$, the vertices $a_i, b_i$ have at least two common neighbors in $G - N[x]$. Let $u_1, u_2$ (resp. $w_1, w_2$) be two distinct common neighbors of $a_1$ and $b_1$ (resp. $a_2$ and $b_2$) in $G - N[x]$. By Menger's Theorem, $G - N[x]$ contains two disjoint paths from $\{u_1, u_2\}$ to $\{w_1, w_2\}$ and so $G > N[x] + a_1a_2 + b_1b_2 > K_9$, a contradiction.

Thus we may assume that $G - N[x]$ has at least three vertices and is not 2-connected. If $G - N[x]$ is disconnected, let $H_1 = K$ and $H_2$ be another connected component of $G - N[x]$. If $G - N[x]$ has a cut-vertex, say $w$, let $H_1$ be a connected component of $G - N[x] - w$ and let $H_2 = G - N[x] - V(H_1)$. In either case, $H_1$ and $H_2$ are disjoint connected subgraphs of $G - N[x]$ such that $M \subseteq N(H_1) \cup N(H_2)$ (because we have shown that $M \subseteq N(K)$). For $i = 1, 2$ let $A_i = N(H_i) \cap N(x)$. By Lemmas 4.2 and 4.1, $A_i$ is not complete and $|A_i| \geqslant 5$ for $i = 1, 2$. By property (B), $A_1$ and $A_2$ satisfy properties (B1), (B2) or (B3).

Suppose first that $A_1$ and $A_2$ satisfy property (B1). Then there exist $a_i \in A_i$ such that $N(x) + \{a_1a : a \in A_1 - \{a_1\}\} + \{a_2a : a \in A_2 - \{a_2\}\} > K_8$. By contracting the connected sets $V(H_1) \cup \{a_1\}$ and $V(H_2) \cup \{a_2\}$ to single vertices, we see that $G > K_9$, a contradiction. Suppose next that $A_1$ and $A_2$ satisfy property (B2). Then there exist $a_1 \in A_1 - A_2$ and $a_2 \in A_2 - A_1$ such that $a_1a_2 \in E(G)$ and the vertices $a_1$ and $a_2$ have at most five common neighbors in $N(x)$. Thus $a_1, a_2 \in M$ by Lemma 4.3, and by another application of the same lemma there exists a common neighbor $u \in V(G) - N[x]$ of $a_1$ and $a_2$. But $a_1 \notin A_2$ and $a_2 \notin A_1$, and hence

$u \notin V(H_1) \cup V(H_2)$. Thus $G - N[x]$ is disconnected and $H_1 = K$. But then $a_2 \in M \subseteq N(K) = N(H_1)$, a contradiction. Thus we may assume that $A_1$ and $A_2$ satisfy (B3), and hence $A_i \subseteq A_{3-i}$ for some $i \in \{1, 2\}$. As $M \subseteq A_1 \cup A_2$, we have $M \subseteq N(H_{3-i})$. Since $A_i$ is not complete, let $a, b \in A_i$ be distinct and not adjacent. By property (B3), $N(x) + ab > K_7 \cup K_1$. Let $P$ be an $a$-$b$ path with interior in $H_i$. By contracting all but one of the edges of the path $P$ and by contracting $H_{3-i}$ similarly as above, we see that $G > K_9$, a contradiction. □

**Lemma 4.7.** *$G - N[x]$ is disconnected for every vertex $x \in V(G)$ of degree at most* 13.

**Proof.** If $G - N[x]$ is not null, then it is disconnected by Lemma 4.6. Thus we may assume that $x$ is adjacent to every other vertex of $G$. Let $H = G - x$. Then $e(H) = e(G) - n + 1 = 7n - 27 - n + 1 = 6|H| - 20$. By Theorem 1.2 applied to $H$ the graph $G$ has a $K_9$ minor or is a $(K_{1,2,2,2,2,2}, 6)$-cockade, a contradiction. □

**Lemma 4.8.** $\delta(G) \geqslant 10$.

**Proof.** Let $x \in V(G)$ be such that $d(x) = \delta(G) = 9$. By Lemma 4.3, $\delta(N(x)) \geqslant 7$, and hence $N(x) > K_{1,2,2,2,2,2}$. Let $K$, $K'$ be two components of $G - N[x]$. By Lemma 4.2, $N(K)$ and $N(K')$ contain distinct pairs of nonadjacent vertices of $N(x)$, say $a, b$ and $c, d$, respectively. Thus $G > N[x] + ab + cd > K_9$ by the existence of internally disjoint $a$-$b$ and $c$-$d$ paths with interiors in $K$, $K'$, respectively, a contradiction. □

**Lemma 4.9.** *Let $x \in V(G)$ be such that $10 \leqslant d(x) \leqslant 13$. Then there is no component $K$ of $G - N[x]$ such that $d_G(y) \geqslant 14$ for every vertex $y \in V(K)$.*

**Proof.** Assume that such a component $K$ exists. Let $G_1 = G - K$ and $G_2 = G[K \cup N(K)]$. Let $d_1$ be defined as in the paragraph prior to (3). Let $G_2'$ be a graph with $V(G_2') = V(G_2)$ and $e(G_2') = e(G_2) + d_1$ edges obtained from $G$ by contracting edges in $G_1$. By Lemma 4.8, $|G_2'| \geqslant 11$. If $e(G_2') \geqslant 7|G_2'| - 26$, then by induction $G > G_2' > K_9$, a contradiction. Thus

$$e(G_2) = e(G_2') - d_1 \leqslant 7|G_2| - 27 - d_1 = 7|N(K)| + 7|K| - 27 - d_1.$$

By contracting the edge $xz$, where $z \in N(K)$ has minimum degree in $N(K)$, we see that $d_1 \geqslant |N(K)| - d - 1$, and hence

$$e(G_2) \leqslant 6|N(K)| + 7|K| - 26 + d. \tag{a}$$

Let $t = e_G(N(K), K)$ and $d = \delta(N(K))$. We have $e(G_2) = e(K) + t + e(N(K))$ and

$$2e(K) \geqslant 14|K| - t, \tag{b}$$

and hence

$$e(G_2) \geqslant 7|K| + t/2 + d|N(K)|/2. \tag{c}$$

Since $N(x)$ has minimum degree at least seven, it follows that the subgraph $N(K)$ of $N(x)$ has minimum degree at least $7 - (d(x) - |N(K)|)$. Thus $d \geqslant 7 - (d(x) - |N(K)|) \geqslant |N(K)| - 6$. From (a) and (c) we get

$$-t/2 \geqslant 26 - 6|N(K)| + d(|N(K)| - 2)/2 \geqslant -18, \tag{d}$$

where the second inequality holds with equality only when $|N(K)| = 10$. Since $G$ is not contractible to $K_9$, we deduce from (b) by induction that $|K| < 9$. The inequality $e(K) \geqslant 7|K| - 18$ implies $|K| \leqslant 3$. But every vertex of $K$ has degree at least 14 and $N(K)$ is a proper subgraph of $N(x)$, and hence $|K| = 3$, $|N(K)| = 12$ and (d) holds with equality, contrary to our earlier observation that (d) holds with equality only when $|N(K)| = 10$. □

By Lemma 4.8 and the fact that $e(G) = 7n - 27$ there is a vertex $x$ of degree 10, 11, 12 or 13 in $G$. Choose such a vertex $x$ so that $G - N[x]$ has a component $K$ of minimum order. Then choose a vertex $y \in V(K)$ of least degree in $G$. Thus $10 \leqslant d_G(y) \leqslant 13$ by Lemmas 4.8 and 4.9. Let $L$ be the component of $G - N[y]$ containing $x$. We claim that $N(L)$ contains all vertices of $N(y)$ that are not adjacent to all other vertices of $N(y)$. Indeed, let $z \in N(y)$ be not adjacent to some vertex of $N(y) - \{z\}$. We may assume that $z \notin N(x)$, for otherwise $z \in N(L)$. Thus $z \in V(K)$, and hence $d_G(z) \geqslant d_G(y)$ by the choice of $y$. Thus $z$ has a neighbor $z' \in N[x] \cup K - N[y]$. Then $z' \in V(L)$, for otherwise the component of $G - N[y]$ containing $z'$ would be a proper subgraph of $K$. Thus $z \in N(L)$. This proves our claim that $N(L)$ contains all vertices $z$ as above, contrary to Lemma 4.6. This contradiction completes the proof of Theorem 1.3.

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