Some remarks on interval colorings of complete tripartite and biregular graphs

Puning Jing, Zhengke Miao*
Department of Mathematics and Statistics
Jiangsu Normal University
Xuzhou 221116, China
and
Zi-Xia Song
Department of Mathematics
University of Central Florida
Orlando, FL 32816, USA

January 3, 2019

Abstract

A proper edge-coloring of a graph with positive integers is an *interval coloring* if the colors on the edges incident to any vertex are consecutive. It is NP-complete to determine whether a graph has an interval coloring. A bipartite graph is (a, b)-*biregular* if every vertex in one part has degree a and every vertex in the other part has degree b. It has been conjectured that all such graphs have interval colorings. In this paper, we obtain several sufficient conditions for complete tripartite graphs to have interval colorings. In particular, we consider the case of complete tripartite graphs where one part has size 2. We also obtain two results on interval colorings of (a, b)-biregular graphs by extending known proof techniques.

Keywords: interval edge coloring, edge coloring, biregular graph, path factor

1 Introduction

All graphs in this paper are finite and simple. For any positive integer k, let $[k] := \{1, 2, \ldots, k\}$. For a graph G, let $c : E(G) \to [k]$ be a proper edge-coloring of G. For any $v \in V(G)$, let $c(v)$ denote the set of colors on the edges incident to v under the coloring c. Then c is an *interval k-coloring* of G (or G has an *interval k-coloring*) if for all $v \in V(G)$, $c(v)$ is an interval of integers, i.e. the colors in $c(v)$ are consecutive; and G has an *interval coloring* or *consecutive coloring* if G admits an interval k-coloring for some positive integer k.

*Corresponding author.
Email addresses: jingpuning@163.com; zkmiao@jsnu.edu.cn; Zixia.Song@ucf.edu

The notion of interval colorings was introduced in 1987 by Asratian and Kamalian [3, 4], motivated by the problem of constructing school timetables without "gaps" for teachers and classes, namely, the lectures of each teacher and each class are scheduled in consecutive periods. Hansen [14] introduced another scenario: a school wishes to schedule parent-teacher conferences in time slots so that every person's conferences occur in consecutive slots. A solution exists if and only if the bipartite graph with vertices for the parents and teachers and edges for the required meetings has an interval coloring.

Every k-regular bipartite graph admits an interval k-coloring, since such a graph has a proper k-edge coloring. It is proved in [4] that if a graph G has an interval k-coloring, then G must have a proper $\Delta(G)$-edge coloring (obtained from the interval k-coloring by taking colors modulo $\Delta(G)$). Therefore, not every graph has an interval coloring. It is shown that deciding whether a graph has an interval coloring is NP-complete even for regular graphs [3, 4] and bipartite graphs [19]. The first example of a bipartite graph without an interval coloring was given in [19]. Since then, a number of smaller examples of bipartite graphs have been found (see [16]), in particular, one such example due to Erdős is constructed as follows: given a finite projective plane, say P, of order $q \geq 3$, let X and Y be the set of q^2+q+1 points and the set of q^2+q+1 lines in P, respectively. Let H be the bipartite graph with bipartition (X, Y) such that for any $x \in X$ and any $y \in Y$, $xy \in E(H)$ if and only if the point x lies on the line y in P. Let G be a bipartite graph obtained from H by adding a new vertex adjacent to all vertices in Y. One can easily see that G has no interval coloring.

Giaro [10] showed that one can decide in polynomial time whether bipartite graphs with maximum degree 4 have interval 4-colorings. Trees [4, 14], complete bipartite graphs [3, 14], grids [11], and bipartite outplanar graphs [6, 12] all have interval colorings. Hansen [14] showed that all bipartite graphs of maximum degree 3 have interval 4-colorings. Let a and b be positive integers. A bipartite graph $G = (X \cup Y, E)$ is (a, b)-*biregular* if $X \cup Y = V(G)$, $d(x) = a$ for each $x \in X$ and $d(y) = b$ for each $y \in Y$. It has been conjectured in [14, 16, 20] that:

Conjecture 1.1 *Every (a, b)-biregular graph has an interval coloring.*

By results of [14, 15], all $(2, b)$-biregular graphs admit interval coloring (the case when b is odd was proved independently by Kostochka [17]). The smallest unsolved case of Conjecture 1.1 is $(a, b) = (3, 4)$. Several sufficient conditions for a $(3, 4)$-biregular graph to admit an interval 6-coloring have been obtained [5, 9, 18, 21]. In particular, Pyatkin [18] proved that if a $(3, 4)$-biregular graph has a cubic subgraph covering all vertices of degree 4, then it has an interval 6-coloring. To state the main result obtained in [5], we need to introduce more notation.

Let $G = (X \cup Y, E)$ be a $(3, 4)$-biregular graph. We say that F is a *path-factor* of G if F is a spanning subgraph of G whose components are paths with two distinct endpoints in X; and that

F is a *path-factor with lengths in* $\{\ell_1, \ell_2, \ldots, \ell_k\}$ if each component of F has length ℓ_i for some $i \in \{1, 2, \ldots, k\}$, where $2 \leq \ell_1 \leq \ell_2 \leq \cdots \leq \ell_k$. Under these circumstances, if $k = 1$, we simply say that F is a P_{ℓ_1+1}-factor, i.e., every component of F is isomorphic to a path on $\ell_1 + 1$ vertices.

Theorem 1.2 ([5]) *If a* $(3, 4)$*-biregular graph has a path-factor with lengths in* $\{2, 4, 6, 8\}$*, then it has an interval* 6*-coloring.*

Asratian and Casselgren [2] proved that every $(3, 4)$-biregular graph has a path-factor. Casselgren [7] further proved that every $(3, 4)$-biregular graph has a path-factor with each path of length at most 22. Casselgren and Toft [8] recently proved that Conjecture 1.1 is true for $(3, 6)$-biregular graphs. They proved the following two results.

Theorem 1.3 ([8]) *Every* $(3, 6)$*-biregular graph has an interval* 7*-coloring.*

Corollary 1.4 ([8]) *If* G *is a* $(3, 9)$*-biregular graph having a cubic subgraph covering all vertices of degree* 9*, then* G *has an interval* 10*-coloring.*

Grzesik and Khachatrian [13] recently studied interval colorings of complete tripartite graphs. They proved the following.

Theorem 1.5 ([13]) *Let* $\ell \geq 2, m, n$ *be positive integers.*

(a) $K_{1,m,n}$ *has an interval coloring if and only if* $gcd(m + 1, n + 1) = 1$*.*

(b) $K_{\ell,m,n}$ *admits no interval coloring if* ℓ, m *and* n *are odd.*

(c) $K_{\ell,m,\ell+m}$ *admits an interval coloring.*

They also made the following conjecture.

Conjecture 1.6 ([13]) *Let* $\ell \leq m \leq n$ *be positive integers.*

(a) For $n > \ell + m$*,* $K_{\ell,m,n}$ *has an interval coloring if and only if* $K_{\ell,m,n-\ell-m}$ *has an interval coloring.*

(b) For $n \leq \ell + m$*,* $K_{\ell,m,n}$ *has an interval coloring if and only if* $\ell + m + n$ *is even.*

In this paper, we first study interval colorings of complete tripartite graphs in Section 2. We obtain several sufficient conditions for a complete tripartite graph to admit an interval coloring, in particular, we consider the case of complete tripartite graphs where one part has size 2. We then study interval colorings of (a, b)-biregular graphs in Section 3. Using proof techniques developed in [5, 9, 1, 8], we obtain two results on interval colorings of (a, b)-biregular graphs.

2 Interval coloring complete tripartite graphs

In this section, we study interval colorings of complete tripartite graphs. We provide several sufficient conditions for such graphs to admit internal colorings. In particular, we consider the case of complete tripartite graphs where one part has size 2. It is worth noting that Theorem 2.1 below generalizes Theorem 10 from [13].

Theorem 2.1 *Let $\ell \geq 2$, $m \geq 2$, $k \geq 1$ be integers. Then $K_{\ell,\, m,\, k(\ell+m)}$ has an interval p-coloring, where $p = (k+1)(\ell+m) - 1$.*

Proof. Let ℓ, m, k, p be given. Let U, V, W be the three parts of $G := K_{\ell,m,k(\ell+m)}$ with $U := \{u_1, u_2, ..., u_\ell\}$, $V := \{v_1, v_2, ..., v_m\}$ and W being partitioned into $W_1, \ldots, W_k$ such that for all $q \in [k]$, $W_q := \{w_1^q, \ldots, w_{\ell+m}^q\}$. Let c be a p-edge-coloring of G defined as follows.

We first color the edges between U and V in G. For all $i \in [\ell]$ and $j \in [m]$, let $c(u_i v_j) := i + j - 1$. We then color the edges between U and W in G. For all $q \in [k]$ and $i \in [\ell]$, let $c(u_i w_j^q) := q(\ell+m) + i + j - 1$ when $j \in [m]$ and $c(u_i w_j^q) := (q-1)(\ell+m) + i + j - 1$ when $j \in \{m+1, \ldots, \ell+m\}$. Finally, we color the edges between V and W in G. For all $i \in [m]$, $q \in [k]$ and $j \in [\ell+m]$, let $c(v_i w_j^q) = (q-1)(\ell+m) + \ell + i + j - 1$.

It can be checked that for all $i \in [\ell]$, $c(u_i) = \{i, \ldots, k(\ell+m)+m+i-1\}$; for all $j \in [m]$, $c(v_j) = \{j, \ldots, k(\ell+m)+\ell+j-1\}$; and for all $q \in [k]$, $c(w_s^q) = \{(q-1)(\ell+m)+\ell+s, \ldots, q(\ell+m)+\ell+s-1\}$ when $s \in [m]$, and $c(w_s^q) = \{(q-1)(\ell+m)+s, \ldots, q(\ell+m)+s-1\}$ when $s \in \{m+1, \ldots, \ell+m\}$. Hence c is an interval p-coloring of G and so $K_{\ell,m,k(\ell+m)}$ admits an interval p-coloring, where $p = (k+1)(\ell+m) - 1$. ∎

Theorem 2.2 *For all $k \geq 1$, $K_{2,\, 3k+1,\, 3k+4}$ has an interval $(6k+6)$-coloring.*

Proof. Let W, U, V be the three parts of $G := K_{2,3k+1,3k+4}$ with $W := \{w_1, w_2\}$, U being partitioned into $U_1, \ldots, U_{k+1}$ and V being partitioned into $V_1, \ldots, V_{k+1}$ such that $U_{k+1} = \{u_1^{k+1}\}$, $V_{k+1} := \{v_1^{k+1}, v_2^{k+1}, v_3^{k+1}, v_4^{k+1}\}$, and for all $q \in [k]$, $U_q := \{u_1^q, u_2^q, u_3^q\}$ and $V_q := \{v_1^q, v_2^q, v_3^q\}$. We define a $(6k+6)$-edge-coloring c of G below.

We first color the edges between W and $U \cup V$ in G. For all $q \in [k]$ and $j \in [3]$, let $c(w_1 u_j^q) := 3q+j-3$, $c(w_1 u_1^{k+1}) := 3k+1$, $c(w_1 v_j^q) := 3k+3q+j-1$, $c(w_1 v_j^{k+1}) := 6k+2+j$, $c(w_1 v_4^{k+1}) := 3k+2$, $c(w_2 u_j^q) := 3q + j - 2$, $c(w_2 u_1^{k+1}) := 3k + 2$, $c(w_2 v_j^q) := 3k + 3q + j + 1$, $c(w_2 v_1^{k+1}) := 6k + 5$, $c(w_2 v_2^{k+1}) := 3k + 3$, $c(w_2 v_3^{k+1}) := 3k + 4$, and $c(w_2 v_4^{k+1}) := 1$. It can be easily checked that $c(w_1) = c(w_2) = \{1, 2, \ldots, 6k+5\}$.

We then color the edges between $\{u_1^k, u_2^k, u_3^k, u_1^{k+1}\}$ and V in G. For all $\ell \in [k]$ and $j \in [3]$, let $c(u_1^k v_j^\ell) := 3\ell + 3k + j - 3$, $c(u_1^k v_j^{k+1}) := 6k + j$, $c(u_2^k v_j^\ell) := 3\ell + 3k + j - 2$, $c(u_2^k v_j^{k+1}) := 6k + j + 1$, $c(u_3^k v_1^\ell) := 3\ell + 3k + 3$, $c(u_3^k v_2^\ell) := 3\ell + 3k + 4$, $c(u_3^k v_3^\ell) := 3\ell + 3k - 1$, $c(u_3^k v_1^{k+1}) := 3k + 3$, $c(u_3^k v_2^{k+1}) := 3k + 4$, $c(u_3^k v_3^{k+1}) := 6k + 2$, $c(u_1^{k+1} v_j^\ell) := 3k + 3\ell + j$, $c(u_1^{k+1} v_j^{k+1}) := 6k + j + 3$, $c(u_1^k v_4^{k+1}) := 3k$, $c(u_2^k v_4^{k+1}) := 3k+1$, $c(u_3^k v_4^{k+1}) := 3k-1$, and $c(u_1^{k+1} v_4^{k+1}) := 3k+3$. It follows that $c(u_1^k) = \{3k-2, \ldots, 6k+3\}$, $c(u_2^k) = c(u_3^k) = \{3k-1, \ldots, 6k+4\}$, $c(u_1^{k+1}) = \{3k+1, \ldots, 6k+6\}$. In particular, when $k = 1$, this yields an interval 12-coloring of G.

When $k \geq 2$, we color the edges between V and $U_1 \cup \cdots \cup U_{k-1}$ in G as follows. For all $\ell \in [k]$, $q \in [k-1]$ and $j \in [3]$, $c(v_j^\ell u_1^q) := 3\ell + 3q + j - 3$, $c(v_j^\ell u_2^q) := 3\ell + 3q + j - 2$, $c(v_j^\ell u_1^{k+1}) := 3k + 3\ell + j$, $c(v_1^\ell u_3^q) = 3\ell + 3q$, $c(v_2^\ell u_3^q) = 3\ell + 3q + 1$, $c(v_j^{k+1} u_1^q) := 3k + 3q + j$, $c(v_j^{k+1} u_2^q) := 3k + 3q + j + 1$, $c(v_j^{k+1} u_1^{k+1}) := 6k + 3 + j$, $c(v_1^{k+1} u_3^q) = 3k + 3q + 3$, $c(v_2^{k+1} u_3^q) = 3k + 3q + 4$, $c(v_3^\ell u_3^q) = 3\ell + 3q - 1$, Finally, let $c(v_4^{k+1} u_1^q) := 3q$, $c(v_4^{k+1} u_2^q) := 3q + 1$, $c(v_4^{k+1} u_3^q) := 3q - 1$. It can be easily checked that for all $q \in [k-1]$, $c(u_1^q) = \{3q-2, \ldots, 3k+3q+3\}$ and $c(u_2^q) = c(u_3^q) = \{3q-1, \ldots, 3k+3q+4\}$. For all $\ell \in [k]$, $c(v_1^\ell) = \{3\ell+1, \ldots, 3k+3\ell+3\}$, $c(v_2^\ell) = c(v_3^\ell) = \{3\ell+2, \ldots, 3k+3\ell+4\}$, $c(v_1^{k+1}) = c(v_2^{k+1}) = \{3k+3, \ldots, 6k+5\}$, $c(v_3^{k+1}) = \{3k+4, \ldots, 6k+6\}$, and $c(v_4^{k+1}) = \{1, \ldots, 3k+3\}$.

It follows from above that c is an interval $(6k+6)$-coloring of G, as desired. ∎

Theorem 2.3 *For all $k \geq 1$ and $m \geq 2$,*

(a) $K_{2,m,m}$ has an interval $(2m+1)$-coloring.

(b) $K_{2,m,m+k(m+2)}$ has an interval p-coloring, where $p = (k+1)(m+2) + m - 1$.

(c) $K_{2,2,2(m-1)}$ has an interval $(2m+1)$-coloring.

Proof. To prove (a), let $m \geq 2$ be given, and let W, U, V be the three parts of $G := K_{2,m,m}$ with $W := \{w_1, w_2\}$, $U := \{u_1, u_2, ..., u_m\}$ and $V := \{v_1, v_2, ..., v_m\}$. We next define an interval $(2m+1)$-coloring c of G. We first color the edges between U and V in G. For all $i \in \{2, 3, \ldots, m\}$ and $j \in [m]$, let $c(u_1 v_j) := m + 1 + j$ and $c(u_i v_j) := i + j - 1$. We then color the edges between W and $U \cup V$ in G. Let $c(w_1 u_1) := m$ and for all $i \in \{2, \ldots, m\}$, let $c(w_1 u_i) := i - 1$; and then for all $j \in [m]$, let $c(w_1 v_j) = c(w_2 u_j) := m + j$ and $c(w_2 v_j) := j$. One can easily check that c is an interval $(2m+1)$-coloring of G. Thus $K_{2,m,m}$ has an interval $(2m+1)$-coloring. This proves (a).

To prove (b), let k, m, p be given. Let W, U, V be the three parts of $G := K_{2,m,m+k(m+2)}$ with $W := \{w_1, w_2\}$, $U := \{u_1, u_2, ..., u_m\}$ and V being partitioned into $V_0, V_1, \ldots, V_k$ such that $V_0 := \{v_1, v_2, ..., v_m\}$ and for all $i \in [k]$, $V_i := \{v_1^i, \ldots, v_{m+2}^i\}$. Note that $G[U \cup W \cup V_0] = K_{2,m,m}$. Let c_0 be an interval $(2m+1)$-coloring of $G[U \cup W \cup V_0]$ given as in the proof of Theorem 2.3(a).

Note that $c_0(w_1) = c_0(w_2) = \{1, \ldots, 2m\}$, $c_0(u_1) = \{m, \ldots, 2m+1\}$, and for all $i \in \{2, \ldots, m\}$, $c_0(u_i) = \{i-1, \ldots, m+i\}$, and for all $j \in [m]$, $c_0(v_j) = \{j, \ldots, m+j+1\}$. We next construct an interval p-coloring c of G from c_0 by coloring the edges between $V_1 \cup \cdots \cup V_k$ and $U \cup W$.

For each $q \in [k]$, we color the edges between V_q and $U \cup W$ in G as follows: for all $j \in [m+2]$, let $c(u_1v_j^q) := q(m+2) + (m-1) + j$; for all $i \in \{2, \ldots, m\}$ and $j \in [m+2]$, let $c(u_iv_j^q) := q(m+2) + (i-2) + j$; and finally, for all $j \in [m-1]$, let $c(w_1v_j^q) := q(m+2) + m + j$, $c(w_2v_j^q) := q(m+2) + m + 1 + j$, $c(w_1v_m^q) := q(m+2) + 2m$, $c(w_1v_{m+1}^q) := q(m+2) + m - 1$, $c(w_1v_{m+2}^q) := q(m+2)+m$, $c(w_2v_m^q) := q(m+2)+m-1$, $c(w_2v_{m+1}^q) := q(m+2)+m$, $c(w_2v_{m+2}^q) := q(m+2) + m + 1$.

It can be easily checked that $c(w_1) = c(w_2) = \{1, \ldots, p-1\}$, $c(u_1) = \{m, \ldots, p\}$, for all $i \in \{2, \ldots, m\}$, $c(u_i) = \{i-1, \ldots, k(m+2)+m+i\}$, for all $j \in [m]$, $c(v_j) = \{j, \ldots, m+1+j\}$, and finally, for all $q \in [k]$ and $j \in [m-1]$, $c(v_j^q) = \{q(m+2)+j, \ldots, q(m+2)+m+1+j\}$, $c(v_m^q) = \{q(m+2)+m-1, \ldots, q(m+2)+2m\}$, $c(v_{m+1}^q) = \{q(m+2)+m-1, \ldots, q(m+2)+2m\}$, and $c(v_{m+2}^q) = \{q(m+2)+m, \ldots, q(m+2)+2m+1\}$. This proves that c is an internal p-coloring of G, as desired. This proves (b).

It remains to prove (c). By Theorem 2.3(a), the statement is true when $m = 2$. We may assume that $m \geq 3$. Then $2(m-1) = 4\ell$ or $2(m-1) = 4\ell + 2$ for some integer $\ell \geq 1$. Assume first that $2(m-1) = 4\ell + 2$. By Theorem 2.3(b), $K_{2,2,4\ell+2}$ has an interval $(2m+1)$-coloring. It remains to consider the case when $2(m-1) = 4\ell$. By Theorem 2.1, $K_{2,2,4\ell}$ has an interval $(2m+1)$-coloring.

This completes the proof of Theorem 2.3. ∎

3 Some remarks on interval colorings of biregular graphs

In this section, we study interval colorings of (a,b)-biregular graphs. Using a method developed in [5, 9], we begin with a result which provides a sufficient condition for any $(3,4)$-biregular graph to have an interval 6-coloring. Before doing so, we need one more notation.

Let $G = (X \cup Y, E)$ denote a $(3,4)$-biregular graph. For any path-factor F of G, we write $P \in F$ if P is a component of F. We use $P = x_1y_1 \ldots x_iy_ix_{i+1}$ to denote a path $P \in F$ with vertices $x_1, y_1, \ldots, x_i, y_i, x_{i+1}$ in order, where $x_1, \ldots, x_{i+1} \in X$, $y_1, \ldots, y_i \in Y$, and $2i = |P| - 1$. Note that $G \backslash E(F)$ consists of disjoint even cycles and even paths with both endpoints in X. Thus $\chi'(G \backslash E(F)) = 2$. Furthermore, for any $P = x_1y_1 \ldots x_iy_ix_{i+1} \in F$ with $i \geq 2$, each of $x_2, \ldots, x_{i-1}$ is an endpoint of a path component of $G \backslash E(F)$.

Theorem 3.1 *Let $G = (X \cup Y, E)$ be a $(3,4)$-biregular graph. If G has a path-factor F such that $G\backslash E(F)$ has a proper 2-edge-coloring c with the property that for any $P = x_1y_1 \dots x_iy_ix_{i+1} \in F$ with $i \geq 4$, either $c(x_2) = \cdots = c(x_i)$ or there exists an integer s with $2 \leq s \leq i-1$ such that $c(x_2) = \cdots = c(x_s) \neq c(x_{s+1}) = \cdots = c(x_i)$, then G has an interval 6-coloring.*

Proof. Let G, F and c be given as in the statement. We may assume that the edges of $G\backslash E(F)$ are colored by colors in $\{3,4\}$ under the coloring c. We next extend c to $E(G)$ to obtain an interval 6-coloring of G by coloring the edges of each component of F using colors in $\{1,2,5,6\}$.

Let $P \in F$ be any path with vertices $x_1, y_1, \dots, x_i, y_i, x_{i+1}$ in order, where $2i = |P| - 1$. Then $i \geq 1$. Assume first that $i = 1$. Let $c(x_1y_1) = 5$ and $c(y_1x_2) = 2$. Then $c(x_1) = \{3,4,5\}$, $c(y_1) = \{2,3,4,5\}$ and $c(x_2) = \{2,3,4\}$. Next assume that $i \geq 2$ and $c(x_2) = \cdots = c(x_i)$. If $c(x_2) = \cdots = c(x_i) = \{3\}$, then color the edge x_1y_1 by color 5 and the edges of the subpath $y_1x_2 \dots x_iy_ix_{i+1}$ alternatively using colors in $\{1,2\}$ by letting $c(y_1x_2) = 2$. If $c(x_2) = \cdots = c(x_i) = \{4\}$, then color the edge x_1y_1 by color 2 and the edges of the subpath $y_1x_2 \dots x_iy_ix_{i+1}$ alternatively using colors in $\{5,6\}$ by letting $c(y_1x_2) = 5$. In either case, one can easily check that for any $v \in \{x_1, y_1, \dots, x_i, y_i, x_{i+1}\}$, all the edges incident to v are properly colored in G and $c(v)$ is an interval of integers.

It remains to consider the case when $i \geq 2$ and $c(x_2) = \cdots = c(x_s) \neq c(x_{s+1}) = \cdots = c(x_i)$ for some integer integer s with $2 \leq s \leq i-1$. Then $i \geq 3$. By renaming the vertices of P if necessary, we may further assume that $c(x_2) = \cdots = c(x_s) = \{3\}$ and $c(x_{s+1}) = \cdots = c(x_i) = \{4\}$. Let $c(x_1y_1) = 2$ and $c(y_sx_{s+1}) = 5$. Now color the edges of the subpath $x_1y_1x_2 \dots x_sy_s$ alternatively using colors in $\{1,2\}$ and the edges of the subpath $y_sx_{s+1} \dots x_iy_ix_{i+1}$ alternatively using colors in $\{5,6\}$. Clearly, $c(x_1) = \{2,3,4\}$, $c(x_2) = \cdots = c(x_s) = \{1,2,3\}$, $c(x_{s+1}) = \cdots = c(x_i) = \{4,5,6\}$, $c(x_{i+1}) = \{3,4,5\}$, and $c(y_1) = \cdots = c(y_{s-1}) = \{1,2,3,4\}$, $c(y_s) = \{2,3,4,5\}$, $c(y_{s+1}) = \cdots = c(y_i) = \{3,4,5,6\}$. By the arbitrary choice of P, we obtain an interval 6-coloring of G.

This completes the proof of Theorem 3.1. ∎

We next study interval colorings of (a, ka)-biregular graphs, where a, k are positive integers. Using a method developed in [1, 8], we prove the following result. Note that Proposition 3.2(c) generalizes Theorem 1.4.

Proposition 3.2 *Let $G = (X \cup Y, E)$ be an (a, ka)-biregular graph, where a, k are positive integers.*

(a) G has an interval ka-coloring if and only if it admits a decomposition into k edge-disjoint a-regular subgraphs covering Y.

(b) If $a = 3$, $k \geq 3$ and X can be partitioned into $X_1, X_2, \ldots, X_{k-1}$ such that $G[X_i \cup Y]$ is 3-regular for all $i \in [k-2]$, then G has an interval $(3k+1)$-coloring.

(c) If $a \geq 3$, $k = 3$ and there exists $X_1 \subset X$ such that $G[X_1 \cup Y]$ is a-regular and $G \backslash X_1$ has an interval $(2a+1)$-coloring, then G admits an interval $(3a+1)$-coloring.

Proof. Let G, k, and a be given as in the statement. To prove (a), assume G has an interval ka-coloring. Let c be an interval ka-coloring of G using colors $1, 2, \ldots, ka$. For every $j \in [k]$, let E_j be the set of edges of G colored by color $a(j-1)+1$. Since c is an interval coloring of G and G is (a, ka)-biregular, we see that $|E_j| = |Y|$ for all $j \in [k]$ and $V(E_i) \cap V(E_j) = Y$ for all $i, j \in [k]$ with $i \neq j$. Moreover, for all $j \in [k]$, $G_j := G[V(E_j)]$ must be a-regular and so $G_1, G_2, \ldots, G_k$ are k edge-disjoint a-regular subgraphs of G, each covering Y. Conversely, assume G has a decomposition into k edge-disjoint a-regular subgraphs, say $G_1, G_2, \ldots, G_k$, such that each G_j is a-regular and covers Y. We obtain an interval ka-coloring of G by properly coloring the edges of G_j with colors in $\{a(j-1)+1, \ldots, a(j-1)+a\}$ for all $j \in [k]$. This proves (a).

We next prove (b). Assume $a = 3$, $k \geq 3$ and X can be partitioned into $X_1, X_2, \ldots, X_{k-1}$ such that $G[X_i \cup Y]$ is 3-regular for all $i \in [k-2]$. We proceed the proof by induction on k. By Corollary 1.4, the statement is true when $k = 3$. So we may assume that $k \geq 4$. Let $H := G \backslash X_1$. Then H is a $(3, 3(k-1))$-biregular graph such that each of $H[X_2 \cup Y], \ldots, H[X_{k-2} \cup Y]$ is 3-regular. By the induction hypothesis, H has an interval $(3k-2)$-coloring, say c^*, using colors $1, 2, \ldots, 3k-2$. Clearly, $c^*(y) = \{1, 2, \ldots, 3k-3\}$ or $c^*(y) = \{2, 3, \ldots, 3k-2\}$ for all $y \in Y$. Let M_1, M_2, M_3 be pairwise disjoint perfect matchings of $G[X_1 \cup Y]$. Let c be obtained from c^* as follows:

$$c(xy) = \begin{cases} 3k-2 & \text{if} \quad xy \in M_1 \quad \textit{and} \quad c^*(y) = \{1, 2, \ldots, 3k-3\} \\ 3k+1 & \text{if} \quad xy \in M_1 \quad \textit{and} \quad c^*(y) = \{2, 3, \ldots, 3k-2\} \\ 3k-1 & \text{if} \quad xy \in M_2 \\ 3k & \text{if} \quad xy \in M_3 \\ c^*(e) & \text{if} \quad xy \in E(H) \end{cases}$$

Then c is an interval $(3k+1)$-coloring of G, as desired. This proves (b).

To prove (c), assume $a \geq 3$, $k = 3$ and there exists $X_1 \subset X$ such that $G[X_1 \cup Y]$ is a-regular and $G \backslash X_1$ has an interval $(2a+1)$-coloring. Clearly, $G \backslash X_1$ is $(a, 2a)$-biregular. By assumption, let c^* be an interval $(2a+1)$-edge-coloring of $G \backslash X_1$ using colors $1, 2, \ldots, 2a+1$. Clearly, $c^*(y) = \{1, 2, \ldots, 2a\}$ or $c^*(y) = \{2, 3, ..., 2a+1\}$ for all $y \in Y$. Let $M_1, M_2, \ldots, M_a$ be pairwise disjoint perfect matchings

of $G[X_1 \cup Y]$. Let c be an edge-coloring of G obtained from c^* as follows:

$$c(xy) = \begin{cases} 2a+1 & \text{if} \quad xy \in M_1 \quad \textit{and} \quad c^*(y) = \{1, 2, \ldots, 2a\} \\ 3a+1 & \text{if} \quad xy \in M_1 \quad \textit{and} \quad c^*(y) = \{2, 3, \ldots, 2a+1\} \\ 2a+i & \text{if} \quad xy \in M_i, \text{ where } i \in \{2, 3, \ldots, a\} \\ c^*(e) & \text{if} \quad xy \in E(G \backslash X_1) \end{cases}$$

Then c is an interval $(3a+1)$-coloring of G, as desired. This proves (c). ■

Acknowledgments. The authors thank both referees for their careful reading and many helpful comments.

The third author would like to thank Zhengke Miao and Jiangsu Normal University for hospitality and support during her visit in May 2016 and June 2017.

Puning Jing and Zhengke Miao are partially supported by the National Natural Science Foundation of China (11571149 and 11571168).

References

[1] A. S. Asratian and C. J. Casselgren, On interval edge colorings of (α, β)-biregular bipartite graphs, Discrete Math. 307 (2007) 1951–1956.

[2] A. S. Asratian and C. J. Casselgren, On path factors of $(3, 4)$-biregular bigraphs, Graphs and Combin. 24 (2008) 405–411.

[3] A. S. Asratian and R. R. Kamalian, Interval colorings of edges of a multigraph (in Russian), Appl Math 5 (1987) 25–34.

[4] A. S. Asratian and R. R. Kamalian, Investigation on interval edge-colorings of graphs, J. Combin. Theory, Ser. B 62 (1994) 34–43.

[5] A. S. Asratian, C. J. Casselgren, J. Vandenbussche and D. B. West, Proper path-factors and interval edge-coloring of $(3, 4)$-biregular bigraphs, J. Graph Theory 61(2009) 88–97.

[6] M. A. Axenovich, On interval colorings of planar graphs, Congr Numer 159 (2002) 77–94.

[7] C. J. Casselgren, A note on path factors of $(3, 4)$-biregular bipartite graphs, Electronic J. of Combin. 18, p#218, 2011.

[8] C. J. Casselgren and B. Toft, On Interval edge colorings of biregular bipartite graphs with small vertex degrees, J.Graph Theory 80 (2015) 83–97.

[9] C. J. Casselgren, Some results on interval edge colorings of bipartite graphs. Master Thesis, Linköping University, Linköping, Sweden, 2005.

[10] K. Giaro, The complexity of consecutive Δ-coloring of bipartite graphs: 4 is easy, 5 is hard, Ars Combinatoria 47 (1997) 287–298.

[11] K. Giaro and M. Kubale, Consecutive edge-colorings of complete and incomplete cartesian products of graphs, Congressus Numerantium 128 (1997) 143–149.

[12] K. Giaro and M. Kubale, Compact scheduling of zero-one time operations in multi-stage systems, Discrete Appl Math 145 (2004) 95–103.

[13] A. Grzesik and H. Khachatrian, Interval edge-colorings of $K_{\ell,m,n}$, Discrete Appl Math. 174 (2014) 140–145.

[14] H. M. Hansen, Scheduling with minimum waiting periods (in Danish), Master Thesis, Odense University, Odense, Denmark, 1992.

[15] D. Hanson, C. O. M. Loten and B. Toft, On interval colourings of bi-regular bipartite graphs, Ars Combin 50 (1998) 23–32.

[16] T. R. Jensen and B. Toft, *Graph Coloring Problems*, Wiley-Interscience, Wiley, New York, 1995.

[17] A. V. Kostochka, unpublished manuscript, 1995.

[18] A. V. Pyatkin, Interval coloring of $(3, 4)$-biregular bipartite graphs having large cubic subgraphs, J. Graph Theory 47 (2004) 122–128.

[19] S. V. Sevastjanov, Interval colorability of the edges of a bipartite graph (in Russian). Metody Diskretnogo Analiza 50 (1990) 61–72.

[20] M. Stiebitz, D. Scheide, B. Toft and L. M. Favrholdt, Graph Edge Coloring: Vizing's Theorem and Goldberg s Conjecture, Wiley Interscience, New York, 2012.

[21] F. Yang and X. Li, Interval coloring of $(3, 4)$-biregular bigraphs having two $(2, 3)$-biregular bipartite subgraphs, Appl Math Lett 24 (2011) 1574–1577.