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# Saturation numbers for Ramsey-minimal graphs

Martin Rolek, Zi-Xia Song *

*Department of Mathematics, University of Central Florida, Orlando, FL 32816, United States*

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ABSTRACT

Given graphs $H_1, \ldots, H_t$, a graph $G$ is $(H_1, \ldots, H_t)$-*Ramsey-minimal* if every $t$-coloring of the edges of $G$ contains a monochromatic $H_i$ in color $i$ for some $i \in \{1, \ldots, t\}$, but any proper subgraph of $G$ does not possess this property. We define $\mathcal{R}_{\min}(H_1, \ldots, H_t)$ to be the family of $(H_1, \ldots, H_t)$-Ramsey-minimal graphs. A graph $G$ is $\mathcal{R}_{\min}(H_1, \ldots, H_t)$-*saturated* if no element of $\mathcal{R}_{\min}(H_1, \ldots, H_t)$ is a subgraph of $G$, but for any edge $e$ in $\overline{G}$, some element of $\mathcal{R}_{\min}(H_1, \ldots, H_t)$ is a subgraph of $G + e$. We define $sat(n, \mathcal{R}_{\min}(H_1, \ldots, H_t))$ to be the minimum number of edges over all $\mathcal{R}_{\min}(H_1, \ldots, H_t)$-saturated graphs on $n$ vertices. In 1987, Hanson and Toft conjectured that $sat(n, \mathcal{R}_{\min}(K_{k_1}, \ldots, K_{k_t})) = (r-2)(n-r+2)+\binom{r-2}{2}$ for $n \ge r$, where $r = r(K_{k_1}, \ldots, K_{k_t})$ is the classical Ramsey number for complete graphs. The first non-trivial case of Hanson and Toft's conjecture for sufficiently large $n$ was settled in 2011, and is so far the only settled case. Motivated by Hanson and Toft's conjecture, we study the minimum number of edges over all $\mathcal{R}_{\min}(K_3, \mathcal{T}_k)$-saturated graphs on $n$ vertices, where $\mathcal{T}_k$ is the family of all trees on $k$ vertices. We show that for $n \ge 18$, $sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_4)) = \lfloor 5n/2 \rfloor$. For $k \ge 5$ and $n \ge 2k + (\lceil k/2 \rceil + 1)\lceil k/2 \rceil - 2$, we obtain an asymptotic bound for $sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_k))$ by showing that $\left(\frac{3}{2} + \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil\right) n - c \le sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_k)) \le \left(\frac{3}{2} + \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil\right) n + C$, where $c = \left(\frac{1}{2}\left\lceil \frac{k}{2} \right\rceil + \frac{3}{2}\right) k - 2$ and $C = 2k^2 - 6k + \frac{3}{2} - \left\lceil \frac{k}{2} \right\rceil \left(k - \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil - 1\right)$.



## 1. Introduction

All graphs considered in this paper are finite and without loops or multiple edges. For a graph $G$, we will use $V(G)$ to denote the vertex set, $E(G)$ the edge set, $|G|$ the number of vertices, $e(G)$ the number of edges, $\delta(G)$ the minimum degree, $\Delta(G)$ the maximum degree, and $\overline{G}$ the complement of $G$. Given vertex sets $A, B \subseteq V(G)$, we say that $A$ is *complete to* (resp. *anti-complete to*) $B$ if for every $a \in A$ and every $b \in B$, $ab \in E(G)$ (resp. $ab \notin E(G)$). The subgraph of $G$ induced by $A$, denoted $G[A]$, is the graph with vertex set $A$ and edge set $\{xy \in E(G) : x, y \in A\}$. We denote by $B \setminus A$ the set $B - A$, $e_G(A, B)$ the number of edges between $A$ and $B$ in $G$, and $G \setminus A$ the subgraph of $G$ induced on $V(G) \setminus A$, respectively. If $A = \{a\}$, we simply write $B \setminus a$, $e_G(a, B)$, and $G \setminus a$, respectively. For any edge $e \in E(\overline{G})$, we use $G + e$ to denote the graph obtained from $G$ by adding the new edge $e$. The *join* $G \vee H$ (resp. *union* $G \cup H$) of two vertex disjoint graphs $G$ and $H$ is the graph having vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$ (resp. $E(G) \cup E(H)$). Given two isomorphic graphs $G$ and $H$, we may (with a slight but common abuse of notation) write $G = H$. For an integer $t \ge 1$ and a graph $H$, we define $tH$ to be the union of $t$ disjoint copies of $H$. We use $K_n$, $K_{1,n-1}$, $C_n$, $P_n$ and $T_n$ to denote the complete graph, star, cycle, path and a tree on $n$ vertices, respectively.

Given graphs $G, H_1, \ldots, H_t$, we write $G \to (H_1, \ldots, H_t)$ if every $t$-edge-coloring of $G$ contains a monochromatic $H_i$ in color $i$ for some $i \in \{1, 2, \ldots, t\}$. The classical *Ramsey number* $r(H_1, \ldots, H_t)$ is the minimum positive integer $n$ such that

* Corresponding author.
*E-mail addresses:* msrolek@wm.edu (M. Rolek), Zixia.Song@ucf.edu (Z.-X. Song).

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$K_n \to (H_1, \ldots, H_t)$. A graph $G$ is $(H_1, \ldots, H_t)$-*Ramsey-minimal* if $G \to (H_1, \ldots, H_t)$, but for any proper subgraph $G'$ of $G$, $G' \not\to (H_1, \ldots, H_t)$. We define $\mathcal{R}_{\min}(H_1, \ldots, H_t)$ to be the family of $(H_1, \ldots, H_t)$-Ramsey-minimal graphs. It is straightforward to prove by induction that a graph $G$ satisfies $G \to (H_1, \ldots, H_t)$ if and only if there exists a subgraph $G'$ of $G$ such that $G'$ is $(H_1, \ldots, H_t)$-Ramsey-minimal. Ramsey's theorem [18] implies that $\mathcal{R}_{\min}(H_1, \ldots, H_t) \neq \emptyset$ for all integers $t$ and all finite graphs $H_1, \ldots, H_t$. As pointed out in a recent paper of Fox, Grinshpun, Liebenau, Person, and Szabó [12], "it is still widely open to classify the graphs in $\mathcal{R}_{\min}(H_1, \ldots, H_t)$, or even to prove that these graphs have certain properties". Some properties of $\mathcal{R}_{\min}(H_1, \ldots, H_t)$ have been studied, such as the minimum degree $s(H_1, \ldots, H_t) := \min\{\delta(G) : G \in \mathcal{R}_{\min}(H_1, \ldots, H_t)\}$, which was first introduced by Burr, Erdős, and Lovász [4]. Recent results on $s(H_1, \ldots, H_t)$ can be found in [12,13]. For more information on Ramsey-related topics, the readers are referred to a very recent informative survey due to Conlon, Fox, and Sudakov [6].

In this paper, we study the following problem. A graph $G$ is $\mathcal{R}_{\min}(H_1, \ldots, H_t)$-*saturated* if no element of $\mathcal{R}_{\min}(H_1, \ldots, H_t)$ is a subgraph of $G$, but for any edge $e$ in $\overline{G}$, some element of $\mathcal{R}_{\min}(H_1, \ldots, H_t)$ is a subgraph of $G + e$. This notion was initiated by Nešetřil [16] in 1986 when he asked whether there are infinitely many $\mathcal{R}_{\min}(H_1, \ldots, H_t)$-saturated graphs. This was answered in the positive by Galluccio, Siminovits, and Simonyi [14]. We define $sat(n, \mathcal{R}_{\min}(H_1, \ldots, H_t))$ to be the minimum number of edges over all $\mathcal{R}_{\min}(H_1, \ldots, H_t)$-saturated graphs on $n$ vertices. This notion was first discussed by Hanson and Toft [15] in 1987 when $H_1, \ldots, H_t$ are complete graphs. They proposed the following conjecture.

**Conjecture 1.1.** *Let $r = r(K_{k_1}, \ldots, K_{k_t})$ be the classical Ramsey number for complete graphs. Then*

$$sat(n, \mathcal{R}_{\min}(K_{k_1}, \ldots, K_{k_t})) = \begin{cases} \binom{n}{2} & n < r \\ (r-2)(n-r+2) + \binom{r-2}{2} & n \geq r \end{cases}$$

Chen, Ferrara, Gould, Magnant, and Schmitt [5] proved that $sat(n, \mathcal{R}_{\min}(K_3, K_3)) = 4n - 10$ for $n \geq 56$. This settles the first non-trivial case of Conjecture 1.1 for sufficiently large $n$, and is so far the only settled case. Ferrara, Kim, and Yeager [11] proved that $sat(n, \mathcal{R}_{\min}(m_1K_2, \ldots, m_tK_2)) = 3(m_1 + \cdots + m_t - t)$ for $m_1, \ldots, m_t \geq 1$ and $n > 3(m_1 + \cdots + m_t - t)$. The problem of finding $sat(n, \mathcal{R}_{\min}(K_3, T_k))$ was also explored in [5].

**Proposition 1.2.** *Let $k \geq 2$ and $t \geq 2$ be integers. Then*

$$sat(n, \mathcal{R}_{\min}(K_t, T_k)) \leq n(t-2)(k-1) - (t-2)^2(k-1)^2 + \binom{(t-2)(k-1)}{2} + \left\lfloor \frac{n}{k-1} \right\rfloor \binom{k-1}{2} + \binom{r}{2},$$

*where $r = n \pmod{k-1}$.*

It was conjectured in [5] that the upper bound in Proposition 1.2 is asymptotically correct. Note that there is only one tree on three vertices, namely, $P_3$. A slightly better result was obtained for $\mathcal{R}_{\min}(K_3, P_3)$-saturated graphs in [5].

**Theorem 1.3.** *For $n \geq 11$, $sat(n, \mathcal{R}_{\min}(K_3, P_3)) = \lfloor \frac{5n}{2} \rfloor - 5$.*

Motivated by Conjecture 1.1, we study the following problem. Let $\mathcal{T}_k$ be the family of all trees on $k$ vertices. Instead of fixing a tree on $k$ vertices as in Proposition 1.2, we will investigate $sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_k))$, where a graph $G$ is $(K_3, \mathcal{T}_k)$-*Ramsey-minimal* if for any 2-coloring $c : E(G) \to \{\text{red}, \text{blue}\}$, $G$ has either a red $K_3$ or a blue tree $T_k \in \mathcal{T}_k$, and we define $\mathcal{R}_{\min}(K_3, \mathcal{T}_k)$ to be the family of $(K_3, \mathcal{T}_k)$-Ramsey-minimal graphs. By Theorem 1.3, we see that $sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_3)) = \lfloor 5n/2 \rfloor - 5$ for $n \geq 11$. In this paper, we prove the following two main results. We first establish the exact bound for $sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_4))$ for $n \geq 18$, and then obtain an asymptotic bound for $sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_k))$ for all $k \geq 5$ and $n \geq 2k + (\lceil k/2 \rceil + 1)\lceil k/2 \rceil + 2$.

**Theorem 1.4.** *For $n \geq 18$, $sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_4)) = \lfloor \frac{5n}{2} \rfloor$.*

**Theorem 1.5.** *For any integers $k \geq 5$ and $n \geq 2k + (\lceil k/2 \rceil + 1)\lceil k/2 \rceil - 2$, there exist constants $c = \left(\frac{1}{2}\left\lceil \frac{k}{2} \right\rceil + \frac{3}{2}\right)k - 2$ and $C = 2k^2 - 6k + \frac{3}{2} - \left\lceil \frac{k}{2} \right\rceil \left(k - \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil - 1\right)$ such that*

$$\left(\frac{3}{2} + \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil\right) n - c \leq sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_k)) \leq \left(\frac{3}{2} + \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil\right) n + C.$$

The constants $c$ and $C$ in Theorem 1.5 are both quadratic in $k$. We believe that the true value of $sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_k))$ is closer to the upper bound in Theorem 1.5. To establish the desired lower and upper bounds for each of Theorems 1.4 and 1.5, we need to introduce more notation and prove a useful lemma (see Lemma 1.6). Given a graph $H$, a graph $G$ is $H$-*free* if $G$ does not contain $H$ as a subgraph. For a graph $G$, let $c : E(G) \to \{\text{red}, \text{blue}\}$ be a 2-edge-coloring of $G$ and let $E_r$ and $E_b$ be the color classes of the coloring $c$. We use $G_r$ and $G_b$ to denote the spanning subgraphs of $G$ with edge sets $E_r$ and $E_b$, respectively.

We define $c$ to be a *bad* 2-*coloring* of $G$ if $G$ has neither a red $K_3$ nor a blue $T_k \in \mathcal{T}_k$, that is, if $G_r$ is $K_3$-free and $G_b$ is $T_k$-free for any $T_k \in \mathcal{T}_k$. For any $v \in V(G)$, we use $d_r(v)$ and $N_r(v)$ to denote the degree and neighborhood of $v$ in $G_r$, respectively. Similarly, we define $d_b(v)$ and $N_b(v)$ to be the degree and neighborhood of $v$ in $G_b$, respectively.

**Remark.** One can see that if $G$ is $\mathcal{R}_{\min}(K_3, \mathcal{T}_k)$-saturated, then $G$ admits at least one bad 2-coloring but, for any edge $e \in E(\overline{G})$, $G + e$ admits no bad 2-coloring.

We will utilize the following Lemma 1.6(a) to force a unique bad 2-coloring of certain graphs in order to establish an upper bound for $sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_k))$. Lemma 1.6(b) and (c) will be applied to establish a lower bound for $sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_k))$.

**Lemma 1.6.** *For any integer $k \geq 3$, let $c : E(G) \to \{red, blue\}$ be a bad 2-coloring of a graph $G$ on $n \geq k + 2$ vertices.*

(a) *If $e \in E(G)$ belongs to at least $2k - 3$ triangles in $G$, then $e \in E_b$.*
(b) *If $G$ is $\mathcal{R}_{\min}(K_3, \mathcal{T}_k)$-saturated and $D_1, \ldots, D_p$ are the components of $G_b$ with $|D_i| < k/2$ for all $i \in \{1, \ldots, p\}$, then $p \leq 2$. Moreover, if $p = 2$, then $V(D_1)$ is complete to $V(D_2)$ in $G_r$.*
(c) *If $G$ is $\mathcal{R}_{\min}(K_3, \mathcal{T}_k)$-saturated, and among all bad 2-colorings of $G$, $c$ is chosen so that $|E_r|$ is maximum, then $\Delta(G_r) \leq n - 3$ and $G_r$ is 2-connected.*

**Proof.** To prove (a), suppose that there exists an edge $e = uv \in E_r$ such that $e$ belongs to at least $2k - 3$ triangles in $G$. Since $G_r$ is $K_3$-free, we see that either $d_b(u) \geq k - 1$ or $d_b(v) \geq k - 1$. In either case, $G_b$ contains $K_{1,k-1}$ as a subgraph, a contradiction.

To prove (b), let $D_1, \ldots, D_p$ be given as in (b). We next show that $p \leq 2$. Since $G$ is $\mathcal{R}_{\min}(K_3, \mathcal{T}_k)$-saturated, we see that, for any edge $e$ in $\overline{G}$, $G + e$ admits no bad 2-coloring. We claim that, for any $i, j \in \{1, \ldots, p\}$ with $i \neq j$, $V(D_i)$ is complete to $V(D_j)$ in $G_r$. Suppose that there exist vertices $u \in V(D_i)$ and $v \in V(D_j)$ such that $uv \notin E_r$. Then $uv \notin E(G)$ and so we obtain a bad 2-coloring of $G + uv$ from $c$ by coloring the edge $uv$ blue, a contradiction. Thus $V(D_i)$ is complete to $V(D_j)$ in $G_r$ for any $i, j \in \{1, \ldots, p\}$ with $i \neq j$. Since $G_r$ is $K_3$-saturated, it follows that $p \leq 2$.

It remains to prove (c). By the choice of $c$, $G_r$ is $K_3$-free but $G_r + e$ contains a $K_3$ for any $e \in E(\overline{G_r})$, and $G_b$ is $T_k$-free for any $T_k \in \mathcal{T}_k$. Note that $G_b$ is disconnected and every component of $G_b$ contains at most $k - 1$ vertices. Since $G$ is $\mathcal{R}_{\min}(K_3, \mathcal{T}_k)$-saturated, we see that, for any edge $e$ in $\overline{G}$, $G + e$ admits no bad 2-coloring. Suppose that $\Delta(G_r) \geq n - 2$. Let $x \in V(G)$ with $d_r(x) = \Delta(G_r)$ and let $v$ be the unique non-neighbor of $x$ in $G_r$ if $d_r(x) = n - 2$. Since $G_r$ is $K_3$-free, we see that $N_r(x)$ is an independent set in $G_r$. By the choice of $c$, $v$ must be complete to $N_r(x)$ in $G_r$. Since $n \geq k+2$, we have $|N_r(x)| \geq k$. Let $u \in N_r(x)$ and let $H$ be the component of $G_b$ containing $u$. Then $|H| \leq k - 1$ and $V(H) \subset N_r(x)$. Let $w \in N_r(x) \setminus V(H)$. Clearly, $uw \notin E(G)$. We obtain a bad 2-coloring of $G + uw$ from $c$ by coloring the edge $uw$ red, and then recoloring all edges incident with $u$ in $G_r$ blue and all edges incident with $u$ in $G_b$ red, a contradiction. This proves that $\Delta(G_r) \leq n - 3$.

Finally, we show that $G_r$ is 2-connected. Suppose that $G_r$ is not 2-connected. Since $G_r$ is $K_3$-free but $G_r + e$ contains a $K_3$ for any $e \in E(\overline{G_r})$, we see that $G_r$ is connected and must have a cut vertex, say $u$. Since $\Delta(G_r) \leq n - 3$, $u$ has a non-neighbor, say $v$, in $G_r$. Let $G_1$ and $G_2$ be two components of $G_r \setminus u$ with $v \in V(G_2)$. Let $w \in V(G_1)$. By the choice of $c$, $wv \notin E_b$, otherwise we obtain a bad 2-coloring of $G$ from $c$ by recoloring the blue edge $wv$ red. Thus $wv \notin E(G)$ and then we obtain a bad 2-coloring of $G + wv$ from $c$ by coloring the edge $wv$ red, a contradiction. Therefore $G_r$ is 2-connected.

This completes the proof of Lemma 1.6. ■

The remainder of this paper is organized as follows. In Section 2, we discuss $K_3$-saturated graphs with a specified minimum degree and prove a structural result which we shall use in the proof of Theorem 1.4. We then prove Theorem 1.4 in Section 3 and Theorem 1.5 in Section 4.

## 2. $K_3$-saturated graphs

In this section we list known results and establish new ones on $K_3$-saturated graphs that we shall need to prove our main results.

Given a graph $H$, a graph $G$ is $H$-*saturated* if $G$ is $H$-free but, for any edge $e \in E(\overline{G})$, $G + e$ contains a copy of $H$ as a subgraph. We define $sat(n, H)$ to be the minimum number of edges over all $H$-saturated graphs on $n$ vertices. This notion was introduced by Erdős, Hajnal, and Moon [9] in 1964. Results on $H$-saturated graphs can be found in surveys by either Faudree, Faudree, and Schmitt [10] or Pikhurko [17]. In this section we are interested in the case when $H = K_t$. Erdős, Hajnal, and Moon [9] showed that if $G$ is a $K_t$-saturated graph on $n$ vertices, then $e(G) \geq (t - 2)n - \binom{t-1}{2}$. Moreover, they showed that the graph $K_{t-2} \vee \overline{K}_{n-t+2}$ is the unique $K_t$-saturated graph with $n$ vertices and $(t - 2)n - \binom{t-1}{2}$ edges. Notice that this extremal graph has minimum degree $t - 2$. One may ask: what is the minimum number of edges in a $K_t$-saturated graph with specified minimum degree? This was first studied by Duffus and Hanson [8] in 1986. They proved the following two results.

**Theorem 2.1.** *If $G$ is a $K_3$-saturated graph on $n \geq 5$ vertices with $\delta(G) = 2$, then $e(G) \geq 2n-5$ edges. Moreover, if $e(G) = 2n-5$, then $G$ can be obtained from $C_5$ by repeatedly duplicating vertices of degree 2.*

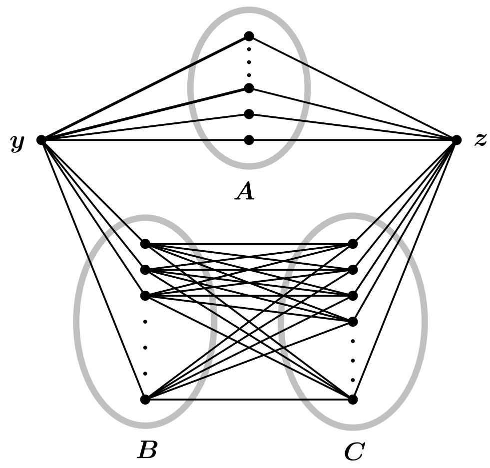


**Fig. 2.1.** The graph $J$.

**Theorem 2.2.** *If $G$ is a $K_3$-saturated graph on $n \geq 10$ vertices with $\delta(G) = 3$, then $e(G) \geq 3n - 15$. Moreover, if $e(G) = 3n - 15$, then $G$ contains the Petersen graph as a subgraph.*

Alon, Erdős, Holzman, and Krivelevich [1] showed that any $K_4$-saturated graph on $n \geq 11$ vertices with minimum degree 4 has at least $4n - 19$ edges. This has recently been generalized by Bosse, the second author, and Zhang [3] by showing that any $K_t$-saturated graph on $n \geq t + 7$ vertices with minimum degree $t \geq 3$ has at least $tn - \binom{t+1}{2} - 9$ edges. Moreover, they showed that the graphs $K_{t-3} \vee H$ are the only $K_t$-saturated graphs with $n$ vertices and $tn - \binom{t+1}{2} - 9$ edges, where $H$ is a $K_3$-saturated graph on $n - t + 3 \geq 10$ vertices with $\delta(H) = 3$. Theorem 2.3 is a result of Day [7] on $K_t$-saturated graphs with prescribed minimum degree. It confirms a conjecture of Bollobás [2] when $t = 3$. It is worth noting that the constant $c$ given in Theorem 2.3 does not have a dependency on $t$. This is a consequence of the fact that every $K_t$-saturated graph has minimum degree at least $t - 2$.

**Theorem 2.3.** *For any integers $p \geq 1$ and $t \geq 3$, there exists a constant $c = c(p)$ such that if $G$ is a $K_t$-saturated graph on $n$ vertices with $\delta(G) \geq p$, then $e(G) \geq pn - c$.*

For our proof of Theorem 1.4, we will need a structural result on $K_3$-saturated graphs with minimum degree at most 2. The graph $J$ depicted in Fig. 2.1 is a $K_3$-saturated graph with minimum degree 2, where $A \neq \emptyset$ and either $B = C = \emptyset$ or $B \neq \emptyset$ and $C \neq \emptyset$; $A$, $B$ and $C$ are independent sets in $J$ and pairwise disjoint; $A$ is anti-complete to $B \cup C$ and $B$ is complete to $C$; $N_J(y) = A \cup B$ and $N_J(z) = A \cup C$; and $|A| + |B| + |C| = |J| - 2$. It is straightforward to check that $e(J) = 2(|J| - 2) + |B||C| - |B| - |C| \geq 2|J| - 5$. Moreover, $e(J) = 2|J| - 5$ when $|B| = 1$ or $|C| = 1$. That is, $e(J) = 2|J| - 5$ when $J$ is obtained from $C_5$ by repeatedly duplicating vertices of degree 2. Lemma 2.4 yields a new proof of Theorem 2.1, and has been generalized for all $K_t$-saturated graphs with minimum degree at most $t - 1$ in [3].

**Lemma 2.4.** *Let $G$ be a $K_3$-saturated graph with $n$ vertices and $\delta(G) = \delta$.*

(a) *If $\delta = 1$, then $G = K_{1,n-1}$.*
(b) *If $\delta = 2$, then $G = J$, where the graph $J$ is depicted in Fig. 2.1. Moreover, $J = K_{2,n-2}$ when $B = C = \emptyset$.*
(c) *If $\delta \geq 3$, then $2e(G) \geq \max\{(\delta+1)n - \delta^2 - 1,\ (\delta+2)n - \delta(\delta+t) - 2\}$, where $t := \min\{d(v) : v$ is adjacent to a vertex of degree $\delta$ in $G\}$.*

**Proof.** Let $x \in V(G)$ be a vertex with $d(x) = \delta$. Since $G$ is $K_3$-saturated, we see that $G$ is connected and $K_3$-free. First assume that $d(x) = 1$. Let $y$ be the neighbor of $x$. If there exists a vertex $z \in V(G)$ such that $yz \notin E(G)$, then $G + xz$ is $K_3$-free, contrary to the fact that $G$ is $K_3$-saturated. Thus $y$ is complete to $V(G) \setminus \{y\}$. Clearly, $N(y)$ is an independent set because $G$ is $K_3$-free. Thus $G = K_{1,n-1}$. This proves (a).

Next assume that $d(x) = 2$. Let $N(x) = \{y, z\}$. Then $yz \notin E(G)$ because $G$ is $K_3$-free. We next show that $N(y) \cup N(z) = V(G) \setminus \{y, z\}$. Suppose there exists a vertex $w \in V(G)$ such that $wy, wz \notin E(G)$. Then $G + xw$ is $K_3$-free, contrary to the fact that $G$ is $K_3$-saturated. Hence $N(y) \cup N(z) = V(G) \setminus \{y, z\}$. Let $A := N(y) \cap N(z)$, $B := N(y) \setminus N(z)$, and $C := N(z) \setminus N(y)$. Then $|A| + |B| + |C| = n - 2$, and $A, B, C$ are pairwise disjoint. Clearly, $x \in A$, and either $B = C = \emptyset$ or $B \neq \emptyset$ and $C \neq \emptyset$ because $\delta(G) = 2$. Since $G$ is $K_3$-free, we see that $A, B, C$ are independent sets in $G$, and $A$ is anti-complete to $B \cup C$. We next show that $B$ must be complete to $C$ when $B \neq \emptyset$ and $C \neq \emptyset$. Suppose there exist vertices $b \in B$ and $c \in C$ such that $bc \notin E(G)$. Then $G + bc$ is $K_3$-free, a contradiction. Thus $G = J$, where $J$ is depicted in Fig. 2.1.

**Table 2.1**
Construction of the graph $J$ determined by $k$.

| $k$ | $e(J)$ | values of $|B|$ and $|C|$ with $|B| \le |C|$ |
|---|---|---|
| 5 | $2n-5$ | $|B|=1$ and $|C|\ge 1$ |
| 4 | $2n-4$ | $|B|=|C|=2$ or $|B|=|C|=0$ |
| 3 | $2n-3$ | $|B|=2$ and $|C|=3$ |
| 2 | $2n-2$ | $|B|=2$ and $|C|=4$ |
| 1 | $2n-1$ | $|B|=2$ and $|C|=5$ or $|B|=|C|=3$ |
| 0 | $2n$ | $|B|=2$ and $|C|=6$ |

It remains to prove (c). Let $\delta \ge 3$ and let $t$ be given as in (c). Then $d(x) \ge 3$. We first show that $2e(G) \ge (\delta+1)n-\delta^2-1$. Since $G$ is $K_3$-saturated, every vertex in $V(G)\setminus N[x]$ has at least one neighbor in $N(x)$, yielding $\sum_{v\in N(x)} d(v) \ge |V(G)\setminus N[x]| + d(x) = n-1$. Therefore

$$\begin{aligned} 2e(G) &= d(x) + \sum_{v\in N(x)} d(v) + \sum_{v\in V(G)\setminus N[x]} d(v) \\ &\ge \delta + n - 1 + \delta(n-\delta-1) \\ &\ge (\delta+1)n - \delta^2 - 1. \end{aligned}$$

We next show that $2e(G) \ge (\delta+2)n - \delta(\delta+t) - 2$. We may assume that there exists a vertex $y \in N(x)$ with $d(y) = t$. Notice that $x$ and $y$ have no common neighbor. Let $M := V(G)\setminus (N(x)\cup N(y))$. Then $|M| = n-\delta-t$. Since $G$ is $K_3$-saturated, each vertex in $M$ has at least one neighbor in $N(x)\setminus y$ and at least one neighbor in $N(y)\setminus x$. Thus $\sum_{v\in N(x)\setminus y} d(v) \ge n-t-1$, and $\sum_{v\in N(y)\setminus x} d(v) \ge n-\delta-1$. Then

$$\begin{aligned} 2e(G) &= d(x) + d(y) + \sum_{v\in N(x)\setminus y} d(v) + \sum_{v\in N(y)\setminus x} d(v) + \sum_{v\in M} d(v) \\ &\ge \delta + t + (n-t-1) + (n-\delta-1) + \delta(n-\delta-t) \\ &= (\delta+2)n - \delta(\delta+t) - 2. \end{aligned}$$

This completes the proof of Lemma 2.4. ∎

**Corollary 2.5.** *Let $G$ be a $K_3$-saturated graph on $n \ge 5$ vertices with $\delta(G) = 2$. If $e(G) = 2n-k$ for some $k \in \{0,1,2,3,4,5\}$, then $G = J$ with $|B||C| - |B| - |C| = 4-k$, where $A, B, C$, and $J$ are as depicted in Fig. 2.1 and the values of $|B|$ and $|C|$ are summarized in Table 2.1.*

**Proof.** Since $\delta(G) = 2$, by Lemma 2.4(b), $G = J$ with $e(G) = 2(n-2) + |B||C| - |B| - |C|$ and either $B = C = \emptyset$ or $B, C \ne \emptyset$, where $A, B, C$, and $J$ are as depicted in Fig. 2.1. We see that $|B||C| - |B| - |C| = 4-k$ because $e(G) = 2n-k$, where $k \in \{0,1,2,3,4,5\}$. Solving the resulting equation in each case of $k$ yields explicit constructions of $J$, which are summarized in Table 2.1. ∎

## 3. Proof of Theorem 1.4

We are now ready to prove Theorem 1.4. We first establish the desired upper bound for $sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_4))$ by constructing an $\mathcal{R}_{\min}(K_3, \mathcal{T}_4)$-saturated graph with the desired number of edges. Let $n \ge 8$ be an integer and let $H = (\lfloor n/2 \rfloor - 4)K_2$. When $n \ge 8$ is even, let $G_{even}$ be the graph obtained from $H$ by adding eight new vertices $y, z, y_1, y_2, y_3, z_1, z_2, z_3$, and then joining: $y$ to all vertices in $V(H)\cup\{y_1, y_2, y_3, z_1, z_2, z_3\}$; $z$ to all vertices in $V(H)\cup\{y_1, y_2, y_3, z_1, z_2\}$; $y_1$ to all vertices in $\{y_2, z_1, z_2, z_3\}$; $y_2$ to all vertices in $\{z_1, z_2, z_3\}$; $z_1$ to $z_2$; and $z_3$ to $y_3$. When $n$ is odd, let $G_{odd}$ be the graph obtained from $H$ by adding nine new vertices $y, z, y_1, y_2, y_3, y_4, z_1, z_2, z_3$, and then joining: $y$ to all vertices in $V(H)\cup\{y_1, z_1, z_2, z_3\}$; $z$ to all vertices in $V(H)\cup\{y_1, y_2, y_3, y_4, z_1, z_2, z_3\}$; $z_1$ to all vertices in $\{y_1, y_2, y_3, y_4, z_2\}$; $z_2$ to all vertices in $\{y_1, y_2, y_3, y_4\}$; $y_2$ to $y_3$; and $y_4$ to $z_3$. The graphs $G_{odd}$ and $G_{even}$ are depicted in Fig. 3.1. It can be easily checked that $e(G_{odd}) = (5n-1)/2$ and $e(G_{even}) = 5n/2$. We next show that $G_{odd}$ and $G_{even}$ are $\mathcal{R}_{\min}(K_3, \mathcal{T}_4)$-saturated.

One can easily check that the coloring $c: E(G) \to \{\text{red, blue}\}$ for each of $G_{odd}$ and $G_{even}$ given in Fig. 3.1 is a bad 2-coloring. We next show that $c$ is the unique bad 2-coloring for each of $G_{odd}$ and $G_{even}$. To find a bad 2-coloring for $G_{odd}$, by Lemma 1.6(a), the edges $zz_1, zz_2, z_1z_2$ must be colored blue and so all the other edges incident with $z, z_1, z_2$ must be red. Then $yy_1, y_2y_3, y_4z_3$ and all edges in $E(H)$ must be blue and all the other edges incident with $y$ must be red. This proves that $G_{odd}$ has a unique bad 2-coloring, as depicted in Fig. 3.1. To find a bad 2-coloring for $G_{even}$, by Lemma 1.6(a), $y_1y_2$ must be colored blue. We next show that $z_1z_2$ must be colored blue. Suppose that $z_1z_2$ is colored red. To avoid a red $K_3$, we may assume that $yz_1$ is colored blue. Then all edges $z_1y_1, z_1y_2, yy_1, yy_2$ must be red, and so $z_2y_1, z_2y_2$ must be blue, which then forces $y_1z$ to be red and $z_1z$ to be blue. Now the edges $z_3y$ and $z_3y_1$ must be colored red, which yields a red $K_3$ with vertices $y, z_3, y_1$. This proves that $z_1z_2$ must be colored blue. Similar to the argument for $G_{odd}$, one can see that the coloring of $G_{even}$, depicted in Fig. 3.1,

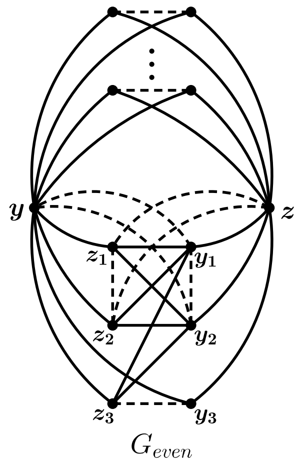


**Fig. 3.1.** Two $\mathcal{R}_{\min}(K_3, \mathcal{T}_4)$-saturated graphs with a unique bad 2-coloring, where dashed lines indicate blue and solid lines indicate red.

is the unique bad 2-coloring of $G_{even}$. It is straightforward to see that both $G_{odd}$ and $G_{even}$ are $\mathcal{R}_{\min}(K_3, \mathcal{T}_4)$-saturated, and so $sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_4)) \leq \lfloor 5n/2 \rfloor$. We next show that $sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_4)) \geq \lfloor 5n/2 \rfloor$.

Let $G$ be an $\mathcal{R}_{\min}(K_3, \mathcal{T}_4)$-saturated graph on $n \geq 18$ vertices. Then, for any edge $e \in E(\overline{G})$, $G + e$ has no bad 2-coloring. Suppose that $e(G) < 5n/2$ if $n$ is even and $e(G) < (5n-1)/2$ if $n$ is odd. Among all bad 2-colorings of $G$, let $c : E(G) \to \{\text{red, blue}\}$ be a bad 2-coloring of $G$ with $|E_r|$ maximum. By the choice of $c$, $G_r$ is $K_3$-saturated. Note that $G_b$ is disconnected and every component of $G$ is isomorphic to $K_1, K_2, P_3$ or $K_3$. By Lemma 1.6(c), we have

(1) $\Delta(G_r) \leq n - 3$ and $G_r$ is 2-connected.

We next show that

(2) $\delta(G_r) = 2$ and so $G_r = J$ with $A \neq \emptyset$, $B \neq \emptyset$, and $C \neq \emptyset$, where $J, A, B, C$ are depicted in Fig. 2.1.

**Proof.** By (1), $\delta(G_r) \geq 2$. Suppose that $\delta(G_r) \geq 3$. We next show that $e(G_r) \geq \lceil (5n-17)/2 \rceil$. This is trivially true if $\delta(G) \geq 5$. So we may assume that $3 \leq \delta(G_r) \leq 4$. By Theorem 2.2 applied to $G_r$ when $\delta(G_r) = 3$ and Lemma 2.4(c) applied to $G_r$ when $\delta(G_r) = 4$, we see that $e(G_r) \geq \lceil (5n-17)/2 \rceil$ because $n \geq 18$. By Lemma 1.6(b), $e(G_b) \geq \lceil (n-2)/2 \rceil$. Thus $e(G) = e(G_r) + e(G_b) \geq \lfloor 5n/2 \rfloor$, a contradiction. Hence $\delta(G_r) = 2$. By Lemma 2.4(b), $G_r = J$, where $J, A \neq \emptyset, B, C$ are depicted in Fig. 2.1. By (1), $B \neq \emptyset$ and $C \neq \emptyset$. ■

For the remainder of the proof, let $J, A, B, C$, and $y, z$ be given as in Fig. 2.1, where $A \neq \emptyset$, $B \neq \emptyset$, and $C \neq \emptyset$. By (2), $G_r = J$. We next show that

(3) $|B| \geq 2$ and $|C| \geq 2$.

**Proof.** Suppose that $|B| = 1$ or $|C| = 1$, say the latter. Let $u$ be the vertex in $C$. If $yz, yu \in E_b$, then $d_b(u) = 1$ because $G_b$ is $\mathcal{T}_4$-free. Now for any $w \in A$, we obtain a bad 2-coloring of $G + uw$ from $c$ by coloring the edge $uw$ red, and then recoloring the edge $zu$ blue. Thus either $yz \notin E(G)$ or $yu \notin E(G)$. We may assume that $yz \notin E(G)$. Then $yu \in E_b$, otherwise, we obtain a bad 2-coloring of $G + yu$ from $c$ by coloring the edge $yu$ blue, and then recoloring the edge $zu$ blue, and all the edges incident with $z$ and $u$ in $G_b$ red. Notice that $d_b(u) = 1$, for otherwise let $w \in A$ be the other neighbor of $u$ in $G_b$ and $v \in B$. Then $d_b(w) = 1$ and so we obtain a bad 2-coloring of $G + wv$ from $c$ by coloring the edge $wv$ red, and then recoloring the edge $yw$ blue. We next claim that $B = N_b(z)$.

Suppose that $B \neq N_b(z)$. Let $w \in B \setminus N_b(z)$, and let $K$ be the component of $G_b$ containing $w$. If $V(K) \subseteq B$, then for any $v \in A$, we obtain a bad 2-coloring of $G + wv$ from $c$ by coloring the edge $wv$ red, and then recoloring the edges $yw, uw$ blue and all edges incident with $w$ in $G_b$ red, a contradiction. Thus $V(K) \cap A \neq \emptyset$. Let $v \in V(K) \cap A$. We claim that $V(K) = \{w, v\}$. Suppose that $|K| = 3$. Let $v'$ be the third vertex of $K$. Then $K$ is isomorphic to $K_3$. If $v' \in A$, then we obtain a bad 2-coloring of $G$ from $c$ by recoloring the edge $yw$ blue, and then recoloring the edges $wv, wv'$ red, contrary to the choice of $c$. Thus $v' \in B$, which again yields a bad 2-coloring of $G$ from $c$ by recoloring the edge $yv$ blue, and then recoloring the edges $vw, vv'$ red, contrary to the choice of $c$. Thus $V(K) = \{w, v\}$, as claimed. For any $v^* \in (A \cup B) \setminus (\{w, v\} \cup N_b(z))$, we obtain a bad 2-coloring of $G + wv^*$ from $c$ by coloring the edge $wv^*$ red, and then recoloring the edge $wv$ red, and the edges $yw, uw$ blue. Thus $B = N_b(z)$, as claimed.

Since $B = N_b(z)$, we have $|B| \leq 2$. Then $yu \in E_b$, otherwise by a similar argument for showing $B = N_b(z)$, we have $|A| = |N_b(u)| \leq 2$ and so $n \leq 7$, a contradiction. Let $v \in B$. If $B = \{v\}$, then by a similar argument for showing $d_b(u) = 1$, we have $d_b(v) = 1$. But then we obtain a bad 2-coloring of $G + yz$ from $c$ by coloring the edge $yz$ blue, and then recoloring the edge $yu$ red, and the edge $yv$ blue. Thus $|B| = 2$. Let $v'$ be the other vertex in $B$. Then $vv' \in E_b$, otherwise we obtain a bad 2-coloring of $G + vv'$ from $c$ by coloring the edge $vv'$ blue. But now we obtain a bad 2-coloring of $G + yz$ from $c$ by coloring the edge $yz$ blue, and then recoloring the edges $yu, vv', zv'$ red, and edges $yv, uv'$ blue, a contradiction. ■

By Lemma 1.6(b), $G_b$ has at most two isolated vertices. Thus $e(G_b) \geq (n-2)/2$. Since $e(G) < 5n/2$, we see that $e(G_r) \leq 2n$. By (3), $|B| \geq 2$ and $|C| \geq 2$. By Corollary 2.5, $e(G_r) \geq 2n - 4$ and $|B| + |C| \leq 8$. Thus $|A| \geq n - 10 \geq 8$. We next show that

(4) If $P_3$ is a component of $G_b \setminus \{y, z\}$ with vertices $x_1, x_2, x_3$ in order, then $x_2 \in A$ and $|\{x_1, x_3\} \cap B| = |\{x_1, x_3\} \cap C| = 1$.

**Proof.** Clearly, $\{x_1, x_2, x_3\} \not\subseteq A \cup B$ or $\{x_1, x_2, x_3\} \not\subseteq A \cup C$, otherwise $x_1x_3 \notin E(G)$ and we obtain a bad 2-coloring of $G + x_1x_3$ from $c$ by coloring the edge $x_1x_3$ blue. Since $y, z \notin \{x_1, x_2, x_3\}$, we see that $x_2 \in A$. Then $|\{x_1, x_3\} \cap B| = |\{x_1, x_3\} \cap C| = 1$. ■

(5) $yz \notin E(G)$.

**Proof.** Suppose that $yz \in E(G)$. Then $yz \in E_b$. Since $G_b$ does not contain a $T_4$, we see that either $d_b(y) = 1$ or $d_b(z) = 1$. We may assume that $d_b(z) = 1$. We claim that $d_b(y) = 1$ as well. Suppose that $d_b(y) = 2$. Let $w \in C$ be the other neighbor of $y$ in $G_b$. Then $d_b(w) = 1$. Let $v \in A$. We obtain a bad 2-coloring of $G + wv$ from $c$ by coloring the edge $wv$ red, and recoloring the edge $zw$ blue. Thus $d_b(y) = d_b(z) = 1$. Since $e(G_r) \leq 2n$ and $|A| \geq n - 10 \geq 8$, by Corollary 2.5 and (4), $G_b$ contains a component, say $K$, such that $V(K) \cap A \neq \emptyset$ and $V(K) \subset A \cup B$ or $V(K) \subset A \cup C$. Let $u \in V(K) \cap A$ and $w \in A \setminus V(K)$. We obtain a bad 2-coloring of $G + uw$ from $c$ by coloring the edge $uw$ red, and then recoloring the edges $yu, zu$ blue, and all the edges incident with $u$ in $G_b$ red, a contradiction. ■

(6) $G_b$ has no isolated vertex.

**Proof.** Suppose for a contradiction that $G_b$ has an isolated vertex, say $u$. Then $d(u) = d_r(u)$. By (1), $d(u) \leq n - 3$. For any $w \in V(G) \setminus N[u]$, adding a blue edge $uw$ to $G$ must yield a blue $T_4$, because $G$ is $\mathcal{R}_{\min}(K_3, \mathcal{T}_4)$-saturated. Hence,

($*$) every vertex of $V(G) \setminus N[u]$ belongs to a $P_3$ or $K_3$ in $G_b$.

We next claim that every vertex of $A \setminus u$ belongs to a $P_3$ or $K_3$ in $G_b$. By ($*$), this is obvious if $u \in A \cup B \cup C$. So we may assume that $u \in \{y, z\}$. By symmetry, we may further assume that $u = z$. By (5), $yz \notin E(G)$. Suppose that there exists a vertex $v \in A$ such that $v$ belongs to a component, say $K$, with $|K| \leq 2$. Then $V(K) \subseteq A \cup B$ or $V(K) \subseteq A \cup C$. Let $w \notin V(K)$ be a vertex in $C$. This is possible because $|C| \geq 2$ by (3). We then obtain a bad 2-coloring of $G + vw$ from $c$ by coloring the edge $vw$ red, and recoloring the edge $vu$ blue, a contradiction. Thus every vertex of $A \setminus u$ belongs to a $P_3$ or $K_3$ in $G_b$, as claimed.

Since $|B| + |C| \leq 8$ and $|A| \geq n - 10 \geq 8$, by (4) and Corollary 2.5, we see that $G_b[A]$ has at least two components isomorphic to $K_3$. By Lemma 1.6(b), $G_b$ has at most two isolated vertices and so $e(G_b) \geq 6 + (n-8)/2$. Since $e(G) < 5n/2$, we have $e(G_r) \leq 2n - 3$. By (3), $|B| \geq 2$ and $|C| \geq 2$. By Corollary 2.5, $2n - 4 \leq e(G_r) \leq 2n - 3$ and $\max\{|B|, |C|\} \leq 3$. Thus $|A| \geq n - 8 \geq 10$. By (4) and Corollary 2.5 again, $G_b[A]$ has at least three components isomorphic to $K_3$. Thus $e(G_b) \geq 9 + \lceil (n-11)/2 \rceil$ and so $e(G) \geq (2n-4) + 9 + \lceil (n-11)/2 \rceil \geq \lfloor 5n/2 \rfloor$, a contradiction. ■

(7) $d_b(y) = d_b(z) = 2$.

**Proof.** Suppose that $d_b(y) \leq 1$ or $d_b(z) \leq 1$. By (6), $d_b(y), d_b(z) \geq 1$. We may assume that $d_b(y) = 1$. By (5), $yz \notin E(G)$. Let $y_1 \in C$ be the unique neighbor of $y$ in $G_b$, and let $z_1 \in B$ be a neighbor of $z$ in $G_b$. We claim that $d_b(y_1) = 1$. Suppose that $d_b(y_1) = 2$. Let $y_1^* \in A \cup C$ be the other neighbor of $y_1$ in $G_b$. Then $y_1^* \in A$, otherwise, we obtain a bad 2-coloring of $G + yy_1^*$ from $c$ by coloring the edge $yy_1^*$ blue. Let $w \in B$. Then we obtain a bad 2-coloring of $G + y_1^*w$ from $c$ by coloring the edge $y_1^*w$ red and recoloring the edge $y_1^*y$ blue. Thus $d_b(y_1) = 1$, as claimed.

By (3), $|B| \geq 2$ and $|C| \geq 2$. We next claim that $N_b(z) = B$. Suppose that there exists a vertex $u \in B$ such that $uz \notin E(G_b)$. Then $uz_1 \notin E_b$, otherwise, we obtain a bad 2-coloring of $G + uz$ from $c$ by coloring the edge $uz$ blue. This implies that $B \setminus N_b(z)$ is anti-complete to $N_b(z)$ in $G_b$. Let $K$ be the component of $G_b$ containing $u$. By (6), $|K| \geq 2$. Since $G_b$ is $T_4$-free, we see that $N_b[z]$ is anti-complete to $V(K)$ in $G_b$. Suppose first that $V(K) \subseteq B$. If $K$ is isomorphic to $K_3$ or $|N_b(z)| = 2$, then $|B| \geq 4$ and $G_b$ contains at least one $K_3$ ($K$ or $G[N_b[z]]$). By Corollary 2.5, $e(G_r) \geq 2n - 2$. By (6), $e(G_b) \geq 3 + \lceil (n-3)/2 \rceil$. Hence $e(G) = e(G_r) + e(G_b) \geq (2n-2) + 3 + \lceil (n-3)/2 \rceil \geq \lfloor 5n/2 \rfloor$, a contradiction. Thus $K$ is isomorphic to $K_2$ and $d_b(z) = 1$. Using a similar argument to show that $d_b(y_1) = 1$, we have $d_b(z_1) = 1$. Let $V(K) = \{u, u'\}$. If $B = \{u, u', z_1\}$, then we obtain a bad 2-coloring of $G + yz$ from $c$ by coloring the edge $yz$ blue, and then recoloring the edges $y_1u, y_1u', yz_1$ blue, and the edge $yy_1$ red. Thus $|B| \geq 4$. By Corollary 2.5, $|C| = 2$. Let $C = \{w, y_1\}$. Let $v \in A$ be such that $v$ and $w$ are not in the same component of $G_b$. This is possible because $|A| \geq 8$. Then we obtain a bad 2-coloring of $G + vw$ from $c$ by coloring the edge $vw$ red, and then recoloring the edges $z_1w, zw$ blue, and all the edges incident with $w$ in $G_b$ red. This proves that $V(K) \not\subseteq B$ and so $V(K) \cap A \neq \emptyset$. Let $v \in V(K) \cap A$. We next show that $V(K) = \{u, v\}$. Suppose that $|K| = 3$. Let $v'$ be the third vertex of $K$. Then $K$ is isomorphic to $K_3$. If $v' \in A$, then we obtain a bad 2-coloring of $G$ from $c$ by recoloring the edge $uy$ blue, and then recoloring the edges $uv, uv'$ red, contrary to the choice of $c$. If $v' \in B$, then we obtain a bad 2-coloring of $G$ from $c$ by recoloring the edge $vy$ blue, and then recoloring the edges $vu, vv'$ red, contrary to the choice of $c$. Thus $v' \in C$. Now for any $w \in A \setminus v$, we obtain a bad 2-coloring of $G + uw$ from $c$ by coloring the edge $uw$ red, and then recoloring the edges $uy, uy_1$

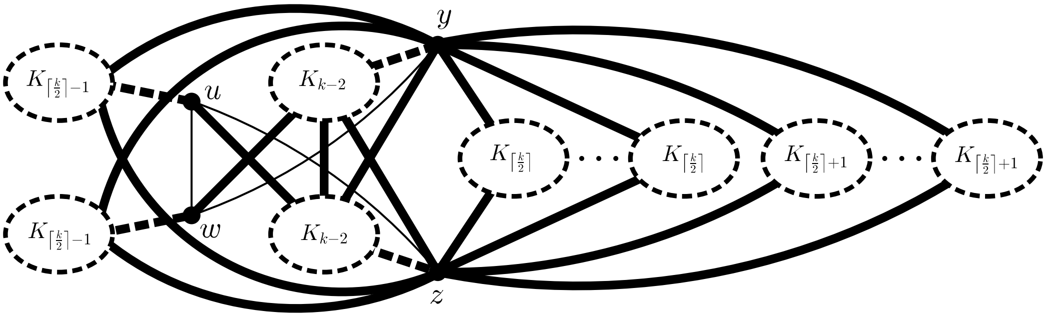


**Fig. 4.1.** An $\mathcal{R}_{\min}(K_3, \mathcal{T}_k)$-saturated graph with a unique bad 2-coloring, where dashed lines indicate blue and solid lines indicate red.

blue, and $uv$ red. Hence $V(K) = \{u, v\}$. For any $v' \in A \setminus v$, we obtain a bad 2-coloring of $G + uv'$ from $c$ by coloring the edge $uv'$ red, and then recoloring the edges $uy$ blue and $uv$ red. Thus $N_b(z) = B$, as claimed.

Since $N_b(z) = B$ and $d_b(z) \le 2 \le |B|$, we see that $|B| = 2$. Let $B = \{z_1, z_2\}$. Then $z_1z_2 \in E(G_b)$, otherwise, we obtain a bad 2-coloring of $G + z_1z_2$ from $c$ by coloring the edge $z_1z_2$ blue. Let $C = \{y_1, \ldots, y_t\}$, where $t = |C|$. Then $y_1y_j \notin E(G_b)$ for all $j \in \{2, \ldots, t\}$ because $d_b(y_1) = 1$. If $t \ge 4$, then by Corollary 2.5, $e(G_r) \ge 2n - 2$. By (6), $e(G_b) \ge 3 + \lceil (n-3)/2 \rceil$. Thus $e(G) \ge (2n-2) + 3 + \lceil (n-3)/2 \rceil \ge \lfloor 5n/2 \rfloor$, a contradiction. Thus $2 \le t \le 3$. Let $v \in A$ be such that $vy_j \notin E(G)$ for all $j \in \{1, 2, \ldots, t\}$. This is possible because $|A| \ge 8$ and $t \le 3$. We obtain a bad 2-coloring of $G + y_2v$ from $c$ by coloring the edge $y_2v$ red, and then when $t = 2$, recoloring the edges $yz_1, z_1y_1, z_2y_2, y_2z$ blue, the edges $z_1z, z_1z_2$, and all the edges incident with $y_2$ in $G_b$ red; when $t = 3$, recoloring the edges $y_1z_1, y_1z_2, zy_2, zy_3$ blue, the edges $yy_1, zz_1, zz_2$, and all the edges between $A$ and $\{y_2, y_3\}$ in $G_b$ red. ■

By (7), $d_b(y) = d_b(z) = 2$. By (5), $yz \notin E(G)$. Let $N_b(y) = \{y_1, y_2\} \subseteq C$ and $N_b(z) = \{z_1, z_2\} \subseteq B$. Then $y_1y_2, z_1z_2 \in E_b$, otherwise, we obtain a bad 2-coloring of $G + e$ from $c$ by coloring the edge $e$ blue, where $e \in \{y_1y_2, z_1z_2\}$. By (6), $e(G_b) \ge 6 + \lceil (n-6)/2 \rceil$. Since $e(G) < \lfloor 5n/2 \rfloor$, by Corollary 2.5, we see that $n$ is even and $|B| = |C| = 2$. Let $v \in A$. We obtain a bad 2-coloring of $G + vz_1$ from $c$ by coloring the edge $vz_1$ red, and then recoloring the edges $yz_1, z_2y_1, z_2y_2$ blue, and edges $yy_1, yy_2, zz_2, z_1z_2$ red, a contradiction.

This completes the proof of Theorem 1.4. ■

## 4. Proof of Theorem 1.5

Finally, we prove Theorem 1.5. We will construct an $\mathcal{R}_{\min}(K_3, \mathcal{T}_k)$-saturated graph on $n \ge 2k + (\lceil k/2 \rceil + 1)\lceil k/2 \rceil - 2$ vertices which yields the desired upper bound in Theorem 1.5.

For positive integers $k, n$ with $k \ge 5$ and $n \ge 2k + (\lceil k/2 \rceil + 1)\lceil k/2 \rceil - 2$, let $t$ be the remainder of $n - 2k - 2\lceil k/2 \rceil + 2$ when divided by $\lceil k/2 \rceil$, and let $H = 2K_{\lceil k/2 \rceil -1} \cup 2K_{k-2} \cup sK_{\lceil k/2 \rceil} \cup tK_{\lceil k/2 \rceil +1}$, where $s \ge 0$ is an integer satisfying $s\lceil k/2 \rceil + t(\lceil k/2 \rceil + 1) = n - 2k - 2\lceil k/2 \rceil + 2$. Let $H_1, H_2$ be the two disjoint copies of $K_{k-2}$, and let $H_3, H_4$ be the two disjoint copies of $K_{\lceil k/2 \rceil -1}$ in $H$, respectively. Finally, let $G$ be the graph obtained from $H$ by adding four new vertices $y, z, u, w$, and then joining: every vertex in $H_1$ to all vertices in $H_2$; $y$ to all vertices in $V(H) \cup \{w\}$; $z$ to all vertices in $V(H) \cup \{u\}$; $u$ to all vertices in $\{w\} \cup V(H_2) \cup V(H_3)$; and $w$ to all vertices in $V(H_1) \cup V(H_4)$, as depicted in Fig. 4.1.

Clearly, the coloring $c : E(G) \to \{\text{red}, \text{blue}\}$ given in Fig. 4.1 is a bad 2-coloring of $G$. We next show that $c$ is the unique bad 2-coloring of $G$. By Lemma 1.6(a), each edge $e \in E(H_1) \cup E(H_2)$ must be colored blue because $e$ belongs to $2k-3$ triangles in $G$. Then all edges between $V(H_1)$ and $V(H_2)$ in $G$ must be colored red and the edge $yv$ must be colored red for some $v \in V(H_1) \cup V(H_2)$, because $G_b$ is $\mathcal{T}_k$-free. Additionally, $y$ can only be joined by a blue edge to a vertex in either $V(H_1)$ or $V(H_2)$ but not both. It follows that $y$ is complete to one of $V(H_1)$ or $V(H_2)$ in $G_r$. We next show that $y$ is complete to $V(H_2)$ in $G_r$. Suppose that $y$ is complete to $V(H_1)$ in $G_r$. Then $y$ is complete to $V(H_2)$ in $G_b$ since $G_r$ is $K_3$-free, and so $yw \in E_r$ since $G_b$ is $\mathcal{T}_k$-free. This implies that $z$ must be complete to $V(H_1)$ in $G_b$. But now $w$ must be complete to $V(H_1)$ in $G_r$, which yields a red $K_3$ on $y, w, v$ for any $v \in V(H_1)$, a contradiction. Hence $y$ is complete to $V(H_2)$ in $G_r$. Then $y$ must be complete to $V(H_1)$ in $G_b$. Since $G_b$ is $\mathcal{T}_k$-free, $y$ is complete to $\{w\} \cup (V(H) \setminus V(H_1))$ in $G_r$, and $z$ is complete to $V(H_1)$ in $G_r$. Since $G_r$ is $K_3$-free, we see that all edges in each component of $H$ must be colored blue, and then $z$ must be complete to $V(H_2)$ in $G_b$ and $w$ must be complete to $V(H_4)$ in $G_b$. By symmetry of $y$ and $z$, it follows that $z$ is complete to $\{u\} \cup (V(H) \setminus V(H_2))$ in $G_r$, and $u$ is complete to $V(H_3)$ in $G_b$. This proves that $c$ is the unique bad 2-coloring of $G$. It is straightforward to see that $G$ is $\mathcal{R}_{\min}(K_3, \mathcal{T}_k)$-saturated. Using the facts that $s\lceil k/2 \rceil + t(\lceil k/2 \rceil + 1) = n - 2k - 2\lceil k/2 \rceil + 2$ and $t \le \lceil k/2 \rceil - 1$, we see that

$$e(G) = 2(n-2) + \binom{2k-4}{2} + (2(k-2)+1) + (s+2)\binom{\lceil k/2 \rceil}{2} + t\binom{\lceil k/2 \rceil + 1}{2}$$
$$= (2n + 2k^2 - 7k + 3) + (s+2)\lceil k/2 \rceil \frac{\lceil k/2 \rceil - 1}{2} + t(\lceil k/2 \rceil + 1)\frac{(\lceil k/2 \rceil - 1) + 1}{2}$$

$$
\begin{aligned}
&= (2n+2k^2-7k+3) + \frac{\lceil k/2\rceil -1}{2}\left((s+2)\lceil k/2\rceil + t(\lceil k/2\rceil+1)\right) + \frac{t}{2}\left(\left\lceil\frac{k}{2}\right\rceil+1\right)\\
&= (2n+2k^2-7k+3) + \frac{\lceil k/2\rceil -1}{2}\left((s\lceil k/2\rceil + t(\lceil k/2\rceil+1)) + 2\lceil k/2\rceil\right) + \frac{t}{2}\left(\left\lceil\frac{k}{2}\right\rceil+1\right)\\
&\le (2n+2k^2-7k+3) + \frac{\lceil k/2\rceil -1}{2}\left(n-2k-2\lceil k/2\rceil+2+2\lceil k/2\rceil\right) + \frac{t}{2}\left(\left\lceil\frac{k}{2}\right\rceil+1\right)\\
&\le \left(\frac{3}{2}+\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right)n + 2k^2-6k+2-(k-1)\left\lceil\frac{k}{2}\right\rceil + \frac{\lceil k/2\rceil -1}{2}\left(\left\lceil\frac{k}{2}\right\rceil+1\right)\\
&\le \left(\frac{3}{2}+\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right)n + 2k^2-6k+\frac{3}{2}-\left\lceil\frac{k}{2}\right\rceil\left(k-\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil-1\right)\\
&= \left(\frac{3}{2}+\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right)n + C,
\end{aligned}
$$

where $C = 2k^2-6k+\frac{3}{2}-\left\lceil\frac{k}{2}\right\rceil\left(k-\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil-1\right)$. Therefore $sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_k)) \le e(G) \le \left(\frac{3}{2}+\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right)n + C$.

Let $c = \left(\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil+\frac{3}{2}\right)k-2$. We next show that $sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_k)) \ge \left(\frac{3}{2}+\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right)n-c$. Let $G$ be an $\mathcal{R}_{\min}(K_3, \mathcal{T}_k)$-saturated graph on $n \ge 2k + (\lceil k/2\rceil+1)\lceil k/2\rceil - 2$ vertices. Then $G+e$ has no bad 2-coloring for any edge $e \in E(\overline{G})$. Among all bad 2-colorings of $G$, let $c : E(G) \to \{\text{red, blue}\}$ be a bad 2-coloring of $G$ with $|E_r|$ maximum. By the choice of $c$, $G_r$ is $K_3$-saturated and $G_b$ is $T_k$-free for any $T_k \in \mathcal{T}_k$. Note that $G_b$ is disconnected and every component of $G_b$ contains at most $k-1$ vertices. By Lemma 1.6(c), we have

(1) $\Delta(G_r) \le n-3$ and $G_r$ is 2-connected.

Let $D_1, D_2, \ldots, D_p$ be the components of $G_b$. Since $n \ge 2k + (\lceil k/2\rceil+1)\lceil k/2\rceil - 2$, we have $p \ge 3$. We next show that

(2) $G[V(D_i)] = K_{|D_i|}$ for all $i \in \{1, 2, \ldots, p\}$.

**Proof.** Suppose that there exists a component of $G_b$, say $D_1$, such that $G[V(D_1)] \ne K_{|D_1|}$. Let $u, v \in V(D_1)$ be such that $uv \notin E(G)$. We obtain a bad 2-coloring of $G+uv$ from $c$ by coloring the edge $uv$ blue, a contradiction. ■

(3) $\sum_{i=1}^{p} e(G[V(D_i)]) \ge \left(\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil-\frac{1}{2}\right)n - \left(\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil-\frac{1}{2}\right)k$

**Proof.** By (2), $G[V(D_i)] = K_{|D_i|}$ for all $i \in \{1, 2, \ldots, p\}$. By Lemma 1.6(b), at most two components $D_i$ have fewer than $k/2$ vertices. Let $t$ be the remainder of $n-k$ when divided by $\lceil k/2\rceil$, and let $s \ge 0$ be an integer such that $n-k = s\lceil k/2\rceil + t(\lceil k/2\rceil+1)$. It is straightforward to see that $\sum_{i=1}^{p} e(G[V(D_i)])$ is minimized when: two of the components, say $D_1, D_2$, are such that $|D_1|, |D_2| < k/2$; $t$ of the components, say $D_3, \ldots, D_{t+2}$, are such that $|D_3| = \cdots = |D_{t+2}| = \lceil k/2\rceil+1$; and $s$ of the components, say $D_{t+3}, \ldots, D_{t+s+2}$, are such that $|D_{t+3}| = \cdots = |D_{t+s+2}| = \lceil k/2\rceil$. Using the facts that $s\lceil k/2\rceil + t(\lceil k/2\rceil+1) = n-k$ and $t \le \lceil k/2\rceil - 1$, it follows that

$$
\begin{aligned}
\sum_{i=1}^{p} e(G[V(D_i)]) &> s\binom{\lceil k/2\rceil}{2} + t\binom{\lceil k/2\rceil+1}{2}\\
&= s\lceil k/2\rceil\frac{\lceil k/2\rceil-1}{2} + t(\lceil k/2\rceil+1)\frac{(\lceil k/2\rceil-1)+1}{2}\\
&= \left(\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil-\frac{1}{2}\right)\left(s\left\lceil\frac{k}{2}\right\rceil+t\left(\left\lceil\frac{k}{2}\right\rceil+1\right)\right)+\frac{t}{2}\left(\left\lceil\frac{k}{2}\right\rceil+1\right)\\
&\ge \left(\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil-\frac{1}{2}\right)(n-k)\\
&\ge \left(\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil-\frac{1}{2}\right)n - \left(\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil-\frac{1}{2}\right)k. \quad ■
\end{aligned}
$$

Assume that $G_b[V(D_i)] = K_{|D_i|}$ for all $i \in \{1, 2, \ldots, p\}$. By (3), $|E_b| \ge \left(\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil-\frac{1}{2}\right)n - \left(\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil-\frac{1}{2}\right)k$. By Lemma 1.6(b) and Theorem 2.2, $|E_r| \ge 2n-5$. Therefore $e(G) = |E_r| + |E_b| \ge \left(\frac{3}{2}+\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right)n - \left(\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil-\frac{1}{2}\right)k - 5 \ge \left(\frac{3}{2}+\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right)n - c$, where $c = \left(\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil+\frac{3}{2}\right)k-2$, as desired. So we may assume that $G_b[V(D_i)] \ne K_{|D_i|}$ for some $i \in \{1, 2, \ldots, p\}$, say $i=1$. Let $u_1, u_2 \in V(D_1)$ be such that $u_1u_2 \notin E_b$. By (2), $u_1u_2 \in E_r$. Since $G_r$ is $K_3$-saturated, we have $N_r(u_1) \cap N_r(u_2) = \emptyset$. We next show that

(4) for any $j \in \{2, \ldots, p\}$ and any $w \in V(D_j)$, if $wu_i \notin E_r$ for some $i \in \{1, 2\}$, then $N_r(w) \cap N_r(u_i) \setminus (V(D_1) \cup V(D_j)) \ne \emptyset$.

**Proof.** We may assume that $wu_1 \notin E_r$. Since $G_r$ is $K_3$-saturated, we see that $N_r(w) \cap N_r(u_1) \ne \emptyset$. Note that $wu_1 \notin E(G)$. If $N_r(w) \cap N_r(u_1) \setminus (V(D_1) \cup V(D_j)) = \emptyset$, then we obtain a bad 2-coloring of $G+wu_1$ from $c$ by coloring $wu_1$ red, and then recoloring all red edges incident with $u_1$ in $D_1$ blue and all red edges incident with $w$ in $D_j$ blue, a contradiction. ■

(5) For any $j \in \{2, \ldots, p\}$ and any $w \in V(D_j)$, $|N_r(w) \setminus V(D_j)| \geq 2$.

**Proof.** This is obvious when $wu_1, wu_2 \in E_r$. So we may assume that $wu_1 \notin E_r$. Since $N_r(u_1) \cap N_r(u_2) = \emptyset$, it follows from (4) that either $|N_r(w) \setminus (V(D_1) \cup V(D_j))| \geq 2$ when $wu_2 \notin E(G)$ or $|N_r(w) \setminus V(D_j)| = |N_r(w) \setminus (V(D_1) \cup V(D_j))| + |N_r(w) \cap V(D_1)| \geq 1 + 1 = 2$ when $wu_2 \in E(G)$. In both cases, $|N_r(w) \setminus V(D_j)| \geq 2$, as desired. ■

For each vertex $w \in V(G) \setminus V(D_1)$, since $G_r$ is $K_3$-saturated, we see that either $wu_1 \notin E_r$ or $wu_2 \notin E_r$. Let $P := \{w \in V(G) \setminus V(D_1) : wu_1, wu_2 \notin E_r\}$, $Q := \{w \in V(G) \setminus V(D_1) : wu_1 \notin E_r, wu_2 \in E_r\}$, and $R := \{w \in V(G) \setminus V(D_1) : wu_1 \in E_r, wu_2 \notin E_r\}$. Further, let $Q_1$ denote the set of vertices $w \in Q$ such that $N_r(w) \cap V(D_1) = \{u_2\}$, and let $R_1$ denote the set of vertices $w \in R$ such that $N_r(w) \cap V(D_1) = \{u_1\}$. Let $Q_2 := Q \setminus Q_1$ and $R_2 := R \setminus R_1$. By definition, $P, Q_1, Q_2, R_1, R_2$ are pairwise disjoint and $|P| + |Q| + |R| = n - |V(D_1)| \geq n - k + 1$. Let $H$ be obtained from $G \setminus V(D_1)$ by deleting all edges in $G[V(D_i)]$ for all $i \in \{2, 3, \ldots, p\}$. Then $E(H) \subset E_r$ and for each edge $e$ in $H$, $e$ is not in $G[V(D_i)]$ for any $i \in \{2, 3, \ldots, p\}$. For any $w \in Q_1 \cup R_1$, by (4), $N_H(w) \setminus P \neq \emptyset$. We next show that

(6) for any $w \in Q_1$, if $w$ is adjacent to exactly one vertex, say $v$, in $H \setminus P$, then $v \in R_2$.

**Proof.** We may assume that $w \in V(D_2)$. Since $w \in Q_1$, we have $N_r(w) \cap V(D_1) = \{u_2\}$. By (4), $vu_1 \in E_r$, and we may further assume that $v \in V(D_3)$. Then $vu_2 \notin E_r$ because $G_r$ is $K_3$-free. Since $D_1$ is a component of $G_b$, there must exist a vertex, say $u \in V(D_1)$, such that $uu_2 \in E_b$. Then $wu \notin E_r$ (and so $wu \notin E(G)$) because $N_r(w) \cap V(D_1) = \{u_2\}$. Hence $uv \in E_r$, otherwise, we obtain a bad 2-coloring of $G + wu$ from $c$ by coloring $wu$ red and then recoloring all edges incident with $w$ in $D_2$ blue. Therefore $v \in R_2$. ■

By symmetry, for any $w \in R_1$, if $w$ is adjacent to exactly one vertex, say $v$, in $H \setminus P$, then $v \in Q_2$. We next count the number of edges in $H$. Since $N_r(u_1) \cap N_r(u_2) = \emptyset$, it follows from (4) that for each $w \in P$, $e_H(w, Q \cup R) \geq 2$ and so $e_H(P, Q \cup R) \geq 2|P|$. Let $Q_1^*$ be the set of vertices $w \in Q_1$ such that $w$ is adjacent to exactly one vertex in $H \setminus P$. Similarly, let $R_1^*$ be the set of vertices $w \in R_1$ such that $w$ is adjacent to exactly one vertex in $H \setminus P$. By (6), $e_H(Q_1^*, R_2) \geq |Q_1^*|$ and $e_H(R_1^*, Q_2) \geq |R_1^*|$. Notice that for any $w \in (Q_1 \cup R_1) \setminus (Q_1^* \cup R_1^*)$, $w$ is adjacent to at least two vertices in $H \setminus (P \cup Q_1^* \cup R_1^*)$ and so $e(H \setminus (P \cup Q_1^* \cup R_1^*)) \geq |Q_1 \setminus Q_1^*| + |R_1 \setminus R_1^*| = |Q_1| + |R_1| - |Q_1^*| - |R_1^*|$. Therefore

$$\begin{aligned} e(H) &= e_H(P, Q \cup R) + e_H(Q_1^*, R_2) + e_H(R_1^*, Q_2) + e(H \setminus (P \cup Q_1^* \cup R_1^*)) \\ &\geq 2|P| + |Q_1^*| + |R_1^*| + |Q_1| + |R_1| - |Q_1^*| - |R_1^*| \\ &= 2|P| + |Q_1| + |R_1|. \end{aligned}$$

Note that $e_G(V(D_1), Q \cup R) \geq |Q_1| + 2|Q_2| + |R_1| + 2|R_2| = |Q| + |R| + |Q_2| + |R_2|$. We see that $e(H) + e_G(V(D_1), Q \cup R) \geq (2|P| + |Q_1| + |R_1|) + (|Q| + |R| + |Q_2| + |R_2|) = 2(|P| + |Q| + |R|) \geq 2n - 2k + 2$. By (3),

$$\begin{aligned} e(G) &\geq e(H) + e_G(V(D_1), Q \cup R) + \sum_{i=1}^{p} e(G[V(D_i)]) \\ &\geq (2n - 2k + 2) + \left(\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil - \frac{1}{2}\right)n - \left(\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil - \frac{1}{2}\right)k \\ &= \left(\frac{3}{2} + \frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right)n - \left(\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil + \frac{3}{2}\right)k + 2 \\ &= \left(\frac{3}{2} + \frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right)n - c \end{aligned}$$

where $c = \left(\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil + \frac{3}{2}\right)k - 2$.

This completes the proof of Theorem 1.5. ■

## Conclusion

For the graphs $G_{odd}$ and $G_{even}$ in the proof of Theorem 1.4, we want to point out here that we found the graph $G_{odd}$ when $d_b(y) = 1$, $d_b(z) = 2$, and $G_r = J$ with $|B| = 2$ and $|C| = 4$; and the graph $G_{even}$ when $d_b(y) = d_b(z) = 2$, and $G_r = J$ with $|B| = 3$ and $|C| = 2$. We believe that the method we developed in this paper can be applied to determine $sat(n, \mathcal{R}_{\min}(K_p, T_k))$ for any given tree $T_k$ and any $p \geq 3$.

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## References


[1] N. Alon, P. Erdős, R. Holzman, M. Krivelevich, On *k*-saturated graphs with restrictions on the degrees, J. Graph Theory 23 (1996) 1–20.
[2] B. Bollobás, On generalized graphs, Acta Math. Acad. Sci. Hungar. 16 (1965) 447–452.
[3] C. Bosse, Z.-X. Song, J. Zhang, On $\mathcal{R}_{\min}(K_3, K_4)$-saturated graphs, in preparation.
[4] S.A. Burr, P. Erdős, L. Lovász, On graphs of Ramsey type, Ars Combin. 1 (1976) 167–190.
[5] G. Chen, M. Ferrara, R.J. Gould, C. Magnant, J. Schmitt, Saturation numbers for families of Ramsey-minimal graphs, J. Comb. 2 (2011) 435–455.
[6] D. Conlon, J. Fox, B. Sudakov, Recent developments in graph Ramsey theory, Surv. Combin. 424 (2015) 49–118.
[7] A.N. Day, Saturated graphs of prescribed minimum degree, Combin. Probab. Comput. 26 (2017) 201–207.
[8] D.A. Duffus, D. Hanson, Minimal *k*-saturated and color critical graphs of prescribed minimum degree, J. Graph Theory 10 (1986) 55–67.
[9] P. Erdős, A. Hajnal, J.W. Moon, A problem in graph theory, Amer. Math. Monthly 71 (1964) 1107–1110.
[10] J.R. Faudree, R.J. Faudree, J.R. Schmitt, A survey of minimum saturated graphs and hypergraphs, Electron. J. Combin. 18 (2011) DS19.
[11] M. Ferrara, J. Kim, E. Yeager, Ramsey-minimal saturation numbers for matchings, Discrete Math. 322 (2014) 26–30.
[12] J. Fox, A. Grinshpun, A. Liebenau, Y. Person, T. Szabó, On the minimum degree of minimal Ramsey graphs for multiple colors, J. Combin. Theory Ser. B. 120 (2016) 64–82.
[13] J. Fox, K. Lin, The minimum degree of Ramsey-minimal graphs, J. Graph Theory 54 (2007) 167–177.
[14] A. Galluccio, M. Simonovits, G. Simonyi, On the structure of co-critical graphs, in: Graph Theory, Combinatorics, and Algorithms, vol. 1,2, Wiley-Intersci. Publ., Wiley, New York, 1995, pp. 1053–1071 Kalamazoo, MI, 1992.
[15] D. Hanson, B. Toft, Edge-colored saturated graphs, J. Graph Theory 11 (1987) 191–196.
[16] J. Nešetřil, Problem, in: G. Halász, V.T. Sós (Eds.), Irregularities of Partitions, in: Series Algorithms and Combinatorics, vol. 8, Springer Verlag, 1989, p. P164, Proc. Coll. held at Fertőd, Hungary 1986.
[17] O. Pikhurko, Results and open problems on minimum saturated graphs, Ars Combin. 72 (2004) 111–127.
[18] F.P. Ramsey, On a problem of formal logic, Proc. Lond. Math. Soc. (3) 30 (1930) 264–286.