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# Antimagic orientations of even regular graphs

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**Abstract**

A *labeling* of a digraph $D$ with $m$ arcs is a bijection from the set of arcs of $D$ to $\{1, \dots, m\}$. A labeling of $D$ is *antimagic* if no two vertices in $D$ have the same vertex-sum, where the vertex-sum of a vertex $u \in V(D)$ for a labeling is the sum of labels of all arcs entering $u$ minus the sum of labels of all arcs leaving $u$. Motivated by the conjecture of Hartsfield and Ringel from 1990 on antimagic labelings of graphs, Hefetz, Mütze, and Schwartz [On antimagic directed graphs, J. Graph Theory 64 (2010) 219–232] initiated the study of antimagic labelings of digraphs, and conjectured that every connected graph admits an antimagic orientation, where an orientation $D$ of a graph $G$ is *antimagic* if $D$ has an antimagic labeling. It remained unknown whether every disjoint union of cycles admits an antimagic orientation. In this article, we first answer this question in the positive by proving that every 2-regular graph has an antimagic orientation. We then show that for any integer $d \geq 2$, every connected, $2d$-regular graph has an antimagic orientation. Our technique is new.

**KEYWORDS**

antimagic labeling, antimagic orientation, regular graph

## 1 | INTRODUCTION

All graphs in this article are finite and simple. For a graph $G$, we use $|G|$ and $e(G)$ to denote the number of vertices and edges of $G$, respectively. An *antimagic labeling* of a graph $G$ is a bijection from $E(G)$ to $\{1, 2, \dots, e(G)\}$ such that for any distinct vertices $u$ and $v$, the sum of labels on edges incident to $u$ differs from that for edges incident to $v$. A graph $G$ is *antimagic* if it has an antimagic labeling.

Hartsfield and Ringel [8] introduced antimagic labelings in 1990 and conjectured that every connected graph other than $K_2$ is antimagic. The most significant progress on this problem is a result of Alon et al. [1], which states that there exists an absolute constant $c$ such that every graph on $n$ vertices with minimum degree at least $c \log n$ is antimagic. Eccles [6] recently improved this result by showing that there exists an absolute constant $c_0$ such that if $G$ is a graph with average degree at least $c_0$, and $G$ contains no isolated edge and at most one isolated vertex, then $G$ is antimagic. Cranston [4] proved that any $d$-regular bipartite graph with $d \geq 2$ is antimagic. For nonbipartite regular graphs, Cranston et al. [5] proved that every odd regular graph is antimagic, and later Bérczi et al. [2], and Chang et al. [3], independently, proved that every even regular graph is antimagic. For more information on antimagic labelings of graphs and related labeling problems, see the recent informative survey [7].

Motivated by antimagic labelings of graphs, Hefetz et al. [9] initiated the study of antimagic labelings of digraphs. For a positive integer $k$, we define $[k] := \{1, 2, \ldots, k\}$. Let $D$ be a digraph. We use $A(D)$ and $V(D)$ to denote the set of arcs and vertices of $D$, respectively. A *labeling* of $D$ with $m$ arcs is a bijection from $A(D)$ to $[m]$. A labeling of $D$ is *antimagic* if no two vertices in $D$ have the same vertex-sum, where the vertex-sum of a vertex $u \in V(D)$ for a labeling is the sum of labels of all arcs entering $u$ minus the sum of labels of all arcs leaving $u$. A digraph $D$ is *antimagic* if it has an antimagic labeling. A graph $G$ has an *antimagic orientation* if an orientation of $G$ is antimagic. Hefetz et al. [9] raised the questions " Is every orientation of any connected graph antimagic?" and "Does every graph admit an antimagic orientation?." Except for $K_{1,2}$ and $K_3$, no other counterexamples to the first question are known. They proved an analogous result of Alon et al. [1] that there exists an absolute constant $c$ such that every orientation of any graph on $n$ vertices with minimum degree at least $c \log n$ is antimagic. They also showed that every orientation of the star $S_n$ with $n \neq 2$ is antimagic; every orientation of the wheel $W_n$ is antimagic; and every orientation of $K_n$ with $n \neq 3$ is antimagic. For the second question, they prove the following.

**Theorem 1.1** (**[9]**). *For any integer $d \geq 1$,*

- **(a)** *every $(2d - 1)$-regular graph admits an antimagic orientation.*
- **(b)** *every connected, $2d$-regular graph $G$ admits an antimagic orientation if $G$ has a matching that covers all but at most one vertex of $G$.*

Hefetz et al. [9] asked whether it is true that every orientation of any connected graph on at least four vertices is antimagic. They also pointed out that "It seems hard to discard any of the two conditions in Theorem 1.1(b), that is connectedness and having a matching that covers all vertices but at most one. In fact, we do not even know if every disjoint union of cycles admits an antimagic orientation." They proposed the following conjecture.

**Conjecture 1.2** (**[9]**). *Every connected graph admits an antimagic orientation.*

Recently, Shan and Yu [10] proved that Conjecture 1.2 holds for biregular bipartite graphs. It remained unknown whether every 2-regular graph, that is, every disjoint union of cycles, has an antimagic orientation. In this article, we first answer this question in the positive by proving that every 2-regular graph admits an antimagic orientation. We then prove that for any integer $d \geq 2$, every connected, $2d$-regular graph admits an antimagic orientation. It turns out that finding an antimagic orientation of a 2-regular graph is, indeed, a bit more complicated than finding an antimagic orientation of an odd regular graph (see Theorem 1.3 in [9]) or a connected, even regular graph (see Theorem 3.1 below). Our technique is new and proofs of both results are neat.

We need to introduce more notation. A closed walk in a graph is an *Euler tour* if it traverses every edge of the graph exactly once. The following is a result of Euler.

**Theorem 1.3** (**Euler 1736**). *A connected graph admits an Euler tour if and only if every vertex has even degree.*

Let $D$ be an orientation of a graph $G$ with $m$ edges. For any labeling $c : A(D) \to [m]$ of $D$ and any vertex $u \in V(D)$, we use $s_D(u)$, or simply $s(u)$ when there is no confusion, to denote the vertex-sum of $u$ for the labeling $c$.

## 2 | ANTIMAGIC ORIENTATIONS OF 2-REGULAR GRAPHS

In this section, we study antimagic orientations of 2-regular graphs.

**Theorem 2.1.** *Every 2-regular graph admits an antimagic orientation.*

*Proof.* Let $G$ be a 2-regular graph on $n$ vertices. Then $e(G) = n$ and every component of $G$ is a cycle. Let $C_1, \dots, C_s, C_{s+1}, \dots, C_{s+t}$ be all distinct components of $G$ such that $C_1, \dots, C_s$ are odd cycles and $C_{s+1}, \dots, C_{s+t}$ are even cycles, where $|C_1| \leq \cdots \leq |C_s|$ and $|C_{s+1}| \leq \cdots \leq |C_{s+t}|$. For any $i \in [s+t]$, we may assume that $C_i$ has vertices $v_{i,1}, v_{i,2}, \dots, v_{i,r_i}$ in order, where $r_i := |C_i|$. We first find an orientation $D$ of $G$. Let $M_1$ and $M_2$ be two disjoint matchings of $G \backslash \{v_{1,1}v_{1,r_1}, \dots, v_{s,1}v_{s,r_s}\}$ such that $v_{1,1}v_{1,2}, \dots, v_{s+t,1}v_{s+t,2} \in M_1$ and $M_1 \cup M_2 = E(G \backslash \{v_{1,1}v_{1,r_1}, \dots, v_{s,1}v_{s,r_s}\})$. Let $D$ be the orientation of $G$, obtained by directing the edges of $G$ as follows: for all $i \in [s]$, orient every edge $v_{i,1}v_{i,r_i}$ in $\{v_{1,1}v_{1,r_1}, \dots, v_{s,1}v_{s,r_s}\}$ from $v_{i,r_i}$ to $v_{i,1}$; then for all $i \in [s+t]$, orient every edge $v_{i,j}v_{i,j+1} \in M_1$ from $v_{i,j}$ to $v_{i,j+1}$; and every edge $v_{i,j}v_{i,j+1} \in M_2$ from $v_{i,j+1}$ to $v_{i,j}$, where all arithmetic on the index $j+1$ in $v_{i,j+1}$ for each cycle $C_i$ here and henceforth is done modulo $r_i$. Clearly, $D$ is an orientation of $G$. Let $D_o$ be the above orientation of the odd cycles $C_1, \dots, C_s$.

**Algorithm 1.** Label the edges of $C_1, \dots, C_s$

**Data**: Odd cycles $C_1, \dots, C_s$ with the given orientation $D_o$, $s(v_{i,1}) = -i$ for all $i \in [p]$, $s(v_{j,1}) = j - p$ for all $j \in \{p+1, \dots, s\}$, $c_o(v_{1,r_1}v_{1,1}) = 1$, $c_o(v_{1,1}v_{1,2}) = 2$, $c_o(v_{p+1,r_{p+1}}v_{p+1,1}) = n_s$, and $c_o(v_{p+1,1}v_{p+1,2}) = n_s - 1$

**Result**: An antimagic labeling of $D_o$

**for** $i = 2$ *to* $p$ **do**
- Assign the smallest unused number, say $\alpha$, in $[n_s]$ to the edge entering $v_{i,1}$;
- Assign the value $\alpha - s(v_{i,1})$ to the edge leaving the vertex $v_{i,1}$;

**end**

Set $A$ to be the set of edges in $G$ incident with $v_{i,1}$ for all $i \in [p]$, and set $A^*$ to be $A$;

**while** $A \neq E(C_1) \cup \cdots \cup E(C_p)$ **do**
- Assign the smallest unused number in $[n_s]$ to the edge $e \in (E(C_1) \cup \cdots \cup E(C_p)) \backslash A$ which is adjacent to the edge $e^* \in A^*$ with $c_o(e^*)$ the smallest among the labels on the edges in $A^*$;
- Set $A$ to be $A \cup \{e\}$, and set $A^*$ to be $(A^* \backslash e^*) \cup \{e\}$;

**end**

**for** $j = p+2$ *to* $s$ **do**
- Assign the largest unused number, say $\beta$, in $[n_s]$ to the edge entering $v_{j,1}$;
- Assign the value $\beta - s(v_{j,1})$ to the edge leaving $v_{j,1}$;

**end**

Set $B$ to be the set of edges in $G$ incident with $v_{j,1}$ for all $j \in \{p+1, \dots, s\}$, and set $B^*$ to be $B$;

**while** $B \neq E(C_{p+1}) \cup \cdots \cup E(C_s)$ **do**
- Assign the largest unused number in $[n_s]$ to the edge $e \in (E(C_{p+1}) \cup \cdots \cup E(C_s)) \backslash B$ which is adjacent to the edge $e^* \in B^*$ with $c_o(e^*)$ the largest among the labels on the edges in $B^*$ ;
- Set $B$ to be $B \cup \{e\}$, and set $B^*$ to be $(B^* \backslash e^*) \cup \{e\}$;

**end**

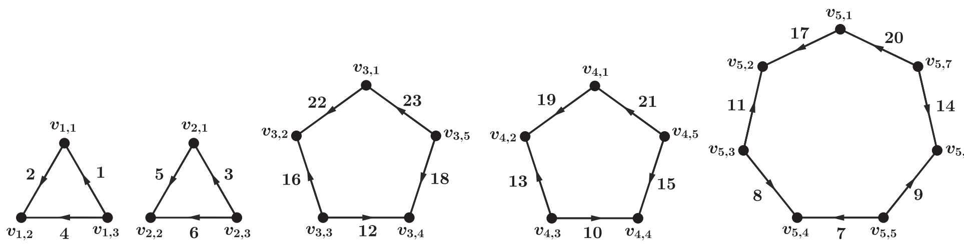


**FIGURE 1** An example of running Algorithm 1 with $s = 5$, $p = 2$, $n_1 = 3$, and $n_5 = 23$

We next find a labeling $c : A(D) \to [n]$ of $D$ such that $c$, together with $D$, is a desired antimagic orientation of $G$. Let $n_0 := 0$ and $n_i := r_1 + \cdots + r_i$ for all $i \in [s+t]$. We first find a bijection $c_e : A(C_{s+1}) \cup \cdots \cup A(C_{s+t}) \to \{n_s + 1, \ldots, n\}$. For any $i \in \{s+1, \ldots, s+t\}$, let $c_e(v_{i,j}v_{i,j+1}) = n_{i-1} + j$ for all $j \in [r_i - 2]$, $c_e(v_{i,r_{i-1}}v_{i,r_i}) = n_{i-1} + r_i$, and $c_e(v_{i,r_i}v_{i,1}) = n_{i-1} + r_i - 1$. We next find a labeling $c_o : A(D_o) \to [n_s]$ of $D_o$ with $s(v_{i,1}) = -i$ for all $i \in [p]$, and $s(v_{p+j,1}) = j$ for all $j \in \{1, \ldots, \lceil \frac{s}{2} \rceil\}$, where $p = \lfloor \frac{s}{2} \rfloor$. Let $c_o(v_{1,r_1}v_{1,1}) = 1$ and $c_o(v_{p+1,r_{p+1}}v_{p+1,1}) = n_s$. Then $c_o(v_{1,1}v_{1,2}) = c_o(v_{1,r_1}v_{1,1}) - s(v_{1,1}) = 2$, and $c_o(v_{p+1,1}v_{p+1,2}) = c_o(v_{p+1,r_{p+1}}v_{p+1,1}) - s(v_{p+1,1}) = n_s - 1$. We then label the remaining edges of $C_1, \ldots, C_s$ recursively as depicted in Algorithm 1 on the next page, where the edges of $C_1, \ldots, C_p$ are labeled from line 1 through line 9, and the edges of $C_{p+1}, \ldots, C_s$ are labeled from line 10 to line 18. Let $c$ be obtained from $c_o$ and $c_e$, that is, label the arcs in $D$ as they are labeled under $c_o$ and $c_e$. Clearly, $c$ is a labeling of $D$. An example on running Algorithm 1 on five odd cycles is depicted in Figure 1.

It remains to verify that $c$ is antimagic. For any $i \in \{s+1, \ldots, s+t\}$, we see that $s(v_{i,1}) = -(2n_{i-1} + r_i)$, $s(v_{i,j}) = (-1)^{\delta_j}(2n_{i-1} + 2j - 1)$ for all $j \in \{2, 3, \ldots, r_i - 2, r_i\}$, and $s(v_{i,r_i-1}) = -(2n_{i-1} + 2r_i - 2)$, where $\delta_j = 0$ if $j$ is even and $\delta_j = 1$ if $j$ is odd. Clearly, no two vertices of $C_{s+1}, \ldots, C_{s+t}$ have the same vertex-sum under $c$. Thus $c$ is an antimagic labeling of $D$ if $s = 0$. So we may assume that $s \geq 1$. Next, for any $u \in V(C_1) \cup \cdots \cup V(C_s)$ and $v \in V(C_{s+1}) \cup \cdots \cup V(C_{s+t})$, we see that $|s(u)| \leq 2n_s - 1$ and $|s(v)| \geq 2n_s + 3$. Thus $s(u) \neq s(v)$. To show that $c$ is antimagic, it suffices to show that $c_0$ given in Algorithm 1 is an antimagic labeling of $D_o$. We do that next.

Let $X := \{v_{1,1}, v_{2,1}, \ldots, v_{s,1}\}$. By the choice of $c_o$, no two vertices in $X$ have the same vertex-sum under $c_o$. Furthermore, for any $u \in (V(C_1) \cup \cdots \cup V(C_s))\backslash X$, by the orientation $D$ of $G$, $|s(u)| = a + b$ for some distinct integers $a, b \in [n_s]$. According to line 1 through line 9 in Algorithm 1, $\alpha \leq n_p$ and for any $u \in (V(C_1) \cup \cdots \cup V(C_p))\backslash X$, either $-(2n_p - 1) \leq s(u) \leq -(p+2)$ or $p + 3 \leq s(u) \leq 2n_p - 1$. Similarly, according to line 10 through line 17 in Algorithm 1, $\beta \geq n_p + 1$ and for any $v \in (V(C_{p+1}) \cup \cdots \cup V(C_s))\backslash X$, either $-(2n_s - 1) \leq s(v) \leq -(2n_p + 3)$ or $2n_p + 3 \leq s(v) \leq 2n_s - 1$. It follows that no vertex in $X$ has the same vertex-sum as any vertex in $(V(C_1) \cup \cdots \cup V(C_s))\backslash X$; and no vertex in $(V(C_1) \cup \cdots \cup V(C_p))\backslash X$ has the same vertex-sum as any vertex in $(V(C_{p+1}) \cup \cdots \cup V(C_s))\backslash X$. It remains to show that no two vertices in $(V(C_1) \cup \cdots \cup V(C_p))\backslash X$ (resp. $(V(C_{p+1}) \cup \cdots \cup V(C_s))\backslash X$) have the same vertex-sum. We only verify below that no two vertices in $(V(C_1) \cup \cdots \cup V(C_p))\backslash X$ have the same vertex-sum, because by the choice of labels stated in line 7 and line 16 in Algorithm 1, a similar argument can be applied to prove that no two vertices in $(V(C_{p+1}) \cup \cdots \cup V(C_s))\backslash X$ have the same vertex-sum.

By Algorithm 1, we see that for all $i \in [p]$ and $u \in V(C_i)\backslash\{v_{i,1}\}$, $|s(u)|$ grows with the distance of $u$ from $v_{i,1}$ on the cycle $C_i$. We observe the following.

**(a)** For any $i, k \in [p]$ with $i < k$, $c(v_{k,r_k-j+1}v_{k,r_k-j}) < c(v_{i,r_i-j}v_{i,r_i-j-1}) < c(v_{k,r_k-j}v_{k,r_k-j-1})$ for any $j = 0, 1, \ldots, \lfloor \frac{r_i}{2} \rfloor - 2$, and $c(v_{i,j}v_{i,j+1}) < c(v_{k,j}v_{k,j+1})$ for any $j = 1, \ldots, \lfloor \frac{r_i}{2} \rfloor$.

**(b)** For any $i \in [p]$, $c(v_{i,r_i-j}v_{i,r_i-j-1}) > c(v_{i,j+1}v_{i,j+2})$ for any $j = 0 \dots, \lfloor \frac{r_i}{2} \rfloor - 2$.

Suppose for a contradiction that there exist two distinct vertices $u, w$ in $(V(C_1) \cup \dots \cup V(C_p))\backslash X$ such that $s(u) = s(v)$. We may assume that $u \in V(C_i)$ and $w \in V(C_k)$ for some $i, k \in [p]$. Clearly, $i \neq k$. We may further assume that $i < k$, $s(u) = a + b$, and $s(w) = x + y$ for some distinct integers $a, b, x, y \in [n_p]$ with $a < b$ and $x < y$. If $a < x$ and $u \neq v_{i,\lfloor \frac{r_i}{2} \rfloor}$, then by line 7 in Algorithm 1 , $b < y$, contrary to the fact that $a + b = x + y$. If $a > x$ and $u \neq v_{i,\lfloor \frac{r_i}{2} \rfloor}$, then by (a) and line 7 in Algorithm 1 , $b > y$, a contradiction. If $a < x$ and $u = v_{i,\lfloor \frac{r_i}{2} \rfloor}$, since $c(v_{i,\lfloor \frac{r_i}{2} \rfloor+2}v_{i,\lfloor \frac{r_i}{2} \rfloor+3}) < a$, then $c(v_{i,\lfloor \frac{r_i}{2} \rfloor+2}v_{i,\lfloor \frac{r_i}{2} \rfloor+3}) < x$, by line 7 in Algorithm 1 , $b < y$, also a contradiction. Thus $a > x$ and $u = v_{i,\lfloor \frac{r_i}{2} \rfloor}$, then $x < a < b < y$. Let $q := \lfloor \frac{r_i}{2} \rfloor$ and $q^* := \lfloor \frac{r_k}{2} \rfloor$. Then $q \leq q^*$ because $r_i \leq r_k$. According to Algorithm 1, we see that $u = v_{i,q+1}$, and $c(v_{i,r_i-q+2}v_{i,r_i-q+1}) < x$. Suppose first that $w = v_{k,j}$ for some $j \in [q^*]$. By the orientation of $G$ and the fact that $a > x$, we see that $j \leq q - 1$. By (a), $c(v_{i,r_i-q+2}v_{i,r_i-q+1}) > c(v_{k,r_k-j+2}v_{k,r_k-j+1})$ because $j \leq q - 1$. By (b), $c(v_{k,r_k-j+2}v_{k,r_k-j+1}) > x$. It follows that $c(v_{i,r_i-q+2}v_{i,r_i-q+1}) > x$, contrary to the fact that $c(v_{i,r_i-q+2}v_{i,r_i-q+1}) < x$. Thus $w = v_{k,r_k-j}$ for some $j \in [q^*]$. Since $a > x$, by the orientation of $G$ and (a), we see that $j \geq q$. By (b), $c(v_{i,r_i-q+2}v_{i,r_i-q+1}) > c(v_{i,q-1}v_{i,q})$. Since $a > x$, by line 7 in Algorithm 1, $c(v_{i,q-1}v_{i,q}) > c(v_{k,r_k-j+2}v_{k,r_k-j+1})$. It follows that $c(v_{i,r_i-q+2}v_{i,r_i-q+1}) > c(v_{k,r_k-j+2}v_{k,r_k-j+1})$, which is impossible because $j \geq p$.

This completes the proof of Theorem 2.1. ■

## 3 | ANTIMAGIC ORIENTATIONS OF EVEN REGULAR GRAPHS

In this section, we first prove a result on antimagic orientations of connected, $2d$-regular graphs, where $d \geq 2$.

**Theorem 3.1.** *For any integer $d \geq 2$, every connected, $2d$-regular graph admits an antimagic orientation.*

*Proof.* For any integer $d \geq 2$, let $G$ be a connected, $2d$-regular graph on $n$ vertices. By Theorem 1.3, let $C^*$ be an Euler tour of $G$. We can regard $C^*$ as a cycle $C$ with $d \geq 2$ copies of each vertex of $G$ on $C$. For each vertex $v$ in $G$, arbitrarily pick one of the $d$ copies of $v$ on $C$ as a real vertex and the remaining $d - 1$ copies of $v$ as imaginary vertices. Then $C$ has $n$ real vertices and $(d - 1)n$ imaginary vertices. Let $V_R = \{v_1, v_2, \dots, v_n\}$ and $V_I = \{u_1, u_2, \dots, u_{(d-1)n}\}$ be the set of real vertices and imaginary vertices of $C$, respectively. Then $V(C) = V_R \cup V_I$. By renaming the vertices in $V_R$ if necessary, we label the vertices of $V_R$ on $C$ with $v_1, v_2, v_4, \dots, v_n, v_{n-1}, v_{n-3}, \dots, v_3$ in order when $n$ is even; and $v_1$, $v_2$, $v_4$, $\dots$, $v_{n-1}$, $v_n$, $v_{n-2}$, $\dots$, $v_3$ in order when $n$ is odd, as depicted in Figure 2. Two vertices $v_i, v_j \in V_R$ are a *good pair on $C$* if there exists a $(v_i, v_j)$-path $P_{i,j}$ along $C$ so that either $v_iv_j \in E(C)$ or all the internal vertices of $P_{i,j}$ are imaginary vertices. Notice that such a path $P_{i,j}$ is unique for any good pair $v_i, v_j \in V_R$. We next find an orientation $D$ of $C$.

When $n$ is even, set $d_D^+(v_i) \in \{0, 2\}$ for any $i \in [n]$, and $d_D^+(u_j) = 1$ for any $j \in [(d - 1)n]$ by first directing the path $P_{1,2}$ from $v_1$ to $v_2$, and then paths $P_{2,4}$ from $v_4$ to $v_2$, $\dots$, and finally $P_{5,3}$ from $v_5$ to $v_3$, and $P_{3,1}$ from $v_1$ to $v_3$. When $n$ is odd, set $d_D^+(v_1) = 1$, $d_D^+(v_i) \in \{0, 2\}$ for any $i \in \{2, 3, \dots, n\}$, and $d_D^+(u_j) = 1$ for any $j \in [(d - 1)n]$ by first directing the paths $P_{1,2}$ from $v_1$ to $v_2$ and $P_{3,1}$ from $v_3$ to $v_1$, and then paths $P_{2,4}$ from $v_4$ to $v_2$, $\dots$, and finally $P_{5,3}$ from $v_3$ to $v_5$. Orientations of $C$ for both cases are depicted in Figure 2.

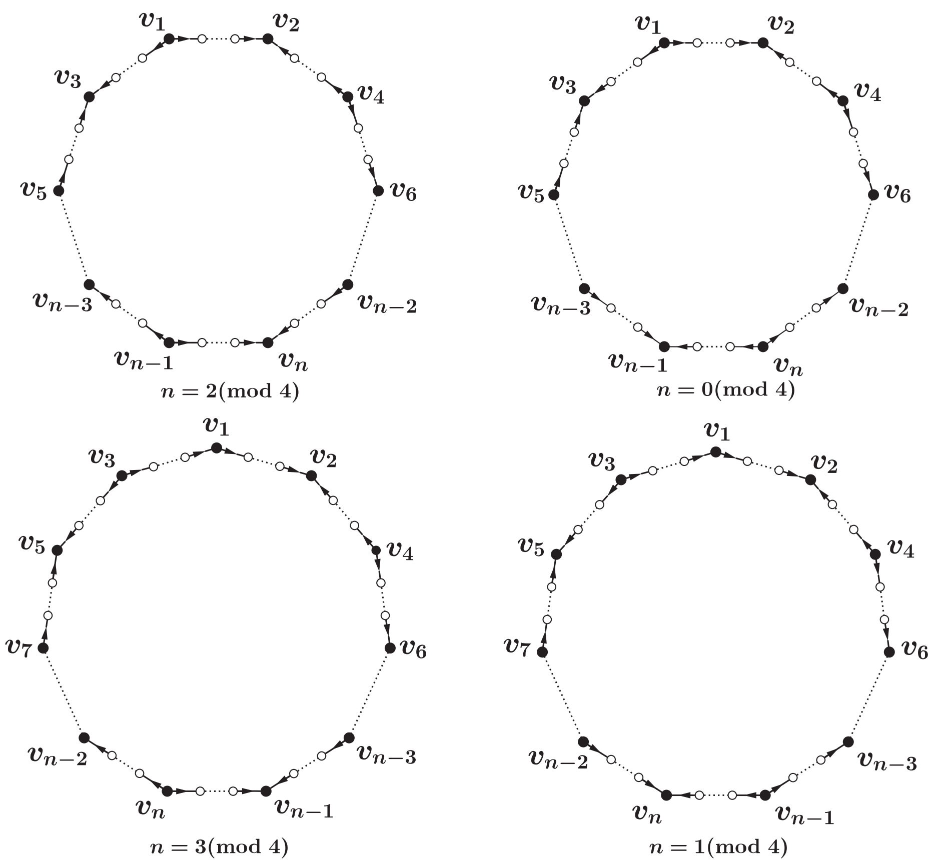


**FIGURE 2** Orientations of $C$ according to the parity of $n$, where real vertices and imaginary vertices of $C$ are denoted by $\bullet$ and $\circ$, respectively

We need to find a labeling $c : A(D) \to [dn]$ such that $c$, together with $D$, is a desired antimagic orientation of $G$. Let $\ell_0$ be the length of $P_{1,2}$. For any $i \in [n-2]$, let $\ell_i$ be the length of $P_{i,i+2}$. Finally, let $\ell_{n-1}$ be the length of $P_{n-1,n}$. Clearly, $\sum_{j=0}^{n-1} \ell_j = dn$. We define a bijection $c : A(D) \to [dn]$ as stated in Algorithm 2 below.

**Algorithm 2.** Label the arcs of $D$

**Data:** Cycle $C$ with the given orientation $D$
**Result:** A bijection $c : A(D) \to [dn]$
Assign the numbers in $[\ell_0]$ to the edges of $P_{1,2}$ in the increasing order along the orientation of $P_{1,2}$ ;
Set $V = \{v_1, v_2\}$ ;
**while** $V \neq \{v_1, \dots, v_n\}$ **do**
  **for** $i = 1$ *to* $n-2$ **do**
    Assign the numbers in $\{\ell_0 + \cdots + \ell_{i-1} + 1, \dots, \ell_0 + \cdots + \ell_{i-1} + \ell_i\}$ to the edges of $P_{i,i+2}$ in the increasing order along the orientation of $P_{i,i+2}$ ;
    Set $V$ to be $V \cup \{v_{i+2}\}$ ;
  **end**
  Assign the numbers in $\{\ell_0 + \cdots + \ell_{n-2} + 1, \dots, \ell_0 + \cdots + \ell_{n-2} + \ell_{n-1}\}$ to the edges of $P_{n-1,n}$ in the increasing order along the orientation of $P_{n-1,n}$ ;
**end**

By Algorithm 2, $s_D(u_j) = -1$ for all $j \in [(d-1)n]$. Let $D^*$ be the corresponding orientation of $C^*$, that is, orient each edge on $C^*$ as it is oriented on $C$. Clearly, $D^*$ is an orientation of $G$. It remains to verify that the bijection $c$ given in Algorithm 2 is an antimagic labeling of $D^*$. We may assume that $V(G) = V_R$. For each $v_i \in V(G)$, $s_{D^*}(v_i) = s_D(v_i) + (d-1)s_D(u_i^*) = s_D(v_i) - (d-1)$, where $u_i^*$ is one of the $d-1$ imaginary vertices of $v_i$. It suffices to show that for any $v_i, v_j \in V_R$ with $i \neq j$, $s_D(v_i) \neq s_D(v_j)$. According to Algorithm 2, when $n$ is even, we see that $s_D(v_1) = -\ell_0 - 2$; $s_D(v_n) = -2(\ell_0 + \cdots + \ell_{n-3}) - \ell_{n-2} - 2$ if $n \equiv 0 \pmod 4$ and $s_D(v_n) = 2(\ell_0 + \cdots + \ell_{n-2}) + \ell_{n-1}$ if $n \equiv 2 \pmod 4$; and for any $2 \le i \le n-1$, $s_D(v_i) = 2(\ell_0 + \cdots + \ell_{i-2}) + \ell_{i-1} + \ell_i$ if $d_D^+(v_i) = 0$, and $s_D(v_i) = -2(\ell_0 + \cdots + \ell_{i-3}) - \ell_{i-2} - \ell_{i-1} - 2$ if $d_D^+(v_i) = 2$. When $n$ is odd, we see that $s_D(v_1) = \ell_0 + \ell_1 - 1$; $s_D(v_n) = -2(\ell_0 + \cdots + \ell_{n-3}) - \ell_{n-2} - 2$ if $n \equiv 3 \pmod 4$ and $s_D(v_n) = 2(\ell_0 + \cdots + \ell_{n-2}) + \ell_{n-1}$ if $n \equiv 1 \pmod 4$; and for any $2 \le i \le n-1$, $s_D(v_i) = 2(\ell_0 + \cdots + \ell_{i-2}) + \ell_{i-1} + \ell_i$ if $d_D^+(v_i) = 0$, and $s_D(v_i) = -2(\ell_0 + \cdots + \ell_{i-3}) - \ell_{i-2} - \ell_{i-1} - 2$ if $d_D^+(v_i) = 2$. It can be easily checked that for any $v_i, v_j \in V_R$ with $i \neq j$, $s_D(v_i) \neq s_D(v_j)$.

This completes the proof of Theorem 3.1. ■

It would be nice if Theorem 3.1 would be true without assuming that $G$ is connected. From the proof of Theorem 3.1, we obtain the following two results, where a component of a graph is *odd* if it has an odd number of vertices.

**Corollary 3.2.** *Let $G$ be a $2d$-regular graph, where $d \ge 2$ is an integer. If $G$ has at most two odd components, then $G$ admits an antimagic orientation.*

*Proof.* Let $G_1, G_2, \ldots, G_q$ be all the components of $G$. For each $i \in [q]$, edges of $G_i$ are oriented as given in the proof of Theorem 3.1. We then label the edges of $G_1$ according to Algorithm 2 with labels in $\{1, \ldots, e(G_1)\}$, for all $i \in \{2, \ldots, q\}$, label the edges of $G_i$ according to Algorithm 2 with labels in $\{e(G_1) + e(G_2) + \cdots + e(G_{i-1}) + 1, \ldots, e(G_1) + e(G_2) + \cdots + e(G_i)\}$. Let $D$ be the resulting orientation of $G$. Clearly, the labeling of $D$ is antimagic if $G$ has at most one odd component. Without loss of generality, we assume that both $G_1$ and $G_2$ are odd. Let $v_1, \ldots, v_n$ be the real vertices of an Euler tour of $G_1$, and $u_1, u_2, \ldots, u_m$ be the real vertices of an Euler tour of $G_2$. From the proof of Theorem 3.1, no two vertices of $D$ has the same vertex-sum, except that $s_D(u_1)$ may be the same as the vertex-sum of some vertex in $G_1$. To avoid this, we relabel the edges on the paths $P_{1,2}$ and $P_{1,3}$ in the orientation of $G_2$ only as follows, where $P_{1,2}$, $P_{1,3}$, $\ell_0$, and $\ell_1$ are defined as in the proof of Theorem 3.1: assign the numbers in $\{e(G_1) + 1, \ldots, e(G_1) + \ell_1\}$ to the edges of $P_{1,3}$ in the increasing order along the orientation of $P_{1,3}$, then assign the numbers in $\{e(G_1) + \ell_1 + 1, \ldots, e(G_1) + \ell_1 + \ell_0\}$ to the edges of $P_{1,2}$ in the increasing order along the orientation of $P_{1,2}$. Then $s_D(u_1) = -1$. One can easily check that the resulting labeling of $D$ is antimagic. ■

**Corollary 3.3.** *Let $d \ge 2$ be an integer. If every vertex of a connected graph $G$ has degree $2d$ or $2d-2$, then $G$ admits an antimagic orientation.*

*Proof.* Let $G$ be a connected graph such that every vertex of $G$ has degree $2d$ or $2d-2$. Let $C^*, C, D, D^*$ be defined as in the proof of Theorem 3.1. Then $C^*$ contains $d$ or $d-1$ copies of each vertex of $G$. From the proof of Theorem 3.1, we see that $|s_D(u) - s_D(v)| \ge 2$ for any two distinct vertices $u, v$ in $D$; and $s_{D^*}(u) = s_D(u) - (d-1)$ or $s_{D^*}(u) = s_D(u) - (d-2)$ for any $u$ in $D^*$. Thus $s_{D^*}(u) \neq s_{D^*}(v)$ for any two distinct vertices $u, v$ in $D^*$, and so the labeling of $D^*$ is antimagic. ■

It seems hard to prove that if each of $G_1$ and $G_2$ has an antimagic orientation, then the disjoint union of $G_1$ and $G_2$ also has an antimagic orientation. But we know of no counterexamples. With the support of Theorem 2.1, we believe the following is true.

**Conjecture 3.4.** *Every graph admits an antimagic orientation.*

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