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Planar Turán number and planar anti-Ramsey number of graphs*

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Abstract The *planar Turán number* of a graph G, denoted $ex_{\mathcal{P}}(n,G)$, is the maximum number of edges in a planar graph on n vertices without containing G as a subgraph. Given a positive integer n and a plane graph H, let $\mathcal{T}_n(H)$ be the family of all plane triangulations T on n vertices such that T contains H as a subgraph. The *planar anti-Ramsey number* of H, denoted $ar_{\mathcal{P}}(n,H)$, is the maximum number k such that no edge-coloring of any plane triangulation in $\mathcal{T}_n(H)$ with k colors contains a rainbow copy of H. The study of these two topics was initiated around 2015, and has attracted extensive attention. This paper surveys results about planar Turán number and planar anti-Ramsey number of graphs. The goal is to give a unified and comprehensive presentation of the major results, as well as to highlight some open problems.

Keywords planar Turán number, planar anti-Ramsey number, Theta graph

Chinese Library Classification O157.5, O157.6

2010 Mathematics Subject Classification 05C10, 05C35, 05D10

图的平面 Turán 数和平面 anti-Ramsey 数*

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摘要 在所有顶点数为 n 且不包含图 G 作为子图的平面图中, 具有最多边数的图的边数称为图 G 的平面 Turán 数, 记为 $ex_{\mathcal{P}}(n,G)$。给定正整数 n 以及平面图 H, 用 $\mathcal{T}_n(H)$ 来表示所有顶点数为 n 且不包含 H 作为子图的平面三角剖分图所组成的图集合。设图集合 $\mathcal{T}_n(H)$ 中的任意平面三角剖分图的任意 k 边染色都不包含彩虹子图 H, 则称满足上述条件的 k 的最大值为图 H 的平面 anti-Ramsey 数, 记作 $ar_{\mathcal{P}}(n,H)$。两类问题的研究均始于 2015 年左右, 至今已经引起了广泛关注。全面地综述两类问题的主要研究成果, 以及一些公开问题。

关键词 平面 Turán 数, 平面 anti-Ramsey 数, Theta 图

中图分类号 O157.5, O157.6

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All graphs considered in this paper are finite and simple. We use P_n, C_n and $K_{1,n-1}$ to denote the path, cycle and star on n vertices, respectively. The *join* $G+H$ (resp. *union* $G\cup H$) of two vertex-disjoint graphs G and H is the graph having vertex set $V(G)\cup V(H)$ and edge set $E(G)\cup E(H)\cup\{xy\,|\,x\in V(G), y\in V(H)\}$ (resp. $E(G)\cup E(H)$). For a positive integer t and a graph G , we use tG to denote disjoint union of t copies of G; $\overline{G}$ the *complement* of G; $|G|$ the number of vertices of G; and $e(G)$ the number of edges of G. For any positive integer k, we define $[k]:=\{1,2,\cdots,k\}$.

Let $\mathcal{F}$ be a family of graphs. A graph is $\mathcal{F}$-free if it does not contain any graph in $\mathcal{F}$ as a subgraph. When $\mathcal{F}=\{F\}$ we write F-free. One of the best known results in extremal graph theory is Turán's Theorem[1], which gives the maximum number of edges that a K_k-free graph on n vertices can have. The celebrated Erdős-Stone Theorem[2] then extends this to the case when K_k is replaced by an arbitrary graph H with at least one edge, showing that the maximum number of edges possible is $(1+o(1))(\frac{\chi(H)-2}{\chi(H)-1})\binom{n}{2}$, where $\chi(H)$ denotes the chromatic number of H. This latter result has been called the "fundamental theorem of extremal graph theory"[3]. Turán-type problems when host graphs are hypergraphs are notoriously difficult. A large quantity of work in this area has been carried out in determining the maximum number of edges in a k-uniform hypergraphs on n vertices without containing k-uniform linear paths and cycles (see, for example, [4-6]). Surveys on Turán-type problems of graphs and hypergraphs can be found in [7] and [8].

Dowden[9] in 2016 initiated the study of Turán-type problems when host graphs are planar graphs, i.e., how many edges can an $\mathcal{F}$-free planar graph on n vertices have? The *planar Turán number of* $\mathcal{F}$, denoted $ex_{\mathcal{P}}(n,\mathcal{F})$, is the maximum number of edges in an $\mathcal{F}$-free planar graph on n vertices. When $\mathcal{F}=\{F\}$ we write $ex_{\mathcal{P}}(n,F)$. Dowden[9] observed that it is straightforward to determine the exact values of $ex_{\mathcal{P}}(n,H)$ when H is a complete graph or non-planar graph: for all $n\geqslant 4$, the planar graph $K_2+\overline{K}_{n-2}$ is K_3-free and so $ex_{\mathcal{P}}(n,H)=2n-4$; for all $n\geqslant 6$, the planar triangulation $2K_1+C_{n-2}$ is K_4-free. Hence, $ex_{\mathcal{P}}(n,H)=3n-6$ for all graphs H which contains K_4 as a subgraph and $n\geqslant\max\{|H|,6\}$. In particular, $ex_{\mathcal{P}}(n,K_5^-)=3n-6$ for all $n\geqslant 6$, where K_p^- denotes the graph obtained from the complete graph K_p by deleting one edge. There are now a variety of papers mainly focusing on the planar Turán number of paths, cycles, Theta graphs and graphs H with $ex_{\mathcal{P}}(n,H)=3n-6$.

Motivated by anti-Ramsey number introduced by Erdős, Simonovits and Sós[10] in 1975, Horňák, Jendrol′, Schiermeyer and Soták[11] (under the name of rainbow numbers) in 2015 initiated study of the anti-Ramsey problem when host graphs are plane triangulations. A subgraph of an edge-colored graph is *rainbow* if all of its edges have different colors. Let $\mathcal{F}$ be a family of planar graphs. For the purpose of this paper, we call an edge-coloring that contains no rainbow copy of any graph in $\mathcal{F}$ an $\mathcal{F}$*-free edge-coloring.* Let $n_{\mathcal{F}}$ be the smallest integer such that for any $n\geqslant n_{\mathcal{F}}$, there exists a plane triangulation on n vertices that is not $\mathcal{F}$-free. Such an integer $n_{\mathcal{F}}$ is well-defined, because for any $F\in\mathcal{F}$, we can obtain a plane triangulation from a plane drawing of F by adding a new vertex to each face of size at least four of F and then joining it to all vertices on the boundary of such a face. For each integer

$n \geqslant n_{\mathcal{F}}$, let $\mathcal{T}_n(\mathcal{F})$ be the family of all plane triangulations T on n vertices such that T is not $\mathcal{F}$-free. The *planar anti-Ramsey number of* $\mathcal{F}$, denoted $ar_{\mathcal{P}}(n, \mathcal{F})$, is the maximum number of colors in an $\mathcal{F}$-free edge-coloring of any plane triangulation in $\mathcal{T}_n(\mathcal{F})$. When $\mathcal{F}$ consists of a single graph H, we write n_H, $ar_{\mathcal{P}}(n, H)$ and $ex_{\mathcal{P}}(n, H)$ instead of $n_{\{H\}}$, $ar_{\mathcal{P}}(n, \{H\})$ and $ex_{\mathcal{P}}(n, \{H\})$. It is easy to see that $ar_{\mathcal{P}}(n, \mathcal{F}) < 3n-6$, and this problem becomes trivial if the host plane triangulation on n vertices is $\mathcal{F}$-free, because $3n-6$ colors can be used.

Analogous to the relation between anti-Ramsey number and Turán number proved in [10], the planar Turán number is closely related to the planar anti-Ramsey number of graphs, as was observed by Lan, Shi and Song in [12]. We recall the proof here.

Proposition 1[12] *Given a planar graph H and a positive integer $n \geqslant n_H$,*

$$1 + ex_{\mathcal{P}}(n, \mathcal{H}) \leqslant ar_{\mathcal{P}}(n, H) \leqslant ex_{\mathcal{P}}(n, H),$$

where $\mathcal{H} = \{H - e : e \in E(H)\}$.

Proof Given an edge-coloring c of a host graph T in $\mathcal{T}_n(H)$, we define a representing graph of c to be a spanning subgraph R of T obtained by taking one edge of each color under the coloring c (where R may contain isolated vertices). It is easy to check that, if c is an H-free edge-coloring of T, then R is H-free. Thus $ar_{\mathcal{P}}(n, H) \leqslant ex_{\mathcal{P}}(n, H)$ for any $n \geqslant n_H$. Next, let G be an $\mathcal{H}$-free plane subgraph of a plane triangulation $T \in \mathcal{T}_n(H)$ with $e(G) = ex_{\mathcal{P}}(n, H)$. We then obtain an H-free edge-coloring of T by coloring the edges of G with distinct colors and then coloring the edges in $E(T) \setminus E(G)$ with a new color. Hence, $1 + ex_{\mathcal{P}}(n, \mathcal{H}) \leqslant ar_{\mathcal{P}}(n, H)$ for any $n \geqslant n_H$. □

This paper surveys results about planar Turán number of graphs and planar anti-Ramsey number of graphs in Section 2 and Section 3, respectively. The goal is to give a unified and comprehensive presentation of the major results, as well as to highlight several open problems. We also collect recent results on counting subgraphs in planar graphs in Section 4.

1 Planar Turán number of graphs

How many edges can an H-free planar graph on n vertices have? Since 2016, this topic has attracted extensive attention. In this section, we collect all results on planar Turán number of graphs. We begin with results in [13] on graphs H satisfying $ex_{\mathcal{P}}(n, H) = 3n-6$, which demonstrate that it is quite non-trivial to determine $ex_{\mathcal{P}}(n, H)$ when H is a planar subcubic graph. Furthermore, the present authors[13] discovered that the chromatic number of H does not play a role in $ex_{\mathcal{P}}(n, H)$, as it does in the celebrated Erdős-Stone Theorem.

1.1 Graphs H with $ex_{\mathcal{P}}(n, H) = 3n-6$

Before we state the main results, we need some definitions. For a graph G and $v \in V(G)$, we use $\chi(G)$ and $\Delta(G)$ to denote the chromatic number and maximum degree of G, respectively; $N_G(v)$ denotes the set of neighbors of v. Let $N_G[x] = N_G(x) \cup \{x\}$. A vertex is a *$k$-vertex* in G if it has degree k. We use $n_k(G)$ to denote the number of k-vertices in G.

For any set $S \subset V(G)$, the subgraph of G induced on S, denoted $G[S]$, is the graph with vertex set S and edge set $\{xy \in E(G) : x, y \in S\}$. We denote by $G \setminus S$ the subgraph of G induced on $V(G) \setminus S$.

As mentioned in the Introduction, $ex_{\mathcal{P}}(n, H) = 3n - 6$ for all graphs H which contains K_4 as a subgraph and $n \geqslant \max\{|H|, 6\}$. Lan, Shi and Song[13] established several sufficient conditions for all K_4-free planar graphs H with $ex_{\mathcal{P}}(n, H) = 3n - 6$.

Theorem 1[13] *Let H be a K_4-free planar graph and let $n \geqslant |H|$ be an integer. Then $ex_{\mathcal{P}}(n, H) = 3n - 6$ if one of the following holds.*

(a) $\chi(H) = 4$ *and* $n \geqslant |H| + 2$,

(b) $\Delta(H) \geqslant 7$,

(c) $\Delta(H) = 6$ *and either* $n_6(H) + n_5(H) \geqslant 2$ *or* $n_6(H) + n_5(H) = 1$ *and* $n_4(H) \geqslant 5$,

(d) $\Delta(H) = 5$ *and either H has at least three 5-vertices or H has exactly two adjacent 5-vertices,*

(e) $\Delta(H) = 4$ *and* $n_4(H) \geqslant 7$,

(f) *H is 3-regular (except for the case $|H| = 6$ and $n \leqslant 9$) or H has at least three vertex-disjoint cycles or H has exactly one vertex u of degree $\Delta(H) \in \{4, 5, 6\}$ such that* $\Delta(H[N(u)]) \geqslant 3$,

(g) $\delta(H) \geqslant 4$ *or H has exactly one vertex of degree at most* 3.

Theorem 1 implies that $ex_{\mathcal{P}}(n, H) = 3n - 6$ for all H with $n \geqslant |H| + 2$ and either $\chi(H) = 4$ or $\chi(H) = 3$ and $\Delta(H) \geqslant 7$. Note that both K_4^- and $K_1 + 2K_2$ have chromatic number 3. Theorem 8(a) and Theorem 2(c) (see below) then demonstrate that the chromatic number of H does not play a role in $ex_{\mathcal{P}}(n, H)$, as it does in the celebrated Erdős-Stone Theorem.

By Theorem 1, $ex_{\mathcal{P}}(n, H)$ remains unknown for K_4-free planar graphs H with $\Delta(H) = 6$, $n_6(H) + n_5(H) = 1$ and $n_4(H) \leqslant 4$; or $\Delta(H) = 5$ and $n_5(H) \leqslant 2$ (and the two 5-vertices are not adjacent when $n_5(H) = 2$); or $\Delta(H) = 4$ and $n_4(H) \leqslant 6$; or $\Delta(H) \leqslant 3$. It seems quite hard to determine $ex_{\mathcal{P}}(n, H)$ when H is planar subcubic graph. In particular, by Theorem 1(f), $ex_{\mathcal{P}}(n, H)$ remains unknown for K_4-free planar graphs H with exactly one vertex, say u, of degree $\Delta(H) \leqslant 6$ and $\Delta(H[N(u)]) \leqslant 2$. It seems quite non-trivial to determine $ex_{\mathcal{P}}(n, H)$ for all such H. Let $W_k := K_1 + C_k$, the *wheel* on $k + 1 \geqslant 5$ vertices. Unlike the classic Turán number of W_k (see [14,15] for more information), the planar Turán number of W_k can be completely determined. Moreover, the exact values of $ex_{\mathcal{P}}(n, K_{1,k})$ have also been settled completely. We summarize all other results from [13] in Theorem 2.

Theorem 2[13] *Let n, k and t be positive integers.*

(a) *For* $n \geqslant k + 1 \geqslant 5$,

$$ex_{\mathcal{P}}(n, K_1 + C_k) = \begin{cases} 3n - 6, & \text{if } k \geqslant 6, \text{ or } k = 5 \text{ and } n \neq 7, \text{ or } k = 4 \text{ and } n \geqslant 12, \\ 3n - 7, & \text{if } k = 4 \text{ and } n \in \{5, 6\}, \text{ or } k = 5 \text{ and } n = 7, \\ 3n - 8, & \text{if } k = 4 \text{ and } 7 \leqslant n \leqslant 11. \end{cases}$$

(b) *For* $n \geqslant t+1 \geqslant 4$,

$$ex_{\mathcal{P}}(n, K_{1,t}) = \begin{cases} 3n-6, & \text{if } t \geqslant 7, \text{ or } t=6 \text{ and } n \in \{7,8,9,10,12\}, \\ 3n-7, & \text{if } t=6 \text{ and } n=11, \\ 3n-8, & \text{if } t=6 \text{ and } n \in \{13,14\}, \text{ or } t=5 \text{ and } n=7, \\ \left\lfloor \dfrac{(t-1)n}{2} \right\rfloor, & \text{if } t \in \{3,4\}, \text{ or } t=5 \text{ and } n \neq 7, \text{ or } t=6 \text{ and } n \geqslant 15. \end{cases}$$

(c) *For* $n \geqslant 5$, $2n-3 \leqslant ex_{\mathcal{P}}(n, K_1+2K_2) \leqslant \frac{19n}{8}-4$. *Furthermore,* $ex_{\mathcal{P}}(n, K_1+2K_2) = \frac{19n}{8}-4$ *if and only if* n *is divisible by* 8.

(d) *For all* $n \geqslant 15$, $\left\lfloor \frac{5n}{2} \right\rfloor \leqslant ex_{\mathcal{P}}(n, K_1+3K_2) < \frac{17n}{6}-4$. *Furthermore,*

$$ex_{\mathcal{P}}(n, K_1+3K_2) = \begin{cases} 3n-6, & \text{if } n \in \{7,8,9,10,12\}, \\ 3n-7, & \text{if } n=11, \\ 3n-8, & \text{if } n \in \{13,14\}. \end{cases}$$

(e) $ex_{\mathcal{P}}(n, K_1+P) \leqslant \frac{13(t-1)n}{4t-2} - \frac{12(t-1)}{2t-1}$ *for all* $n \geqslant t+1$, *where* $4 \leqslant t \leqslant 6$ *and* P *is a disjoint union of paths with* $|P| = t$.

The upper bound in Theorem 2(c) is tight for infinitely many n. To see that, let $n = 8(k+1)$. We recall the construction of F_k in [13] by the illustration given in Figure 1: the graph F_0 is depicted in Figure 1(a), and the graph F_k for all $k \geqslant 1$ is obtained by placing the entire graph F_{k-1} into the center quadrangle of Figure 1(b) (in such a way that the center bold quadrangle of Figure 1(b) is identified with the outer quadrangle of F_{k-1}).

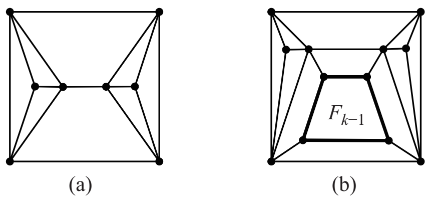


Figure 1 Construction of the sharp upper bound for $ex_{\mathcal{P}}(n, K_1+2K_2)$

Using a similar constructing method, the present authors in [13] also constructed a graph G_k with $n = 24(k+1)$ vertices and $67n/23-4$ edges for all $k \geqslant 0$ such that G_k is K_1+3K_2-free, as depicted in Figure 2. Very recently, Fang, Zhai and Wang[16] proved that graphs G_k for all $k \geqslant 0$ indeed yield the desired tight upper bounds for $ex_{\mathcal{P}}(n, K_1+3K_2)$; they further obtained the sharp upper bounds for $ex_{\mathcal{P}}(n, K_1+P_{t+1})$ for each $t \in \{2,3,4,5\}$. Moreover, each $t \in \{2,3,4,5\}$, the extremal graph for $ex_{\mathcal{P}}(n, K_1+P_{t+1})$ is the graph G_k^* obtained from G_k depicted in Figure 5 by placing K_3, K_4, R_1, R_5 into each 3-face of G_k, respectively, where R_1 is a 4-regular triangulation on 6 vertices and R_5 is depicted in Figure 2(a).

Theorem 3[16] *Let* n *and* t *be positive integers and* $t \in \{2,3,4,5\}$.

(a) $ex_{\mathcal{P}}(n, K_1+3K_2) \leqslant \frac{67n}{24}-4$ *for all* $n \geqslant 13$, *with equality if and only if* n *is divisible by* 24.

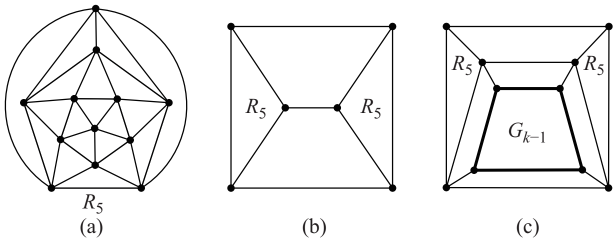


Figure 2 Construction of the sharp upper bound for $ex_{\mathcal{P}}(n, K_1 + 3K_2)$

(b) $ex_{\mathcal{P}}(n, K_1 + P_{t+1}) \leqslant \frac{24t}{7t+6}(n-2)$ *for all* $n \geqslant \frac{12}{6-t} + 1$, *with equality if and only if* $\frac{12(t+2)}{6-t} \equiv n \text{ mod } \frac{28t+24}{6-t}$.

1.2 Planar Turán number of cycles

Wang and Lih[17] in 2007 studied upper bounds on the sizes of $\mathcal{F}$-free graphs that are 2-cell embedded in a surface of nonnegative Euler characteristic to confirm the List Edge Coloring Conjecture for such graphs with maximum degree exceeding prescribed thresholds. We summarize their results on planar graphs below and refer the reader to [17] for further information.

Theorem 4[17] *Let* n *be a positive integer.*
(a) $ex_{\mathcal{P}}(n, C_4) \leqslant 15(n-2)/7$.
(b) $ex_{\mathcal{P}}(n, C_5) \leqslant 12(n-2)/5$.
(c) $ex_{\mathcal{P}}(n, C_6) \leqslant 63(n-2)/25$.
(d) $ex_{\mathcal{P}}(n, C_7) \leqslant (8n-11)/3$.
(e) $ex_{\mathcal{P}}(n, \{C_4, C_5\}) \leqslant 2(n-2)$.

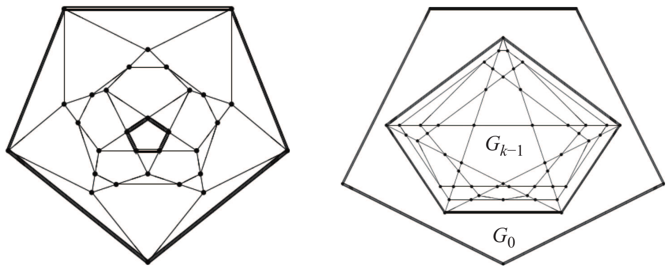
We believe when Dowden[9] in 2016 initiated the study of planar Turán number of graphs, he was unaware of results by Wang and Lih[17]. Dowden[9] studied the upper bounds for $ex_{\mathcal{P}}(n, C_k)$ when $k \in \{4, 5\}$.

Theorem 5[9] *Let* n *be a positive integer.*
(a) $ex_{\mathcal{P}}(n, C_4) \leqslant 15(n-2)/7$, *with equality when* $30 \equiv n \text{mod} 70$.
(b) $ex_{\mathcal{P}}(n, C_5) \leqslant 12(n-2)/5$, *and the bound is sharp for infinitely many* n.

Note that Wang and Lih[17] did not prove that each upper bound in Theorem 4 holds for infinitely many n. Dowden[9] provided clever and involved constructions for both $ex_{\mathcal{P}}(n, C_4)$ and $ex_{\mathcal{P}}(n, C_5)$. For $ex_{\mathcal{P}}(n, C_4)$, the extremal graph G_k with $n = 70k+30$ vertices is depicted in Figure 3. We refer the reader to [9] for their construction of the extremal graphs for C_5. Lan, Shi and Song[18] continued this topic and independently proved the following result.

Theorem 6[18] $ex_{\mathcal{P}}(n, C_6) \leqslant \frac{18}{7}(n-2)$ *for* $n \geqslant 6$, *with equality when* $n = 9$.

All extremal graphs are depicted in Figure 4 when $n = 9$. Very recently, Theorem 6 was subsequently improved by Ghosh, Győri, Martin, Paulos and Xiao[19], who gave a sharp upper bound for $ex_{\mathcal{P}}(n, C_6)$ for all $n \geqslant 18$. We refer the reader to [19] for their construction of the extremal graphs.



(a) G_0 (b) $G_k (k \geqslant 1)$

Figure 3 Construction of the sharp upper bound for $ex_{\mathcal{P}}(n, C_4)$

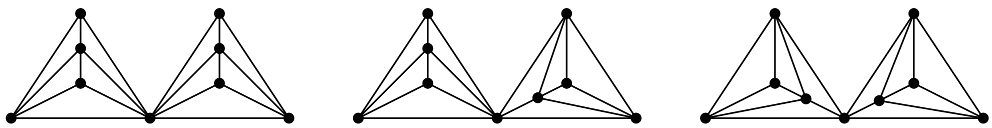


Figure 4 All extremal graphs achieving equality in Theorems 6 and 8(c) when $n = 9$

Theorem 7[19] *$ex_{\mathcal{P}}(n, C_6) \leqslant \frac{5n}{2} - 7$ for $n \geqslant 18$, and the bound is sharp for infinitely many n.*

All proofs of the results are in the same spirit and rely on the Euler's formula for planar graphs, except that Wang and Lih[17] applied the Discharging Method to prove Theorem 4(c) for C_6-free planar graphs.

With the support of Theorem 6, the present authors proposed a conjecture for $ex_{\mathcal{P}}(n, C_k)$ for all $k \geqslant 6$. The extremal graph for the conjectured upper bound can be obtained from vertex-disjoint copies of $T_1 \in \mathcal{T}_{k-1}$ and $T_2 \in \mathcal{T}_{k-1}$ by identifying one vertex in T_1 with a vertex in T_2, see Figure 4 for an example when $k = 6$.

Conjecture 1[18] *Let $n \geqslant k \geqslant 6$ be positive integers. Then $ex_{\mathcal{P}}(n, C_k) \leqslant (3 - \frac{3}{2k-5})(n-2)$, with equality when $n = 2k - 3$.*

Ghosh, Győri, Martin, Paulos and Xiao[19] then posed a better conjecture for $ex_{\mathcal{P}}(n, C_k)$ for all $k \geqslant 7$, which remains open.

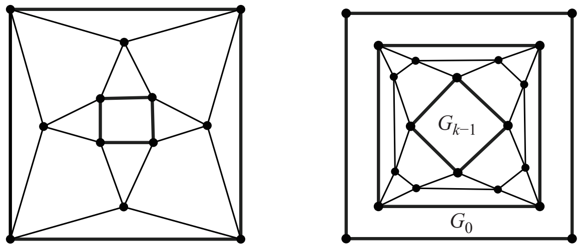
Conjecture 2[19] *Let $n \geqslant k \geqslant 7$ be positive integers. Then there exists an integer $N_0 > 0$ such that $ex_{\mathcal{P}}(n, C_k) \leqslant 3(k-1)n/k - 6(k+1)/k$ for all $n \geqslant N_0$.*

1.3 Planar Turán number of Theta graphs

A graph on at least 4 vertices is a *Theta graph* if it can be obtained from a cycle by adding an additional edge joining two non-consecutive vertices. For each integer $k \geqslant 4$, let Θ_k be the family of non-isomorphic Theta graphs on k vertices. Note that the only graph in Θ_4 is isomorphic to K_4^-, and Θ_5 has only one graph. By abusing notation, we also use Θ_4 and Θ_5 to denote the only graph in Θ_4 and Θ_5, respectively. It is worth noting that every C_k-free graph is also Θ_k-free. Lan, Shi and Song[18] initiated the study of $ex_{\mathcal{P}}(n, \Theta_k)$, which was inspired by a question of Dowden[9] when $k = 4$. They determined the upper bounds for $ex_{\mathcal{P}}(n, \Theta_k)$ for each $k \in \{4, 5, 6\}$. In particular, these bounds are tight for infinitely many n for each $k \in \{4, 5\}$.

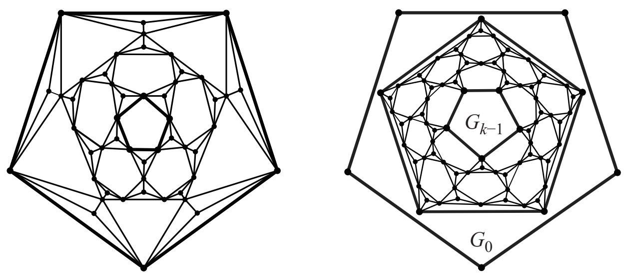
Theorem 8[18] *Let n be a positive integer.*
(a) $ex_{\mathcal{P}}(n, \Theta_4) \leqslant 12(n-2)/5$ *for all* $n \geqslant 4$, *with equality when* $12 \equiv n \mod 20$.
(b) $ex_{\mathcal{P}}(n, \Theta_5) \leqslant 5(n-2)/2$ *for all* $n \geqslant 5$, *with equality when* $50 \equiv n \mod 120$.
(c) $ex_{\mathcal{P}}(n, \Theta_6) \leqslant (18n-36)/7$ *for all* $n \geqslant 6$, *with equality when* $n = 9$.

The extremal graph G_k for $ex_{\mathcal{P}}(n, \Theta_4)$ with $n = 20k + 12$ vertices and $ex_{\mathcal{P}}(n, \Theta_5)$ with $n = 120k + 50$ vertices are given in Figure 5 and Figure 6, respectively. All extremal graphs for $ex_{\mathcal{P}}(n, \Theta_6)$ are depicted in Figure 4 when $n = 9$.



(a) G_0 (b) $G_k\,(k \geqslant 1)$

Figure 5 Construction of the sharp upper bounds for $ex_{\mathcal{P}}(n, \Theta_4)$



(a) G_0 (b) $G_k\,(k \geqslant 1)$

Figure 6 Construction of the sharp upper bounds for Θ_5

Recently, Ghosh, Győri, Paulos, Xiao and Zamora[20] improved further the upper bound in Theorem 8(*c*) for Θ_6. We refer the reader to [20] for their construction of the extremal graphs.

Theorem 9[20] *Let n be a positive integer. Then* $ex_{\mathcal{P}}(n, \Theta_6) \leqslant (18n-48)/7$ *for all* $n \geqslant 14$, *and the bound is sharp for infinitely many* n.

Note that Θ_6 contains exactly two non-isomorphic graphs. We use Θ_6^1 and Θ_6^2 to denote the symmetric (containing no 3-cycle) and asymmetric (containing 3-cycle) Theta graphs in Θ_6. The authors in [20] also proposed asymptotic conjectures for Θ_6^1 and Θ_6^2, respectively. For general Θ_k, this problem remains wide open.

Conjecture 3[20] *Let n be a positive integer.*

(a) $ex_{\mathcal{P}}(n, \Theta_6^1) = 45n/17 + \Theta(1)$.

(b) $ex_{\mathcal{P}}(n, \Theta_6^2) = 18n/7 + \Theta(1)$.

1.4 Planar Turán number of paths and matchings

Let M_n denote a matching with n edges. The classic Turán number for paths and matching were well studied. In 1975, Faudree and Schelp[21] have determined the classical Turán number of paths and all extremal graphs were characterized. Motivated by the classical Turán number of paths, Lan, Shi and Song[22] began the study of $ex_{\mathcal{P}}(n, P_k)$. It is worth noting that for all $k \in \{2, 3, 4, 5\}$, every P_k-free graph must be planar. Hence, when $k \in \{2, 3, 4, 5\}$, the exact values of $ex_{\mathcal{P}}(n, P_k)$ and the extremal graphs have been determined by Faudree and Schelp. The authors in [23] obtained the planar Turán number for P_k with $6 \leqslant k \leqslant 11$.

Theorem 10[23] *If G is a P_6-free planar graph of order $n \geqslant 6$, then $e(G) \leqslant 2n-2$ with equality when $G = 2K_5^-$ if $n = 10$; and $e(G) \leqslant 2n-3$ with equality when $G = K_2 + \overline{K}_{n-2}$ if $n \neq 10$. Moreover, the equality holds when $G \in \{K_5^- \cup K_1, K_5^- \cup K_4, 3K_5^-\}$.*

Theorem 11[23] *Let G be a P_7-free planar graph of order $n \geqslant 7$.*

(a) *If $n = 6t$, then $e(G) \leqslant 2n$ with equality when $G = T_1 \cup \cdots \cup T_t$.*

(b) *If $n = 6t + 5$, then $e(G) \leqslant 2n - 1$ with equality when $G = T_1 \cup \cdots \cup T_t \cup K_5^-$.*

(c) *If $n = 6t + r$ for $r \in [4]$, then $e(G) \leqslant 2n - 2$ with equality when $G \in \{K_2 + (\overline{K}_{n-4} + K_2), T_1 \cup \cdots \cup T_{t-1} \cup (K_2 + (\overline{K}_{2+r} \cup K_2))\}$.*

Moreover, the equality holds when $G \in \{T_1 \cup \cdots \cup T_t \cup K_1, T_1 \cup \cdots \cup T_t \cup K_4, T_1 \cup \cdots \cup T_{t-1} \cup 2K_5^-\}$, where $T_i \in \mathcal{T}_6$ for all $i \in [t]$.

Theorem 12[23] *Let $n \geqslant 3$ be an integer. Let G be a P_8-free planar graph on n vertices. Then $e(G) \leqslant 15n/7$, with equality when $n = 7t$ for any positive integer t and $G = T_1 \cup \cdots \cup T_t$, where $T_i \in \mathcal{T}_7$ for all $i \in [t]$.*

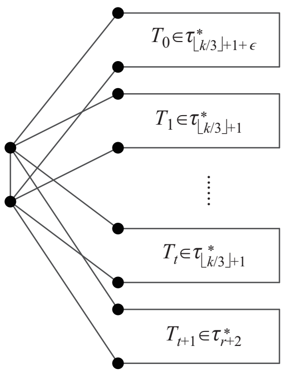


Figure 7 Construction of $\mathcal{G}_{\lfloor k/3 \rfloor+1+\varepsilon,\, n}$

To illustrate the results for P_k with $k \geqslant 9$, we need to introduce more notation. Let $\mathcal{T}_t^* \subseteq \mathcal{T}_t$ denote the family of all plane triangulations with a Hamilton cycle. Given positive integers $k \geqslant 9$, $n = \lfloor k/3 \rfloor - 1 + \varepsilon + t(\lfloor k/3 \rfloor - 1) + r + 2$, $t \geqslant 2$, where $\varepsilon = k \bmod 3$ and $r = (n - \varepsilon - 2) \bmod (\lfloor k/3 \rfloor - 1)$, we next construct $\mathcal{G}_{\lfloor k/3 \rfloor + 1 + \varepsilon,\, n}$, depicted in Figure 7, as follows: let $(a_0, b_0), \cdots, (a_{t+1}, b_{t+1})$ be the edge of one fixed Hamilton cycle of $T_0, T_1, \cdots, T_{t+1}$, respectively, and identify all a_i as a and all b_i as b, where

$$T_0 \in \mathcal{T}^*_{\lfloor k/3 \rfloor + 1 + \varepsilon},\ T_{t+1} \in \mathcal{T}^*_{r+2},\ T_i \in \mathcal{T}^*_{\lfloor k/3 \rfloor + 1} \text{ for any } i \in [t] \text{ when } \varepsilon \in \{0, 1\};$$

$$T_0, T_1 \in \mathcal{T}^*_{\lfloor k/3 \rfloor + 2},\ T_{t+1} \in \mathcal{T}^*_{r+2},\ T_i \in \mathcal{T}^*_{\lfloor k/3 \rfloor + 1} \text{ for any } 2 \leqslant i \leqslant t, \text{ or}$$

$$T_0 \in \mathcal{T}^*_{\lfloor k/3 \rfloor + 3},\ T_{t+1} \in \mathcal{T}^*_{r+2},\ T_i \in \mathcal{T}^*_{\lfloor k/3 \rfloor + 1} \text{ for any } i \in [t] \text{ when } \varepsilon = 2.$$

Clearly, $e(\mathcal{G}_{\lfloor k/3 \rfloor + 1 + \varepsilon,\, n}) = (3 - \frac{1}{\lfloor k/3 \rfloor - 1})n - 5 + \frac{\varepsilon + r + 2}{\lfloor k/3 \rfloor - 1} - (3r + 1 - \max\{3r, 1\})$. For $n \geqslant k - 1$, it is easy to see that the longest path of $\mathcal{G}_{\lfloor k/3 \rfloor + 1 + \varepsilon,\, n}$ has $(|T_0| - 2) + (|T_1| - 2) + (|T_2| - 2) + 2 = k - 1$ vertices and so $\mathcal{G}_{\lfloor k/3 \rfloor + 1 + \varepsilon,\, n}$ is P_k-free, where $\varepsilon = k \bmod 3$.

Theorem 13[23] *Let $n \geqslant 3$ be an integer. Let G be a P_k-free planar graph on n vertices with $k \in \{9, 10, 11\}$. Then $e(G) \leqslant \max\{\frac{3k-9}{k-1}n, \frac{5n-8+\varepsilon}{2}\}$, with equality when $n \geqslant 2k + 1$ are different from k in parity and $G \in \mathcal{G}_{\lfloor k/3 \rfloor + 1 + \varepsilon, n}$, or when $G \in \{T_1, T_1 \cup T_2\}$, where $\varepsilon = k$ mod 3 and $T_1, T_2 \in \mathcal{T}_{k-1}$.*

In view of Theorem 13, we believe that the following conjecture is true.

Conjecture 4 *Let $n \geqslant 3$ be an integer and $k \geqslant 12$. Let G be a P_k-free planar graph on n vertices. Then $e(G) \leqslant \max\{\frac{3k-9}{k-1}n, e(\mathcal{G}_{\lfloor k/3 \rfloor + 1 + \varepsilon, n})\}$, with equality when $G \in \mathcal{G}_{\lfloor k/3 \rfloor + 1 + \varepsilon, n}$ or when $n = s(k-1)$ and $G = T_1 \cup \cdots \cup T_s$, where $T_i \in \mathcal{T}_{k-1}$ for all $i \in [s]$.*

The planar Turán number for matchings has been settled recently by Qin, Lan, Shi and Yue[24].

Theorem 14[24] *Let n, t be positive integers with $n \geqslant 2t \geqslant 8$. Then*

$$ex_{\mathcal{P}}(n, M_t) = \min\{3n - 6, 2n + 3t - 13\}.$$

2 Planar anti-Ramsey number of graphs

We have seen from Proposition 1 that $ar_{\mathcal{P}}(\mathcal{T}_n, H)$ and $ex_{\mathcal{P}}(n, H)$ are intimately related, that is,

$$1 + ex_{\mathcal{P}}(n, \mathcal{H}) \leqslant ar_{\mathcal{P}}(\mathcal{T}_n, H) \leqslant ex_{\mathcal{P}}(n, H),$$

where $\mathcal{H} = \{H - e : e \in E(H)\}$.

We want to point out that finding the exact values of $ar_{\mathcal{P}}(n, H)$ is far from trivial. As observed by Horňák, Jendrol′, Schiermeyer and Soták in [11], an induction argument in general cannot be applied to compute $ar_{\mathcal{P}}(n, H)$ because deleting a vertex from a plane triangulation may result in a graph that is no longer a plane triangulation. Let $ar(W_n, H)$ denote the maximum number of colors in an H-free edge-coloring of W_n. As observed again in [11], the exact value of $ar(W_n, C_k)$ plays a key role in determining the planar anti-Ramsey number of C_k. In this section, we collect results on anti-Ramsey number when host graphs are plane triangulations and wheels. We refer the reader to [25,26] on anti-Ramsey number when host graphs are outer-planar graphs and Halin graphs.

2.1 Planar anti-Ramsey number of cycles

Horňák, Jendrol′, Schiermeyer and Soták[11] initialed the study of planar anti-Ramsey number of cycles under the name of rainbow number. They determined the exact value for $ar_{\mathcal{P}}(n, C_3)$, and obtained lower and upper bounds for $ar_{\mathcal{P}}(n, C_k)$ when $k \in \{4, 5\}$; lower bounds for $ar_{\mathcal{P}}(n, C_k)$ for all $n \geqslant k \geqslant 6$.

Theorem 15[11] *Let n, k be positive integers.*
(a) $ar_{\mathcal{P}}(n, C_3) = \lfloor (3n-6)/2 \rfloor$ *for* $n \geqslant 4$.
(b) $ar_{\mathcal{P}}(n, C_4) \leqslant 2(n-2)$ *for* $n \geqslant 4$.
(c) $ar_{\mathcal{P}}(n, C_5) \leqslant 5(n-2)/2$ *for* $n \geqslant 11$.
(d) $ar_{\mathcal{P}}(n, C_4) \geqslant (9(n-2)-4r)/5$ *for* $n \geqslant 42$ *and* $r = (n-2)\mathrm{mod}20$.
(e) $ar_{\mathcal{P}}(n, C_5) \geqslant (19(n-2)-10r)/9$ *for* $n \geqslant 20$ *and* $r = (n-2)\mathrm{mod}18$.
(f) $ar_{\mathcal{P}}(n, C_k) \geqslant \frac{k-3}{k-2}(3n-6) - \frac{2k-7}{k-2}$ *for* $6 \leqslant k \leqslant n$.

Recently, Lan, Shi and Song[12] obtained upper bounds for $ar_{\mathcal{P}}(n, C_k)$ when $k \in \{6, 7\}$, and improved further lower bounds for $ar_{\mathcal{P}}(n, C_k)$ for all $k \geqslant 5$.

Theorem 16[12] *Let n, k be positive integers.*
(a) $ar_{\mathcal{P}}(n, C_6) \leqslant 17(n-2)/6$ *for* $n \geqslant 8$.
(b) $ar_{\mathcal{P}}(n, C_7) \leqslant (59n-113)/20$ *for* $n \geqslant 13$.
(c) $ar_{\mathcal{P}}(n, C_5) \geqslant (39n-123-21r)/9$ *for* $n \geqslant 119$ *and* $r \equiv (n+7)\mathrm{mod}18$.
(d) $ar_{\mathcal{P}}(n, C_k) \geqslant \left(\frac{k-3}{k-2} + \frac{2}{3(k+1)(k-2)}\right)(3n-6) - \frac{2k^2-5k-5}{k^2-k-2}r$ *for* $k \geqslant 6$ *and* $n \geqslant k^2-k$ *and* $r \equiv (n-2)\mathrm{mod}(k^2-k-2)$.

By Proposition 1, we see that Theorem 5(b) and Theorem 7 yield better upper bounds for $ar_{\mathcal{P}}(n, C_k)$ for $k = 5$ and $k = 6$, respectively.

Corollary 1 *Let n, k be positive integers.*
(a)[9] $ar_{\mathcal{P}}(n, C_5) \leqslant ex_{\mathcal{P}}(n, C_5) \leqslant 12(n-2)/5$ *for all* $n \geqslant 11$.
(b)[19] $ar_{\mathcal{P}}(n, C_6) \leqslant ex_{\mathcal{P}}(n, C_6) \leqslant 5n/2-7$ *for all* $n \geqslant 18$.

For the remainder of this subsection, we focus on $ar(W_n, C_k)$. Horňák, Jendrol′, Schiermeyer and Soták[11] determined the exact values of $ar(W_n, C_k)$ when $k \in \{4, 5\}$.

Theorem 17[11] *Let n be a positive integer.*
(a) $ar(W_n, C_4) = \lfloor 4n/3 \rfloor$ *for any* $n \geqslant 3$.
(b) $ar(W_n, C_5) = \lfloor 3n/2 \rfloor$ *for any* $n \geqslant 4$.

Recently, Lan, Shi and Song[12] proved the exact value of $ar(W_n, C_6)$ for all $n \geqslant 5$, and obtained lower and upper bounds for $ar(W_n, C_k)$ for all $k \geqslant 7$ with lower bounds being also the upper bounds for finite many integers n.

Theorem 18[12] *Let n, k be positive integers with $k \geqslant 5$.*
(a) $ar(W_n, C_6) = \lfloor 5n/3 \rfloor$ *for all* $n \geqslant 5$.
(b) $\lfloor \frac{2k-7}{k-3} n \rfloor \leqslant ar(W_n, C_k) \leqslant \lfloor \frac{2k-5}{k-2} n \rfloor$ *for all* $n \geqslant k-1$.

Very recently, Xu, Lu and Liu proved that the lower bound in Theorem 18(b) is the desired upper bound for $ar(W_n, C_k)$ for all $k \geqslant 7$.

Let C_k^1 denote the graph obtained from C_k by adding one pendent edge. Let C_k^2 (resp. $C_k^{2'}$) be a graph on $k+2$ vertices obtained from C_k^1 by adding one new pendent edge such that it is incident to the vertex of degree three (resp. to a vertex of degree two). Qin, Lei and Li[27] proved the following results.

Theorem 19[27] *Let n be a positive integer.*
(a) $ar_{\mathcal{P}}(n, C_3^1) = \lfloor (3n-2)/2 \rfloor$ *for any* $n \geqslant 4$.
(b) $ar_{\mathcal{P}}(n, C_3^2) \leqslant \lfloor 9n/4 \rfloor - 2$ *for any* $n \geqslant 5$.
(c) $ar_{\mathcal{P}}(n, C_3^{2'}) \leqslant \lfloor (7n-11)/3 \rfloor$ *for any* $n \geqslant 5$.

Theorem 20[27] *Let n be a positive integer.*
(a) $ar(W_n, C_3^1) = n+1$ *for any* $n \geqslant 4$.
(b) $ar(W_n, C_3^2) = n+3$ *for any* $n \geqslant 5$.
(c) $ar(W_n, C_3^{2'}) = \lfloor 4n/3 \rfloor$ *for any* $n \geqslant 5$.

2.2 Planar anti-Ramsey number of paths

Lan, Shi and Song[12] began the study of planar anti-Ramsey number of paths, and obtained lower bounds for $ar_{\mathcal{P}}(n, P_k)$ for all $k \geqslant 8$.

Theorem 21[12] *Let n, k be two positive integers with $n \geqslant k$ and $\varepsilon = k \bmod 2$.*
(a) *If $k \in \{8, 9\}$, then $ar_{\mathcal{P}}(n, P_k) \geqslant (3n + 3\varepsilon - \varepsilon^* - 3)/2$, where $\varepsilon^* = (n+1+\varepsilon) \bmod 2$.*
(b) *If $k \geqslant 10$, then*

$$ar_{\mathcal{P}}(n, P_k) \geqslant \begin{cases} n + 2k - 12, & if \quad k \leqslant n < 3\lfloor k/2 \rfloor + \varepsilon - 5, \\ 2n + k - 14, & if \quad n > 5\lfloor k/2 \rfloor + \varepsilon - 15, \\ (3n + 9\lfloor k/2 \rfloor + 3\varepsilon - 43)/2, & otherwise. \end{cases}$$

Theorem 21 was then improved further by Qin, Li, Lan and Yue[28].

Theorem 22[28] *Let n be a positive integer.*
(a) $\lceil n/3 \rceil \leqslant ar_{\mathcal{P}}(n, P_4) \leqslant n/2$ *for any* $n \geqslant 5$.
(b) $ar_{\mathcal{P}}(n, P_5) = n$ *for any* $n \geqslant 5$.
(c) $n+1 \leqslant ar_{\mathcal{P}}(n, P_6) \leqslant 2n-4$ *for any* $n \geqslant 6$.
(d) $ar_{\mathcal{P}}(n, P_7) = 2n-2$ *for any* $n \geqslant 7$.
(e) $ar_{\mathcal{P}}(n, P_k) \geqslant 3n - 5 - 2\left\lceil \frac{n-k+2\lfloor \frac{k-7}{3} \rfloor + 4}{\lfloor \frac{k-1}{3} \rfloor} \right\rceil$ *for* $n \geqslant k$ *and* $k \geqslant 8$.

The following follows immediately from Theorems 12 and 13.

Theorem 23[28] *Let n be a positive integer.*
(a) $ar_{\mathcal{P}}(n, P_8) \leqslant 15n/7$.
(b) $ar_{\mathcal{P}}(n, P_9) \leqslant \max\{9n/4, 5n/2 - 4\}$.
(c) $ar_{\mathcal{P}}(n, P_{10}) \leqslant \max\{7n/3, (5n-7)/2\}$.
(d) $ar_{\mathcal{P}}(n, P_{11}) \leqslant \max\{12n/5, 5n/2 - 3\}$.

The authors in [28] further obtained the exact values of $ar(W_n, P_k)$ for all $k \in \{4, 5, 6, 7, 8, n+1\}$ and lower and upper bounds for $ar(W_n, P_k)$ for all $9 \leqslant k \leqslant n$.

Theorem 24[28] *Let n, k be positive integers with $k \geqslant 9$.*
(a) $ar(W_n, P_4) = \lfloor n/3 \rfloor + 1$ *for any* $n \geqslant 4$ *and* $ar(W_3, P_4) = 3$.
(b) $ar(W_n, P_5) = n + 1$ *for any* $n \geqslant 4$.
(c) $ar(W_n, P_6) = n + 2$ *for any* $n \geqslant 5$.
(d) $ar(W_n, P_7) = \lfloor 4n/3 \rfloor + 1$ *for any* $n \geqslant 6$.
(e) $ar(W_n, P_8) = \lfloor 3n/2 \rfloor + 1$ *for any* $n \geqslant 7$.
(f) $2n - 1 - 2\left\lceil \frac{n - \lceil \frac{k-1}{2} \rceil}{\lfloor \frac{k-1}{2} \rfloor} \right\rceil \leqslant ar(W_n, P_k) = \left\lfloor \frac{2k-13}{k-6} n \right\rfloor$ *for any* $n \geqslant k - 1$.

2.3 Planar anti-Ramsey number of matchings

Jendrol′, Schiermeyer and Tu[29] first studied planar anti-Ramsey number of matchings. They obtained the exact values of $ar_{\mathcal{P}}(n, M_k)$ for each $k \in \{2, 3, 4\}$, and lower and upper bounds for $ar_{\mathcal{P}}(n, M_k)$ for all $k \geqslant 5$.

Theorem 25[29] *Let n, k be positive integers. Then*
(a) $ar_{\mathcal{P}}(n, M_2) = 2$ *for* $n \geqslant 5$.
(b) $ar_{\mathcal{P}}(n, M_3) = n$ *for* $n \geqslant 7$.
(c) $ar_{\mathcal{P}}(n, M_4) = 2n - 2$ *for* $n \geqslant 8$.
(d) $2n + 2k - 10 \leqslant ar_{\mathcal{P}}(n, M_k) \leqslant 2n + 2k - 7 + 2\binom{2k-2}{3}$ *for all* $n \geqslant 2k \geqslant 10$.

Recently, Qin, Lan and Shi[30] considered the next step. They obtained the exact value of $ar_{\mathcal{P}}(n, M_5)$ for all $n \geqslant 11$, and improved the upper bound for $ar_{\mathcal{P}}(n, M_k)$ for all $k \geqslant 6$.

Theorem 26[30] *Let n, k be positive integers. Then*
(a) $ar_{\mathcal{P}}(n, M_5) = 2n$ *for* $n \geqslant 11$.
(b) $ar_{\mathcal{P}}(n, M_k) \leqslant 2n + 6k - 17$ *for any* $n \geqslant 2k$ *and* $k \geqslant 5$.

Chen, Lan and Song[31] continued the study of planar anti-Ramsey number for matchings. They determined the exact value of $ar_{\mathcal{P}}(n, M_6)$ for all $n \geqslant 30$, and established better lower and upper bounds for $ar_{\mathcal{P}}(n, M_k)$ for all $k \geqslant 6$.

Theorem 27[31] *Let n, k be positive integers. Then*
(a) $ar_{\mathcal{P}}(n, M_6) = 2n + 3$ *for* $n \geqslant 30$.
(b) $2n + 3k - 15 \leqslant ar_{\mathcal{P}}(n, M_k) \leqslant 2n + 4k - 13$ *for any* $n \geqslant 3k - 6$ *and* $k \geqslant 6$.

Qin, Lan, Shi and Yue[24] then proved that the lower bound in Theorem 27(b) is the desired upper bound for $ar_{\mathcal{P}}(n, M_k)$ for all $k \geqslant 7$ and $n \geqslant 9k + 3$. In addition, they significantly improved the upper bound for $ar_{\mathcal{P}}(n, M_k)$ for all $k \geqslant 7$ and n satisfying $3k - 6 \leqslant n < 9k + 3$.

Theorem 28[24] *Let n, k be positive integers. Then*
(a) $ar_{\mathcal{P}}(n, M_k) = 2n + 3k - 15$ *for all* $k \geqslant 7$ *and* $n \geqslant 9k + 3$.
(b) $ar_{\mathcal{P}}(n, M_k) \leqslant 2n + 3k - 14$ *for all* $k \geqslant 7$ *and* $3k - 6 \leqslant n < 9k + 3$.

Finally, we refer the reader to [27] for the exact values of $ar(W_n, M_k)$ for all $k \geqslant 2$.

3 Counting subgraphs in planar graphs

Given a graph H, how many copies of H can a planar graph on n vertices have? Let $N_{\mathcal{P}}(n,H)$ denote the maximum number of copies of H in a planar graph on n vertices. In this section, we shall collect results on $N_{\mathcal{P}}(n,H)$ when H is a path or cycle. We refer the reader to [32-40] for other cases of H or counting subgraphs when host graphs are not planar.

3.1 Counting cycles

Hakimi and Schmeichel[36] began the study of $N_{\mathcal{P}}(n,H)$ and determined the exact values of $N_{\mathcal{P}}(n,H)$ and characterized all extremal graphs when $H = C_3$ or $H = C_4$; they also obtained the order of magnitude of $N_{\mathcal{P}}(n,C_k)$ for all $k \geqslant 5$.

Theorem 29[36] *Let n be a positive integer.*
(a) $N_{\mathcal{P}}(n,C_3) = 3n-8$ *for all* $n \geqslant 6$,
(b) $N_{\mathcal{P}}(n,C_4) = \frac{1}{2}(n^2+3n-22)$ *for all* $n \geqslant 4$,
(c) $N_{\mathcal{P}}(n,C_k) = \Theta(n^{\lfloor k/2 \rfloor})$ *for all* $k \geqslant 5$.

Hakimi and Schmeichel[36] further proposed a conjecture for $N_{\mathcal{P}}(n,C_5)$, which was answered in the positive recently by Győri, Paulos, Salia, Tompkins and Zamora[41].

Theorem 30[41] *Let n be a positive integer. Then*

$$N_{\mathcal{P}}(n,C_5) = \begin{cases} 6, & if\ n=5 \\ 2n^2-10n+12, & if\ n=6\ or\ n \geqslant 8 \\ 41, & if\ n=7. \end{cases}$$

Very recently, Cox and Martin[34] focused on studying $N_{\mathcal{P}}(n,H)$ when H is an even cycle.

Theorem 31[34] *Let n be a positive integer.*
(a) $N_{\mathcal{P}}(n,C_6) = (\frac{n}{3})^3 + O(n^{3-1/5})$,
(b) $N_{\mathcal{P}}(n,C_8) = (\frac{n}{4})^4 + O(n^{4-1/5})$,
(c) $N_{\mathcal{P}}(n,C_{2\ell}) \leqslant \frac{n^\ell}{\ell!} + O(n^{\ell-1/5})$ *for any* $\ell \geqslant 5$.

With the support of Theorem 31(a, b), Cox and Martin[34] proposed the following conjecture for even cycles in general.

Conjecture 5[34] *For all* $\ell \geqslant 3$,

$$N_{\mathcal{P}}(n,C_{2\ell}) = \left(\frac{n}{\ell}\right)^\ell + O(n^{\ell-1/5}).$$

It is worth noting that maximizing the number of induced subgraphs in a graph on n vertices has been one of the most intriguing problems in extremal graph theory. Let $N_{\mathcal{P}}^{\mathrm{ind}}(n,H)$ denote the maximum number of copies of induced H in a planar graph on n vertices. Recently, Ghosh, Győri, Janzer, Paulos, Salia and Zamora[42] determined the asymptotic values of $N_{\mathcal{P}}^{\mathrm{ind}}(n,C_4)$, and obtained asymptotic tight upper bound for $N_{\mathcal{P}}^{\mathrm{ind}}(n,C_5)$.

Theorem 32[42] *Let n be a positive integer. Then*
(a) $N_{\mathcal{P}}^{\text{ind}}(n, C_4) = \frac{1}{2}n^2 + O(n)$, *and*
(b) $N_{\mathcal{P}}^{\text{ind}}(n, C_5) \leqslant \frac{1}{3}n^2 + O(n)$.

3.2 Counting paths

It is easy to see that $N_{\mathcal{P}}(n, P_2) = 3n - 6$. Alon and Caro[33] proved that $N_{\mathcal{P}}(n, P_3) = n^2 + 3n - 16$ for all $n \geqslant 4$. Recently, Győri, Paulos, Salia, Tompkins and Zamora[43] determined the exact values of $N_{\mathcal{P}}(n, P_4)$.

Theorem 33[43] *Let n be a positive integer. Then*

$$N_{\mathcal{P}}(n, P_4) = \begin{cases} 12, & if\ n = 4, \\ 7n^2 - 32n + 27, & if\ n = 5, 6\ or\ n \geqslant 9, \\ 147, & if\ n = 7, \\ 222, & if\ n = 8. \end{cases}$$

Ghosh, Győri, Martin, Paulos, Salia, Xiao and Zamora[44] then determined the asymptotic value of $N_{\mathcal{P}}(n, P_5)$, and posed a conjecture for $N_{\mathcal{P}}(n, P_k)$ for all $k \geqslant 6$.

Theorem 34[44] $N_{\mathcal{P}}(n, P_5) = n^3 + O(n^2)$.

Conjecture 6[44] *For all $\ell \geqslant 3$, we have*
(a) $N_{\mathcal{P}}(n, P_{2\ell+1}) = 4\ell(\frac{n}{\ell})^{\ell+1} + O(n^{\ell})$, *and*
(b) $N_{\mathcal{P}}(n, P_{2\ell}) = 8\ell(\ell - 1)(\frac{n}{\ell-1})^{\ell} + O(n^{\ell-1})$.

Very recently, Conjecture 6 was resolved by Cox and Martin[34] when $k = 7$.

Theorem 35[34] $N_{\mathcal{P}}(n, P_7) = \frac{4}{27}n^4 + O(n^{4-1/5})$.

References

[1] Turán P. Eine Extremalaufgabe aus der Graphentheorie [J]. *Matematikai és Fizikai Lapok*, 1941, **48**: 436-452.
[2] Erdős P, Stone A H. On the structure of linear graphs [J]. *Bulletin of the American Mathematical Society*, 1946, **52**: 1087-1091.
[3] Bollobás B. *Modern Graph Theory* [M]. New York: Springer, 2013.
[4] Füredi Z, Jiang T. Hypergraph Turán numbers of linear cycles [J]. *Journal of Combinatorial Theory, Series A*, 2014, **123**: 252-270.
[5] Füredi Z, Jiang T, Seiver R. Exact solution of the hypergraph Turán problem for k-uniform linear paths [J]. *Combinatorica*, 2014, **34**: 299-322.
[6] Kostochka A, Mubayi D, Verstraëte J. Turán problems and shadows I: Paths and cycles [J]. *Journal of Combinatorial Theory, Series A*, 2015, **129**: 57-79.
[7] Füredi Z. Turán type problems, "Surveys in Combinatorics" [J]. *London Mathematical Society*, 1991, **166**: 253-300.
[8] Keevash P. Hypergraph Turán problems, "Surveys in Combinatorics 2011" [J]. *London Mathematical Society*, 2011, **392**: 83-139.

[9] Dowden C. Extremal C_4-free/C_5-free planar graphs [J]. *Journal of Graph Theory*, 2016, **83**: 213-230.

[10] Erdős P, Simonovits M, Sós V T. Anti-Ramsey theorems [J]. *Colloquia Mathematica Societatis János Bolyai*, 1975, **10**: 633-643.

[11] Horňák M, Jendrol′ S, Schiermeyer I, et al. Rainbow numbers for cycles in plane triangulations [J]. *Journal of Graph Theory*, 2015, **78**: 248-257.

[12] Lan Y, Shi Y, Song Z X. Planar anti-Ramsey numbers of paths and cycles [J]. *Discrete Mathematics*, 2019, **342**: 3216-3224.

[13] Lan Y, Shi Y, Song Z X. Extremal H-free planar graphs [J]. *The Electronic Journal of Combinatorics*, 2019, **26(2)**: #P2.11.

[14] Dzido T. A note on Turán numbers for even wheels [J]. *Graphs and Combinatorics*, 2013, **29**: 1305-1309.

[15] Dzido T, Jastrzębski A. Turán numbers for odd wheels [J]. *Discrete Mathematics*, 2018, **341**: 1150-1154.

[16] Fang L, Zhai M, Wang B. Planar Turán number of intersecting triangles [J]. arXiv: 2007.09650v1.

[17] Wang W, Lih K. On the sizes of graphs embeddable in surfaces of nonnegative Euler characteristic and their applications to edge choosability [J]. *European Journal of Combinatorics*, 2007, **28**: 111-120.

[18] Lan Y, Shi Y, Song Z X. Extremal Theta-free planar graphs [J]. *Discrete Mathematics*, 2019, **342**: 111610.

[19] Ghosh D, Győri E, Martin R, et al. Planar Turán number of the 6-cycle [J]. arXiv: 2004.14094v1.

[20] Ghosh D, Győri E, Paulos A, et al. Planar Turán number of the Θ_6 [J]. arXiv: 2006.00994v1.

[21] Faudree R, Schelp R. Path Ramsey numbers in multicolourings [J]. *Journal of Combinatorial Theory, Series B*, 1975, **19**: 150-160.

[22] Lan Y, Shi Y, Song Z X. Planar Turán numbers for Theta graphs and paths of small order [J]. arXiv: 1711.01614v1.

[23] Lan Y, Shi Y. Planar Turán numbers of short paths [J]. *Graphs and Combinatorics*, 2019, **35**: 1035-1049.

[24] Qin Z, Lan Y, Shi Y, et al. Exact rainbow numbers for matchings in plane triangulations [J]. *Discrete Mathematics*, 2021, **344**(4): 112301.

[25] Jin Z, Ye K. Rainbow number of matchings in planar graphs [J]. *Discrete Mathematics*, 2018, **341**: 2846-2858.

[26] Sang L, Wang H, Ye K. Rainbow number of matchings in Halin graphs [J]. *Journal of Mathematics and Computer Science*, 2019, **9**: 33-45.

[27] Qin Z, Lei H, Li S. Rainbow numbers for small graphs in planar graphs [J]. *Applied Mathematics and Computation*, 2020, **371**: 124888.

[28] Qin Z, Li S, Lan Y, et al. Rainbow numbers for paths in planar graphs [J]. *Applied Mathematics and Computation*, 2021, **397**: 125918.

[29] Jendrol′ S, Schiermeyer I, Tu J. Rainbow numbers for matchings in plane triangulations [J]. *Discrete Mathematics*, 2014, **331**: 158-164.

[30] Qin Z, Lan Y, Shi Y. Improved bounds for rainbow numbers of matchings in plane triangulations [J]. *Discrete Mathematics*, 2019, **342**: 221-225.

[31] Chen G, Lan Y, Song Z X. Planar anti-Ramsey numbers of matchings [J]. *Discrete Mathematics*, 2019, **342**: 2106-2111.

[32] Győri E, Paulos A, Salia N, et al. Generalized planar Turán numbers [J]. arXiv: 2002.04579v2.

[33] Alon N, Caro Y. On the number of subgraphs of prescribed type of planar graphs with a given number of vertices [J]. *Annals Discrete Mathematics*, 1984, **20**: 25-36.

[34] Cox C, Martin R. Counting paths, cycles and blow-ups in planar graphs [J]. arXiv: 2101.05911v1.

[35] Eppstein D. Connectivity, graph minors and subgraph multiplicity [J]. *Journal of Graph Theory*, 1993, **17**(3): 409-416.

[36] Hakimi S, Schmeichel E. On the number of cycles of length k in a maximal planar graph [J]. *Journal of Graph Theory*, 1979, **3**(1): 69-86.

[37] Huynh T, Joret G, Wood D. Subgraph densities in a surface [J]. arXiv: 2003.13777v1.

[38] Matolcsi D, Nagy Z. Generalized outerplanar Turán numbers and maximum number of k-vertex subtrees [J]. arXiv: 2102.11746v1.

[39] Wood D. On the maximum number of cliques in a graph [J]. *Graphs and Combinatorics*, 2007, **23**: 337-352.

[40] Wormald N. On the frequency of 3-connected subgraphs of planar graphs [J]. *Bulletin of the Australian Mathematical Society*, 1986, **34**(2): 309-317.

[41] Győri E, Paulos A, Salia N, et al. The maximum number of pentagons in a planar graph [J]. arXiv: 1909.13532v1.

[42] Ghosh D, Győri E, Janzer O, et al. The maximum number of induced C_5's in a planar graph [J]. arXiv: 2004.01162v1.

[43] Győri E, Paulos A, Salia N, et al. The maximum number of paths of length three in a planar graph [J]. arXiv: 1909.13539v1.

[44] Ghosh D, Győri E, Martin R, et al. The maximum number of paths of length four in a planar graph [J]. *Discrete Mathematics*, 2021, **344**: 112317.