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Planar anti-Ramsey numbers of matchings

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ABSTRACT

Given a positive integer n and a planar graph H, let $\mathcal{T}_n(H)$ be the family of all plane triangulations T on n vertices such that T contains a subgraph isomorphic to H. The *planar anti-Ramsey number of* H, denoted $ar_{\mathcal{P}}(n, H)$, is the maximum number of colors in an edge-coloring of a plane triangulation $T \in \mathcal{T}_n(H)$ such that T contains no rainbow copy of H. In this paper we study planar anti-Ramsey numbers of matchings. For all $t \geq 1$, let M_t denote a matching of size t. We prove that for all $t \geq 6$ and $n \geq 3t - 6$, $2n + 3t - 15 \leq ar_{\mathcal{P}}(n, M_t) \leq 2n + 4t - 14$, which significantly improves the existing lower and upper bounds for $ar_{\mathcal{P}}(n, M_t)$. It seems that for each $t \geq 6$, the lower bound we obtained is the exact value of $ar_{\mathcal{P}}(n, M_t)$ for sufficiently large n. This is indeed the case for M_6. We prove that $ar_{\mathcal{P}}(n, M_6) = 2n + 3$ for all $n \geq 30$.

1. Introduction

All graphs considered in this paper are finite and simple. For a graph G we use $|G|$ and $e(G)$ to the number of vertices and number of edges of G, respectively. For a vertex $x \in V(G)$, we will use $N_G(x)$ to denote the set of vertices in G which are adjacent to x. We define $d_G(x) = |N_G(x)|$. For any $A \subseteq V(G)$, the subgraph of G induced by A, denoted $G[A]$, is the graph with vertex set A and edge set $\{xy \in E(G) : x, y \in A\}$. We denote $G \setminus A$ the subgraph of G induced on $V(G) \setminus A$. If $A = \{a\}$, we simply write $G \setminus a$. For disjoint subsets A, B of $V(G)$, we use $e_G(A, B)$ to denote the number of edges in G with one end in A and the other in B. Since every planar bipartite graph on $n \geq 3$ vertices has at most $2n - 4$ edges, we will frequently use the fact that $e_G(A, B) \leq 2(|A| + |B|) - 4$ when G is planar and $|A \cup B| \geq 3$. Given two isomorphic graphs G and H, we may (with a slight but common abuse of notation) write $G = H$. For any positive integer k, let $[k] := \{1, 2, \ldots, k\}$. We use the convention that "$A :=$" means that A is defined to be the right-hand side of the relation.

Motivated by anti-Ramsey numbers introduced by Erdős, Simonovits and Sós [5] in 1975, we study the anti-Ramsey problem when host graphs are plane triangulations. A subgraph of an edge-colored graph is *rainbow* if all of its edges have different colors. Given a planar graph H and a positive integer $n \geq |H|$, let $\mathcal{T}_n(H)$ be the family of all plane triangulations T on n vertices such that T contains a subgraph isomorphic to H. The *planar anti-Ramsey number of* H, denoted $ar_{\mathcal{P}}(n, H)$, is the maximum number of colors in an edge-coloring of a plane triangulation $T \in \mathcal{T}_n(H)$ such that T contains no rainbow copy of H. Analogous to the relation between anti-Ramsey numbers and Turán numbers proved in [5], planar anti-Ramsey numbers are closely related to planar Turán numbers [12], where the *planar Turán number of* H, denoted $ex_{\mathcal{P}}(n, H)$, is the maximum number of edges of a planar graph on n vertices that contains no subgraph isomorphic to H.

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Proposition 1.1 (*[12]*). *Given a planar graph H and a positive integer $n \geq |H|$,*

$$1 + ex_{\mathcal{P}}(n, \mathcal{H}) \leq ar_{\mathcal{P}}(n, H) \leq ex_{\mathcal{P}}(n, H),$$

where $\mathcal{H} = \{H - e : e \in E(H)\}$.

Dowden [3] began the study of planar Turán numbers (under the name of "extremal" planar graphs). Results on planar Turán numbers of paths and cycles can be found in [3,13]. The study of planar anti-Ramsey numbers was initiated by Horňák, Jendrol', Schiermeyer and Soták [6] (under the name of rainbow numbers). Results on planar anti-Ramsey numbers of paths and cycles can be found in [6,12]. Colorings of plane graphs that avoid rainbow faces have also been studied, see, e.g., [4,7,16,18]. Various results on anti-Ramsey numbers can be found in: [1,2,9–11,14,17] to name a few.

Finding exact values of $ar_{\mathcal{P}}(n, H)$ is far from trivial. As observed in [6], an induction argument in general cannot be applied to compute $ar_{\mathcal{P}}(n, H)$ because deleting a vertex from a plane triangulation may result in a graph that is no longer a plane triangulation. In this paper, we study planar anti-Ramsey numbers of matchings. For all $t \geq 1$, let M_t denote a matching of size t. In [8], the exact value of $ar_{\mathcal{P}}(n, M_t)$ when $t \leq 4$ was determined, and lower and upper bounds for $ar_{\mathcal{P}}(n, M_t)$ were also established for all $t \geq 5$ and $n \geq 2t$. Recently, the exact value of $ar_{\mathcal{P}}(n, M_5)$ was determined in [15] and an improved upper bound for $ar_{\mathcal{P}}(n, M_t)$ was also obtained in [15]. We summarize the results in [8,15].

Theorem 1.2 (*[8]*). *Let n and t be positive integers. Then*

(a) for all $n \geq 7$, $ar_{\mathcal{P}}(n, M_3) = n$.
(b) for all $n \geq 8$, $ar_{\mathcal{P}}(n, M_4) = 2n - 2$.
(c) for all $t \geq 5$ and $n \geq 2t$, $2n + 2t - 10 \leq ar_{\mathcal{P}}(n, M_t) \leq 2n + 2t - 7 + 2\binom{2t-2}{3}$.

Theorem 1.3 (*[15]*). *Let n and t be positive integers. Then*

(a) for all $n \geq 11$, $ar_{\mathcal{P}}(n, M_5) = 2n$.
(b) for all $t \geq 5$ and $n \geq 2t$, $ar_{\mathcal{P}}(n, M_t) \leq 2n + 6t - 17$.

In this paper, we further improve the existing lower and upper bounds for $ar_{\mathcal{P}}(n, M_t)$.

Theorem 1.4. *For all $t \geq 6$ and $n \geq 3t - 6$, $2n + 3t - 15 \leq ar_{\mathcal{P}}(n, M_t) \leq 2n + 4t - 14$.*

Theorem 1.4 significantly improves the lower bound in Theorem 1.2(c) and the new upper bound in Theorem 1.3(b). We believe that for each $t \geq 6$, the lower bound we obtained in Theorem 1.4 is the exact value of $ar_{\mathcal{P}}(n, M_t)$ for sufficiently large n. This is indeed the case for M_6.

Theorem 1.5. *For all $n \geq 30$, $ar_{\mathcal{P}}(n, M_6) = 2n + 3$.*

It seems that the method we developed in the proof of Theorem 1.5 can be applied to close the gap in Theorem 1.4. We prove Theorem 1.4 in Section 2 and Theorem 1.5 in Section 3.

2. Proof of Theorem 1.4

We are ready to prove Theorem 1.4. Let t, n be given as in the statement. We first prove that $ar_{\mathcal{P}}(n, M_t) \geq 2n+3t-15$. Let P be a path with vertices $v_1, v_2, \ldots, v_{t-4}$ in order. Let H be the plane triangulation obtained from P by adding two adjacent vertices x, y and joining each of x and y to all the vertices on P with the outer face of H having vertices x, y, v_1 on its boundary. Then $|H| = t - 2 \geq 4$ and H is hamiltonian. Let T_H be the plane triangulation obtained from H by adding a new vertex to each face F of H and then joining it to all vertices on the boundary of F. Then T_H is a plane triangulation on $(t-2) + (2(t-2) - 4) = 3t - 10$ vertices. Let w be the new vertex added to the outer-face of H. Let T be the plane triangulation on n vertices obtained from T_H by adding $n - (3t - 10) \geq 4$ vertices, say $w_1, w_2, \ldots, w_{n-3t+10}$, to the face of T_H containing x, y, w, such that $ww_1, w_{n-3t+10}x, w_{n-3t+10}y \in E(T)$, and for all $i \in [n - 3t + 9]$, w_i is adjacent to x, y, w_{i+1} in G. The construction of T when $t = 6$ and $n = 13$ is depicted in Fig. 1. Clearly, $T \in \mathcal{T}_n(M_t)$. Let c be an edge-coloring of T by first coloring all the edges $ww_1, w_1w_2, \ldots, w_{n-3t+9}w_{n-3t+10}$ by color 1 and then all the remaining edges of T by distinct colors other than 1. It can be easily checked that T has no rainbow M_t under the coloring c and the total number of colors used by c is $(3n-6)-(n-3t+10)+1 = 2n+3t-15$. This proves that $ar_{\mathcal{P}}(n, M_t) \geq 2n+3t-15$, as desired.

It remains to prove that $ar_{\mathcal{P}}(n, M_t) \leq 2n+4t-14$. Suppose $ar_{\mathcal{P}}(n, M_t) \geq 2n+4t-13$ for some $t \geq 6$ and $n \geq 3t-6$. Then there exists a $T \in \mathcal{T}_n(M_t)$ such that T has no rainbow copy of M_t under some onto mapping $c : E(T) \to [k]$, where $k \geq 2n+4t-13$. We choose such a T with t minimum. Let G be a rainbow spanning subgraph of T with k edges. Then G does not contain M_t because T has no rainbow copy of M_t. By minimality of t and Theorem 1.3(a) (when $t = 6$), G contains a copy of M_{t-1}. Let $M := \{u_iw_i \in E(G) : i \in [t-1]\}$ be a matching of size $t-1$ in G, and let $V(M) := \{u_1, \ldots, u_{t-1}, w_1, \ldots, w_{t-1}\}$. Let $R := V(G) \setminus V(M)$. For each $i \in [t-1]$, we may assume that $|N_G(u_i) \cap R| \leq |N_G(w_i) \cap R|$. Since M is the largest matching in G, we see that G has no M-augmenting path. It follows that $G[R]$ has no edges, and for each $i \in [t-1]$, either $|N_G(u_i) \cap R| = 0$ or $|N_G(u_i) \cap R| = 1$ with $N_G(u_i) \cap R = N_G(w_i) \cap R$. We may further assume that $u_1w_1, \ldots, u_\ell w_\ell$ are such

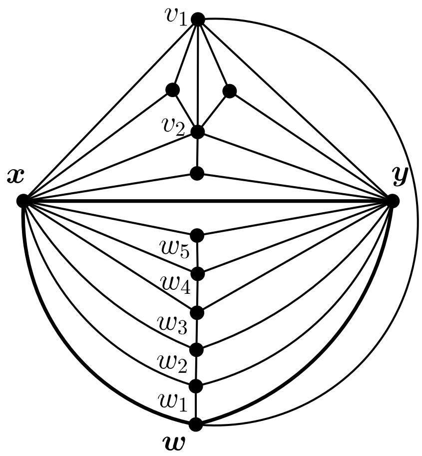

Fig. 1. The construction of T when $t = 6$ and $n = 13$.

that $|N_G(u_i) \cap R| = 1$ for all $0 \le i \le \ell$, and $|N_G(u_j) \cap R| = 0$ for all $j \in \{\ell+1, \dots, t-1\}$, where $0 \le \ell \le t-1$. Then $e_G(\{u_1, \dots, u_\ell, w_1, \dots, w_\ell\}, R) = 2\ell$. Let $L := \{w_{\ell+1}, \dots, w_{t-1}\}$. Then $e_G(L, R) = 0 \le 2n - 2t - 2\ell - 2$ when $\ell = t-1$, and $e_G(L, R) \le 2(n-(t-1)-\ell) - 4 = 2n - 2t - 2\ell - 2$ when $\ell \le t-2$ because $G[L \cup R] \setminus E(G[L])$ is a planar bipartite graph on $n-(t-1)-\ell \ge 3$ vertices. Since G is planar and $|V(M)| = 2(t-1) > 3$, we have $e(G[V(M)]) \le 3(2t-2) - 6 = 6t - 12$. Thus, $e(G) = e(G[V(M)]) + e_G(\{u_1, \dots, u_\ell, w_1, \dots, w_\ell\}, R) + e_G(L, R) \le (6t-12) + 2\ell + (2n - 2t - 2\ell - 2) = 2n + 4t - 14$, contrary to $e(G) = k \ge 2n + 4t - 13$. This completes the proof of Theorem 1.4. ■

Remark. For $t \ge 7$, the condition "$n \ge 3t - 6$" in the statement of Theorem 1.4 can be replaced by "$n \ge 3t - 7$".

3. Proof of Theorem 1.5

We need to introduce more notation that shall be used in this section only. For $n \ge 3$, let $\mathcal{T}_n$ be the set of all plane triangulations on n vertices, and let $\mathcal{T}_n^-$ be the set of all planar graphs with n vertices and $3n-7$ edges. Clearly, every graph in $\mathcal{T}_n^-$ is isomorphic to a plane triangulation on n vertices with one edge removed. By abusing notation, let $e(\mathcal{T}_n) := 3n-6$ and $e(\mathcal{T}_n^-) := 3n - 7$. It is known that every plane triangulation on $n \ge 4$ vertices is 3-connected. It is also known that every plane triangulation on $n \le 10$ vertices has a Hamilton cycle and every plane triangulation on $n \ge 11$ vertices does not necessarily have a Hamilton cycle.[1] We summarize these facts as follows.

Observation 3.1. *Let T be a planar triangulation on $n \ge 4$ vertices. Then*

(a) T is 3-connected.
(b) for every $n \le 10$, T has a Hamilton cycle.
(c) for every $n \ge 11$, T does not necessarily have a Hamilton cycle.

Let $o(H)$ denote the number of odd components in a graph H. We shall make use of the following theorem in the proof of Theorem 1.5.

Theorem 3.2. *[Berge–Tutte Formula] Let G be a graph on n vertices and let d be the size of a maximum matching of G. Then there exists an $S \subseteq V(G)$ with $|S| \le d$ such that*

$$2d = n - o(G \setminus S) + |S|.$$

Moreover, each odd component of $G \setminus S$ is factor-critical.

Proof of Theorem 1.5. Let $n \ge 30$ be an integer. By Theorem 1.4, $ar_{\mathcal{P}}(n, M_6) \ge 2n + 3$. We next show that $ar_{\mathcal{P}}(n, M_6) \le 2n + 3$. Suppose $ar_{\mathcal{P}}(n, M_6) \ge 2n + 4$. Then there exists a $T \in \mathcal{T}_n(M_6)$ such that T has no rainbow M_6 under some onto mapping $c : E(T) \to [k]$, where $k \ge 2n + 4$. Let G be a rainbow spanning subgraph of T with k edges. By Theorem 1.3(a), G has a copy of M_5. Clearly, G has no copy of M_6 because T has no rainbow copy of M_6 under c. By Theorem 3.2, there exists an $S \subseteq V(G)$ with $s := |S| \le 5$ such that $q := o(G \setminus S) = n + s - 10$. Let $H_1, H_2, \dots, H_q$ be

[1] The third author would like to thank Jason Bentley, a Ph.D. student at the University of Central Florida, for his help in carefully verifying these facts with her.

all the odd components of $G \setminus S$. We may assume that $|H_1| \le |H_2| \le \cdots \le |H_q|$. Let $r := \max\{i : |H_i| = 1\}$. Then $n = |G| \ge |S| + (|H_1| + \cdots + |H_r|) + (|H_{r+1}| + \cdots |H_q|) \ge s + r + 3(q - r) = s - 2r + 3(n + s - 10)$. It follows that $r \ge n + 2s - 15 \ge 15$. Let $S := \{v_1, \ldots, v_s\}$ when $s \ge 1$ and $V(H_i) = \{u_i\}$ for all $i \in [r]$. We may further assume that $d_G(u_1) \ge d_G(u_2) \ge \cdots \ge d_G(u_r)$. Let $U := \{u_1, \ldots, u_r\}$ and $W := V(G) \setminus (S \cup U)$. Then $w := |W| = n - s - r$ and $e_G(U, S) \le 2(r + s) - 4$ when $r + s \ge 3$. We next prove several claims.

Claim 1. *If G has two edge-disjoint matchings of size 5, say M' and M'', then $T[V(G) \setminus V(M' \cup M'')]$ has no edges.*

Proof. Suppose $T[V(G) \setminus V(M' \cup M'')]$ has an edge e. We may assume that $c(e) \ne c(e')$ for all $e' \in M'$. But then $M' \cup \{e\}$ is a rainbow M_6 in T under the coloring c, a contradiction. ■

Claim 2. *If $w + s \le 9$, then $H := G[W \cup S \cup \{u_1, \ldots, u_{10-w-s}\}] \notin \mathcal{T}_{10}$.*

Proof. Suppose $H \in \mathcal{T}_{10}$. Then H has a Hamilton cycle by Observation 3.1(b), and thus has two edge-disjoint matchings of size 5. By Claim 1, $T[\{u_{11-w-s}, \ldots, u_r\}]$ has no edges. But then

$$\begin{aligned} e(T) &= e(T[W \cup S \cup \{u_1, \ldots, u_{10-w-s}\}]) + e_T(\{u_{11-w-s}, \ldots, u_r\}, W \cup S \cup \{u_1, \ldots, u_{10-w-s}\}) \\ &\le e(\mathcal{T}_{10}) + (2n - 4) = 24 + (2n - 4), \end{aligned}$$

which implies that $n \le 26$ because $e(T) = 3n - 6$, contrary to $n \ge 30$. ■

Claim 3. $|H_q| \ge 3$.

Proof. Suppose $|H_q| < 3$. Then $r = q$ and so $w + s = n - q = 10 - s$. It follows that

$$\begin{aligned} 2n + 4 \le e(G) &= e(G[W \cup S]) + e_G(U, S) \le (3(w + s) - 6) + (2(n - w) - 4) \\ &= 2n + w + 3s - 10 = 2n + (10 - 2s) + 3s - 10 = 2n + s, \end{aligned}$$

which implies that $s \ge 4$. If $s = 4$, then $w = 2$. But then

$$2n + 4 \le e(G) = e(G[S]) + e(G[W]) + e_G(S, U \cup W) \le e(\mathcal{T}_4) + 1 + 2n - 4 = 2n + 3,$$

which is impossible. Thus $s = 5$, and so $w + s = 5$ and $r = q = n - 5$. By Claim 2, $e(H) \le 23$. Since $2n + 4 \le e(G) = e(G[S]) + e_G(U, S) \le e(G[S]) + 2n - 4$, we see that $e(G[S]) \ge 8$. Thus $G[S] \in \mathcal{T}_5$ or $G[S] \in \mathcal{T}_5^-$. Then $d_G(u_1) \le 12 - e(G[S])$, else $e(G[S \cup \{u_1\}]) \ge 13 > e(\mathcal{T}_6) = 12$, a contradiction. Suppose $G[S] \in \mathcal{T}_5$. Then $d_G(u_1) \le 3$ and $e_G(U, S) = 2n - 5$. It follows that $d_G(u_5) = 3$, else $e_G(U, S) = e_G(\{u_1, u_2, u_3, u_4\}, S) + e_G(\{u_5, \ldots, u_r\}, S) \le 12 + 2(n - 9) = 2n - 6$. But then $e(H) = e(G[S]) + e_G(\{u_1, \ldots, u_5\}, S) = 9 + 3 \times 5 = 24$, contrary to $e(H) \le 23$. This proves that $G[S] \in \mathcal{T}_5^-$. We may assume that $v_1v_2 \notin E(G[S])$. Then $d_G(u_1) \le 12 - e(G[S]) = 4$, and $e_G(U, S) = e(G) - e(G[S]) \ge (2n + 4) - 8 = 2n - 4$, which implies that $e_G(U, S) = 2n - 4$. If $d_G(u_2) \ge 4$, then $e(G[S \cup \{u_1, u_2\}]) = 8 + 8 = 16 > e(\mathcal{T}_7)$, a contradiction. Thus $d_G(u_2) \le 3$. Furthermore, $d_G(u_5) = 3$, else $e_G(U, S) = e_G(\{u_1, u_2, u_3, u_4\}, S) + e_G(\{u_5, \ldots, u_r\}, S) \le (4 + 9) + 2(n - 9) = 2n - 5$, contrary to $e_G(U, S) = 2n - 4$. Since $e(H) \le 23$, we see that $d_G(u_1) = 3$. Then $d_G(u_6) = 3$, else $e_G(U, S) \le 15 + 2(n - 10) = 2n - 5$, a contradiction. Since $G[S \cup \{u_1, \ldots, u_6\}]$ does not contain $K_{3,3}$ as a subgraph, we may assume that $v_1u_6 \in E(G)$. Then $H \in \mathcal{T}_{10}^-$ because $e(H) = e(G[S]) + e_G(\{u_1, \ldots, u_5\}, S) = 8 + 15 = e(\mathcal{T}_{10}^-)$. Note that $H + v_1v_2 \in \mathcal{T}_{10}$. By Observation 3.1(b), H has a hamiltonian path with v_1 as an end. Since $v_1u_6 \in E(G)$, we see that $G[S \cup \{u_1, \ldots, u_6\}]$ has two edge-disjoint matchings of size 5. By Claim 1, $T[\{u_7, \ldots, u_r\}]$ has no edges. But then

$$\begin{aligned} 3n - 6 = e(T) &= e(T[S \cup \{u_1, \ldots, u_6\}]) + e_T(\{u_7, \ldots, u_r\}, S \cup \{u_1, \ldots, u_6\}) \\ &\le e(\mathcal{T}_{11}) + (2n - 4) = 27 + (2n - 4), \end{aligned}$$

which implies that $n \le 29$, contrary to $n \ge 30$. ■

By Claim 3, $w \ge 3$ and $r \le q - 1$. Then $n \ge s + |H_1| + \cdots + |H_q| \ge s + q + 2 = n + 2s - 8$, which implies that $s \le 4$, with $s = 4$ only when $w = |H_q| = 3$.

Claim 4. $|H_{q-1}| \ge 3$.

Proof. Suppose $|H_{q-1}| = 1$. By Claim 3, $r = q - 1$. Thus $w + s = n - r = n - (q - 1) = 11 - s$. It follows that

$$\begin{aligned} 2n + 4 \le e(G) &= e(G[W \cup S]) + e_G(U, S) \le (3(w + s) - 6) + (2(n - w) - 4) \\ &= 2n + w + 3s - 10 = 2n + (11 - 2s) + 3s - 10 = 2n + s + 1, \end{aligned}$$

which implies that $s \ge 3$. If $s = 3$, then $w + s = 8$ and so $d_G(u_1) = 3$, else $e(G) \le e(G[W \cup S]) + e_G(U, S) \le e(\mathcal{T}_8) + 2(n - 8) = 2n + 2$, a contradiction. Since G does not contain $K_{3,3}$ as a subgraph, we see that $d_G(u_3) \le 2$. By Claim 2, $e(H) \le 23$. Thus

$e(G[W \cup S]) = e(H) - d_G(u_1) - d_G(u_2) \le 20 - d_G(u_2)$. Note that $e_G(U, S) \le 3 + d_G(u_2) + 2(n - w - s - 2) = 2n - 17 + d_G(u_2)$. But then

$$2n + 4 \le e(G) = e(G[W \cup S]) + e_G(U, S) \le (20 - d_G(u_2)) + (2n - 17 + d_G(u_2)) = 2n + 3,$$

which is impossible. Thus $s = 4$. Then $w = 3$. It follows that $W = V(H_q)$ and $G[W] = K_3$ because $G[W]$ is factor-critical. Then $d_G(u_2) \ge 3$, else $e_G(U, S) \le 4 + 2(n-8) = 2n - 12$ and so $2n + 4 \le e(G) = e(G[W \cup S]) + e_G(U, S) \le e(\mathcal{T}_7) + (2n - 12) = 2n + 3$, a contradiction. Since $e_G(U, S) \le 2(n - 3) - 4 = 2n - 10$, we see that $e(G[W \cup S]) = e(G) - e_G(U, S) \ge (2n + 4) - (2n - 10) \ge 14$. Thus $G[W \cup S] \in \mathcal{T}_7$ or $G[W \cup S] \in \mathcal{T}_7^-$. Suppose $G[W \cup S] \in \mathcal{T}_7$. Then $d_G(u_i) \le 3$ for all $i \in [r]$. Since $e_G(U, S) = e(G) - e(\mathcal{T}_7) \ge 2(n - 7) + 3$, we see that $d_G(u_1) = d_G(u_2) = d_G(u_3) = 3$. But then $e(H) = 24$, contrary to $e(H) \le 23$. This proves that $G[W \cup S] \in \mathcal{T}_7^-$. Then $d_G(u_2) \le 3$, else $e(G[W \cup S \cup \{u_1, u_2\}]) = e(\mathcal{T}_7^-) + 8 \ge 14 + 8 > e(\mathcal{T}_9)$, a contradiction. Since $e_G(U, S) = e(G) - e(\mathcal{T}_7^-) \ge 2(n - 7) + 4$, we see that $d_G(u_2) = d_G(u_3) = 3$. Then $d_G(u_1) = 3$ because $e(H) \le 23$. Since $G[W \cup S] \in \mathcal{T}_7^-$ and $G[W] = K_3$, we may assume that $G[W \cup S] + v_1 v \in \mathcal{T}_7$ for some $v \in W \cup S$ with $v \ne v_1$. Note that $G[S \cup \{u_2, u_3, u_4\}]$ does not contain $K_{3,3}$ as a subgraph. We may further assume that $v_1 u_4 \in E(G)$. Then $H + v_1 v \in \mathcal{T}_{10}$. By Observation 3.1(b), H has a hamiltonian path with v_1 as an end. Since $v_1 u_4 \in E(G)$, we see that $G[W \cup S \cup \{u_1, u_2, u_3, u_4\}]$ has two edge-disjoint matchings of size 5. By Claim 1, $T[\{u_5, \dots, u_r\}]$ has no edges. But then

$$\begin{aligned} 3n - 6 = e(T) &= e(T[W \cup S \cup \{u_1, u_2, u_3, u_4\}]) + e_T(\{u_5, \dots, u_r\}, W \cup S \cup \{u_1, u_2, u_3, u_4\}) \\ &\le e(\mathcal{T}_{11}) + (2n - 4) = 27 + (2n - 4), \end{aligned}$$

which implies that $n \le 29$, contrary to $n \ge 30$. ■

By Claim 4, $w \ge 6$ and $r \le q - 2$. Then $n \ge s + |H_1| + \dots + |H_q| \ge s + q + 4 = n + 2s - 6$, which implies that $s \le 3$, with $s = 3$ only when $w = |H_q| + |H_{q-1}| = 3 + 3 = 6$ and $r = q - 2$. Suppose $s = 3$. Then $w = 6$. By Claim 2, $e(H) \le 23$. Since $G[S \cup \{u_1, u_2, u_3\}]$ does not contain $K_{3,3}$ as a subgraph, we see that $d_G(u_3) \le 2$. Then $d_G(u_2) = 3$ and $H \in \mathcal{T}_{10}^-$, else either

$$e(G) = e(H) + e_G(\{u_2, \dots, u_r\}, S) \le 23 + 2(n - 10) = 2n + 3, \text{ or}$$

$$e(G) = e(H) + e_G(\{u_2, \dots, u_r\}, S) \le 22 + 3 + 2(n - 11) = 2n + 3,$$

contrary to $e(G) \ge 2n + 4$ in both cases. Let $V(H_{q-1}) = \{x, y, z\}$ and $V(H_q) = \{x', y', z'\}$. Since $H \in \mathcal{T}_{10}^-$, we may assume that $xv_2, x'v_3 \in E(G)$. It follows that $G[W \cup S \cup \{u_1, u_2\}]$ has two edge-disjoint matchings of size 5, namely, $\{v_1u_1, v_2x, v_3u_2, yz, x'y'\}$ and $\{v_1u_2, v_2u_1, v_3x', xy, y'z'\}$. By Claim 1, $T[\{u_3, \dots, u_r\}]$ has no edges. But then

$$\begin{aligned} 3n - 6 = e(T) &= e(T[W \cup S \cup \{u_1, u_2\}]) + e_T(\{u_3, \dots, u_r\}, W \cup S \cup \{u_1, u_2\}) \\ &\le e(\mathcal{T}_{11}) + (2n - 4) = 27 + (2n - 4), \end{aligned}$$

which implies that $n \le 29$, contrary to $n \ge 30$. Thus $s \le 2$. Then $d_G(u_i) \le s$ for all $i \in [r]$ and S is a vertex-cut of $G[S \cup W]$ because $r \le q - 2$. Then $e_G(U, S) \le rs$ and $e(G[S \cup W]) \le e(\mathcal{T}_{w+s}) - 1 = 3(w + s) - 7 = 3(n - r) - 7$. Recall that $r \ge n + 2s - 15$. If $r \ge n + 2s - 14$, then

$$\begin{aligned} 2n + 4 \le e(G) &= e_G(U, S) + e(G[S \cup W]) \\ &\le rs + 3(n - r) - 7 \\ &= 3n - (3 - s)r - 7 \\ &\le 3n - (3 - s)(n + 2s - 14) - 7, \end{aligned}$$

which is impossible because $s \le 2$ and $n \ge 30$. Thus $r = n + 2s - 15 \le q - 3$, and so $e(G[S \cup W]) \le e(\mathcal{T}_{w+s}) - 2 = 3(w + s) - 8 = 3(n - r) - 8$. But then

$$\begin{aligned} 2n + 4 \le e(G) &= e_G(U, S) + e(G[S \cup W]) \\ &\le rs + 3(n - r) - 8 \\ &= 3n - (3 - s)r - 8 \\ &= 3n - (3 - s)(n + 2s - 15) - 8, \end{aligned}$$

which is impossible because $s \le 2$ and $n \ge 30$. This completes the proof of Theorem 1.5. ■

Remark. In the proof of Theorem 1.5, Claim 1 is applied to two vertex-disjoint matchings, instead of edge-disjoint matchings. It seems that the method we developed in the proof of Theorem 1.5 can be used to close the gap in Theorem 1.4.

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Conflict of interest statement

None.

Declaration of conflicting interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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