The Erdős-Lovász Tihany Conjecture – a survery

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Abstract

Let $s \geq 2$ and $t \geq 2$ be integers. A graph G is (s,t)*-splittable* if $V(G)$ can be partitioned into two sets S and T such that $\chi(G[S]) \geq s$ and $\chi(G[T]) \geq t$. The famous Erdős-Lovász Tihany Conjecture from 1968 states that every graph G with $\omega(G) < \chi(G) = s+t-1$ is (s,t)-splittable. We provide a survey on the Erdős-Lovász Tihany Conjecture and its related problems.

1 Introduction

All graphs considered in this paper are finite and without loops or multiple edges. For a graph G, we use $V(G)$ to denote the vertex set, $E(G)$ the edge set, $|G|$ the number of vertices, $e(G)$ the number of edges, $\delta(G)$ the minimum degree, $\Delta(G)$ the maximum degree, $\alpha(G)$ the independence number, $\omega(G)$ the clique number, $\chi(G)$ the chromatic number, $\overline{G}$ the complement of G. For a vertex $x \in V(G)$, we use $N(x)$ to denote the set of vertices in G which are adjacent to x. We define $N[x] = N(x) \cup \{x\}$ and $d(x) = |N(x)|$. If $A, B \subseteq V(G)$ are disjoint, we say that A is *complete* to B if each vertex in A is adjacent to all vertices in B; and A is *anti-complete* to B if no vertex in A is adjacent to any vertex in B. If $A = \{a\}$, we simply say a is complete to B or a is anti-complete to B. The subgraph of G induced by A, denoted $G[A]$, is the graph with vertex set A and edge set $\{xy \in E(G) : x, y \in A\}$. We denote by $B \setminus A$ the set $B - A$, and $G \setminus A$ the subgraph of G induced on $V(G) \setminus A$, respectively. If $A = \{a\}$, we simply write $B \setminus a$ and $G \setminus a$, respectively. A set $K \subseteq V(G)$ is a *clique* of G if vertices in K are pairwise adjacent in G; a k*-clique* if, in addition, $|K| = k$. A graph H is an *induced subgraph* of G if $V(H) \subseteq V(G)$ and $H = G[V(H)]$. We say G is *claw-free* if G does not contain $K_{1,3}$ as an induced subgraph, and G is k*-chromatic* if $\chi(G) = k$. For any positive integer n, we write $[n]$ for the set $\{1, 2, \ldots, n\}$. We use the convention "A :=" to mean that A is defined to be the right-hand side of the relation.

Let $s \geq 2$ and $t \geq 2$ be integers. A graph G is (s,t)*-splittable* if $V(G)$ can be partitioned into two sets S and T such that $\chi(G[S]) \geq s$ and $\chi(G[T]) \geq t$. At the Colloquium held at Tihany, Hungary, September 1966, Erdős [Erd68] proposed the following.

> Trivially every $3k$-chromatic graph contains k odd vertex independent circuits. Perhaps every $3k-1$ chromatic critical graph having more than $n_0(k)$ vertices contains k odd

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> vertex independent circuits. In particular, is it true that every 5-chromatic critical graph having sufficiently many vertices contains two odd vertex independent circuits? GALLAI showed that this is false for 4-chromatic graphs.
>
> LOVÁSZ in trying to prove this made the following conjecture: Let G be a k-chromatic graph which does not contain a complete k-gon and let $a > 1$ and $b > 1$ be arbitrary positive integers satisfying $a + b = k + 1$. Then we can split the vertices of G into two classes so that the graph spanned by the vertices of the first class has chromatic number $> a$ and the graph spanned by the vertices of the second class has chromatic number $> b$. By taking $a = 3$ we obtain from LOVÁSZ's conjecture that every graph of chromatic number $3k - 1$ which does net contain a complete $(3k - 1)$-gon contains k vertex-independent odd circuits. LOVÁSZ further remarks that even the following special case ($a = 2$) does not seem to be easy to prove. Every k-chromatic graph G which does not contain a complete k-gon contains two vertices, x_1 and x_2,which are joined by an edge so that $G - x_1 - x_2$ has chromatic number $\geq k - 1$.

The above conjecture of Lovász from 1966 was then published in 1968 in the proceedings of the conference, and is now known as the Erdős-Lovász Tihany Conjecture from 1968. The name Erdős-Lovász Tihany Conjecture was introduced by Jensen and Toft [JT95, Problem 5.12].

Conjecture 1.1 (Erdős-Lovász Tihany Conjecture [Erd68])**.** *Let G be a graph with $\omega(G) < \chi(G) = s + t - 1$, where $s \geq 2$ and $t \geq 2$ are integers. Then G is (s, t)-splittable.*

To date, Conjecture 1.1 has been shown to be true only for the pairs $(s, t) \in \{(2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (3, 5)\}$. The case $(2, 2)$ is trivial. The cases $(2, 3)$ and $(3, 2)$ were shown by Brown and Jung in 1969 [BJ69]. Mozhan [Moz87] and Stiebitz [Sti87] each independently showed the case $(2, 4)$ in 1987. The cases $(3, 4)$ and $(3, 5)$ was settled by Stiebitz in 1987 [Sti88]. We recall the proofs of these known cases in Section 3. It is worth noting that the case $(3, 3)$ of Conjecture 1.1 answers the question of Erdős in the positive. To see this, let G be a $(3k-1)$-chromatic graph with $\omega(G) < 3k - 1$ and let $V_1, \ldots, V_{3k-1}$ be the color classes of a proper $(3k - 1)$-coloring of G. Then $\chi(G[V_1 \cup \cdots \cup V_5]) = 5$; $\chi(G[V_{3\ell} \cup V_{3\ell+1} \cup V_{3\ell+2}]) = 3$ and so $G[V_{3\ell} \cup V_{3\ell+1} \cup V_{3\ell+2}]$ has an odd cycle for each $\ell \in \{2, \ldots, k - 1\}$. By the result of Brown and Jung for the case $(3, 3)$ of Conjecture 1.1, $G[V_1 \cup \cdots \cup V_5]$ contains two vertex-disjoint odd cycles, and so G has $2 + (k - 2) = k$ vertex-disjoint odd cycles, as desired. It is not hard to see that the same result of Brown and Jung yields that every $5k$-chromatic graph G with $\omega(G) < 5k$ has $2k$ vertex-disjoint odd cycles; the same result of Brown and Jung and the case $(3, 5)$ by Stiebitz imply that every $(3k - 2)$-chromatic graph G with $\omega(G) < 3k - 2$ has k vertex-disjoint odd cycles for all $k \geq 3$. Stiebitz pointed out in [Sti17] that Conjecture 1.1 "seems difficult to attack and might even be false".

Recent work on the Erdős-Lovász Tihany Conjecture has focused on proving the conjecture for certain classes of graphs. A vertex v of a graph G is *bisimplicial* if $N(v)$ is the union of two cliques; a graph is *quasi-line* if every vertex is bisimplicial. Note that every line graph is quasi-line and every quasi-line graph is claw-free [CS12]. A *hole* in a graph is an induced cycle of length at least four; a hole is *even* if it has an even length. Quasi-line graphs have attracted more attention (see [CO07, CF08, CS12]). In particular, Chudnovsky and Seymour [CS12] gave a constructive characterization of quasi-line graphs. Kostochka and Stiebitz [KS08] showed that Conjecture 1.1 holds for line graphs. Balogh, Kostochka, Prince, and Stiebitz [BKPS09] then showed that Conjecture 1.1 holds for quasi-line graphs, and all graphs G with $\alpha(G) = 2$. The present author [Son19] further proved that Conjecture 1.1 holds for all graphs G with $\alpha(G) \geq 3$ and no hole of length between 4 and $2\alpha(G) - 1$.

Theorem 1.2 (Kostochka and Stiebitz [KS08])**.** *If G is the line graph of some multigraph and $\omega(G) < \chi(G) = s + t - 1$, where $t \geq s \geq 2$ are integers, then G contains an s-clique K such that $\chi(G \setminus K) \geq t$. In particular, G is (s,t)-splittable.*

Theorem 1.3 (Balogh, Kostochka, Prince and Stiebitz [BKPS09])**.** *Let G be a graph with $\omega(G) < \chi(G) = s + t - 1$, where $t \geq s \geq 2$ are integers. Then the following hold.*

(a) If G is a quasi-line graph, then G contains an s-clique K such that $\chi(G \setminus K) \geq t$. In particular, G is (s,t)-splittable.

(b) If $\alpha(G) = 2$, then G is (s,t)-splittable.

Theorem 1.4 (Song [Son19])**.** *Let G be a graph with $\alpha(G) \geq 3$ and $\omega(G) < \chi(G) = s + t - 1$, where $s \geq 2$ and $t \geq 2$ are integers. If G has no hole of length between 4 and $2\alpha(G) - 1$, then G is (s,t)-splittable.*

We want to point out that the proof of Theorem 1.3(a) does not rely on the structural result of quasi-line graphs [CS12]; the proof of Theorem 1.4 relies on Theorem 1.3(a) and the Strong Perfect Graph Theorem [CRST06]. Recently, Wang and Yu [WY20b, WY20a] extended the ideas of Theorem 1.2 and Theorem 1.3(b) and proved that if the line graph $L(G)$ of a multigraph G satisfies $\omega(L(G)) < \chi(L(G)) = s+t-1$ and $t \geq s \geq 7\ell/2$ for some integer $\ell \geq 0$, then G is $(s, t+\ell)$-splittable; every graph G with $\alpha(G) = 2$ and $1 + \omega(G) < \chi(G) = s + t - 1$ is (s,t)-splittable. It is worth noting that Conjecture 1.1 remains open for claw-free graphs. Chudnovsky, Fradkin and Plumettaz [CFP] proved the following weakening of Conjecture 1.1 for claw-free graphs, the proof of which is long and relies heavily on the structure theorem for claw-free graphs developed by Chudnovsky and Seymour [CS05]. It is not hard to see that Theorem 1.5 does not completely settle Conjecture 1.1 for all claw-free graphs.

Theorem 1.5. *Let G be a claw-free graph with $\chi(G) > \omega(G)$. Then there exists a clique K with $|K| \leq 5$ such that $\chi(G \setminus K) > \chi(G) - |K|$.*

The least number k such that G has a vertex enumeration in which each vertex is preceded by fewer than k of its neighbors is called the *coloring number* $\mathrm{col}(G)$ of G. The enumeration shows that $\mathrm{col}(G) \leq \max_{H \subseteq G} \delta(H) + 1$. But for $H \subseteq G$, we have $\mathrm{col}(G) \geq \mathrm{col}(H)$ and $\mathrm{col}(H) \geq \delta(H) + 1$, since the "back-degree" of the last vertex in any enumeration of H is just its ordinary degree in H, which is at least $\delta(H)$. It follows that every graph G satisfies the following:

$$\chi(G) \leq \mathrm{col}(G) = \max\{\delta(H) \mid H \subseteq G\} + 1.$$

The coloring number as an upper bound for the chromatic number was first defined and studied by Erdős and Hajnal [EH66] in 1966. Using the chromatic number and coloring number of a graph, Stiebitz [Sti17] proved a nice relaxation of Conjecture 1.1.

Theorem 1.6 (Stiebitz [Sti17]). *Every graph G satisfying $\omega(G) < \chi(G) = s + t - 1$ has two vertex-disjoint subgraphs G_1 and G_2 such that*

$$\chi(G_1) \geq s \text{ and } col(G_2) \geq t \quad \text{or} \quad col(G_1) \geq s \text{ and } \chi(G_2) \geq t.$$

We end this section with a recent result towards Conjecture 1.1 on even-hole-free graphs, where a graph is *even-hole-free* if it contains no even hole. Very recently, Chudnovsky and Seymour [CS20] proved a structural result on even-hole-free graphs.

Theorem 1.7 (Chudnovsky and Seymour [CS20]). *Let G be a non-empty even-hole-free graph. Then G has a bisimplicial vertex and $\chi(G) \leq 2\omega(G) - 1$.*

Using Theorem 1.7 and the properties of minimal counterexamples to Conjecture 1.1 which are recalled in Section 2, the present author recently [Son21] proved that every even-hole-free graph G with $\omega(G) < \chi(G) = s + t - 1$ satisfies Conjecture 1.1 provided that $t \geq s > \chi(G)/3$. We recall the proof here as it is short and the method has new ingredient. We begin with a useful lemma.

Lemma 1.8 (Song [Son21]). *Let G be a graph and $x \in V(G)$ with $p := \chi(G[N(x)]) \geq 2$. Let $V_1, \ldots, V_p$ be the color classes of a proper p-coloring of $G[N(x)]$ with $|V_1| \geq \cdots \geq |V_p| \geq 1$. If $|V_r \cup \cdots \cup V_p| \leq \chi(G) - r - 1$ for some $r \in [p]$ with $2 \leq r \leq p$, then $p \leq \chi(G) - 2$ and G is $(r, \chi(G) + 1 - r)$-splittable.*

Proof. Let $G, p, r, V_1, \ldots, V_p$ be as given in the statement. Note that $p - r + 1 \leq |V_r \cup \cdots \cup V_p| \leq \chi(G) - r - 1$ and so $p \leq \chi(G) - 2$ and $V(G) \setminus N[x] \neq \emptyset$. Let $W := V_1 \cup \cdots \cup V_{r-1}$. Then $\chi(G[\{x\} \cup W]) = r$ and $\chi(G \setminus W) \geq \chi(G) - (r - 1) = \chi(G) + 1 - r$. It suffices to show that $\chi(G \setminus (\{x\} \cup W)) \geq \chi(G \setminus W)$. Let $q := \chi(G \setminus (\{x\} \cup W)) \geq \chi(G \setminus W) - 1 \geq \chi(G) - r \geq 2$ and let $U_1, \ldots, U_q$ be the color classes of a proper q-coloring of $G \setminus (\{x\} \cup W)$. Since x is adjacent to $|V_r \cup \cdots \cup V_p| \leq \chi(G) - r - 1 \leq q - 1$ vertices in $G \setminus W$, we see that x is anti-complete to U_i for some $i \in [q]$. We may assume that $i = 1$. Then $U_1 \cup \{x\}, U_2, \ldots, U_q$ form the color classes of a proper q-coloring of $G \setminus W$. Therefore, $\chi(G \setminus (\{x\} \cup W)) = q \geq \chi(G \setminus W) \geq \chi(G) - r + 1$, as desired. □

Theorem 1.9 (Song [Son21]). *Let G be an even-hole-free graph with $\omega(G) < \chi(G) = s + t - 1$, where $t \geq s \geq 2$. If $s > \chi(G)/3$, then G is (s, t)-splittable.*

Proof. Suppose the assertion is false. Let G be a counterexample with $|G|$ minimum. Then G is vertex-critical; in addition, G is an (s, t)-graph. Thus $\delta(G) \geq \chi(G) - 1 = s + t - 2$. By Theorem 2.2, $\omega(G) \leq t - 1$. Since G is even-hole-free, by Theorem 1.7, G has a bisimplicial vertex v such that $N(v)$ is the union of two cliques. Thus $\alpha(G[N(v)]) \leq 2$, $\omega(G[N(v)]) \leq t - 2$ and

$$s + t - 2 = \chi(G) - 1 \leq \delta(G) \leq d(v) \leq 2\omega(G[N(v)]) \leq 2t - 4.$$

It follows that $t \geq s+2 \geq 4$ and $\chi(G) = s+t-1 \geq 2s+1$. We next claim that $\Delta(G) \leq |G|-2$. Suppose there exists $x \in V(G)$ such that $d(x) = |G|-1$. Then

$$\chi(G \setminus x) = \chi(G) - 1 = s + (t-1) - 1 > \omega(G) - 1 = \omega(G \setminus x) \text{ and } t-1 > s > \chi(G \setminus x)/3.$$

By the minimality of G, $G \setminus x$ is $(s, t-1)$-splittable and thus G is (s,t)-splittable, a contradiction. Thus $\Delta(G) \leq |G|-2$, as claimed. It follows that $V(G) \setminus N[v] \neq \emptyset$ and so $\chi(G[N[v]]) \leq \chi(G)-1$. Let $p := \chi(G[N(v)])$. Then $p = \chi(G[N[v]]) - 1 \leq \chi(G) - 2$. Note that

$$p \geq \omega(G[N(v)]) \geq d(v)/2 \geq (\chi(G)-1)/2 \geq ((2s+1)-1)/2 = s \geq 2.$$

Let $V_1, \ldots, V_p$ be the color classes of a proper p-coloring of $G[N(v)]$ with $2 \geq |V_1| \geq \cdots \geq |V_p| \geq 1$. Suppose $p \geq t-1$. Then $|V_{t-2}| = 1$ because $d(v) \leq 2t-4$. Therefore,

$$|V_t \cup \cdots \cup V_p| = p - t + 1 \leq (\chi(G) - 2) - t + 1 = \chi(G) - t - 1.$$

By Lemma 1.8 applied to G and v with $r = t$, we see that G is (s,t)-splittable, a contradiction. Thus $s \leq p \leq t-2$. Next, if $|V_s \cup \cdots \cup V_p| \leq \chi(G) - s - 1$, then G is (s,t)-splittable by applying Lemma 1.8 to G and v with $r = s$, a contradiction. Hence, $|V_s \cup \cdots \cup V_p| \geq \chi(G) - s = t-1 \geq 3$. Note that $p - s + 1 \leq (t-2) - 2 + 1 = t-3$, and so $|V_s| = 2$ and

$$d(v) = (|V_1| + \cdots + |V_{s-1}|) + |V_s \cup \cdots \cup V_p| \geq 2(s-1) + t - 1 = 2s + t - 3.$$

It follows that $t - 2 \geq \omega(G[N(v)]) \geq d(v)/2 \geq (2s+t-3)/2$, which implies that $t \geq 2s+1$. Thus $\chi(G) = s+t-1 \geq 3s$, contrary to the assumption that $3s > \chi(G)$. □

The remainder of the paper is organized as follows. We recall the properties of minimal counterexamples to Conjecture 1.1 in Section 2; the proofs of known cases of Conjecture 1.1 in Section 3. We then survey the results on: the special case $s = 2$ of Conjecture 1.1 (see Conjecture 4.1) in Section 4; a weakening conjecture of Conjecture 1.1 (see Conjecture 5.6) in Section 5; a weakening conjecture of Conjecture 4.1 (see Conjecture 6.1) in Section 6.

2 Properties of (s,t)-graphs

For the sake of readers, we recall the properties of minimal counterexamples to Conjecture 1.1 due to Stiebitz [Sti88]. An (s,t)-*graph* is a connected $(s+t-1)$-chromatic graph which does not contain two vertex-disjoint subgraphs with chromatic number s and t, respectively.

Lemma 2.1 (Stiebitz [Sti88])**.** *Let G be a k-chromatic graph and let $V_1, \ldots, V_k$ be the color classes of a proper k-coloring of G. Then for each $i \in [k]$, there exists $v_i \in V_i$ such that $N(v_i) \cap V_j \neq \emptyset$ for all $j \in [k]$ with $j \neq i$.*

Proof. Suppose there exists an $i \in [k]$, say $i = k$, such that every vertex of V_k is anti-complete to V_j for some $j \in [k-1]$. Let $U_1 := \{v \in V_k \mid v \text{ is anti-complete to } V_1 \text{ in } G\}$. For each $j \in [k-1]$ with $j \geq 2$, let $U_j := \{v \in V_k \setminus (U_1 \cup \cdots \cup U_{j-1}) \mid v \text{ is anti-complete to } V_j \text{ in } G\}$. Then $U_1 \cup V_1, \ldots, U_{k-1} \cup V_{k-1}$ form the color classes of a proper $(k-1)$-coloring of G, a contradiction. □

Theorem 2.2 (Stiebitz [Sti88]). *Let G be an (s,t)-graph with $t \geq s \geq 2$. Then*

(i) for every s-clique K of G and every proper $(t-1)$-coloring of $G \setminus K$ with color classes $V_1, \ldots, V_{t-1}$, there exists $v_i \in V_i$ for each $i \in [t-1]$ such that v_i is complete to K in G. In particular, every s-clique K of G is contained in at least $t-1$ cliques each of order $s+1$.

(ii) for every t-clique K of G and every proper $(s-1)$-coloring of $G \setminus K$ with color classes $V_1, \ldots, V_{s-1}$, there exists $v_i \in V_i$ for each $i \in [s-1]$ such that v_i is complete to K in G. In particular, every t-clique of G is contained in at least $s-1$ cliques each of order $t+1$.

(iii) $\omega(G) \geq \chi(G)$ if K_t is a subgraph of G.

Proof. Let G be an (s,t)-graph with $t \geq s \geq 2$. Then $\chi(G) = s+t-1$. To prove (i), let $K := \{v_1, \ldots, v_s\}$ and let $V_1, \ldots, V_{t-1}$ be as given in the statement. Then $\{v_1\}, \ldots, \{v_s\}, V_1, \ldots, V_{t-1}$ form the color classes of a proper $(s+t-1)$-coloring of G. By Lemma 2.1, there exists $v_i \in V_i$ for each $i \in [t-1]$ such that v_i is complete to K in G. In particular, K is contained in at least $t-1$ cliques each of order $s+1$. Similarly, one can prove (ii).

To prove (iii), suppose G contains K_t as a subgraph. Let $X_0 := \{x_1, \ldots, x_t\}$ be a t-clique of G. Let $y_1, \ldots, y_r$ be a longest sequence of vertices in $G \setminus X_0$ such that for each $i \in [r]$,

(a) y_i is complete to X_{i-1} in G, and

(b) $X_i := \{y_1, \ldots, y_i, x_{i+1}, \ldots, x_t\}$ is a t-clique of G.

By Theorem 2.2(i), we have $r \geq 1$. Note that X_r is a t-clique by (a). Thus $G[\{y_1, \ldots, y_r\}] = K_r$. It follows that $r \leq s-1$ because G is an (s,t)-graph. Hence $1 \leq r \leq s-1$. By Theorem 2.2(ii), X_r belongs to at least $s-1$ cliques each of order $t+1$. By the maximality of r, no vertex in $V(G) \setminus (X_0 \cup X_r)$ is complete to X_r in G. It follows that $\{x_1, \ldots, x_r\}$ is complete to X_r in G and $r = s-1$. Therefore, $G[X_0 \cup \{y_1, \ldots, y_r\}] = K_{t+r} = K_{s+t-1}$, and so $\omega(G) \geq \chi(G)$, as desired. □

3 The known cases of Conjecture 1.1

In this section, we recall the proofs of Brown and Jung [BJ69] that Conjecture 1.1 is true for the values of $(s,t) \in \{(2,2), (2,3), (3,3)\}$, and the proofs of Stiebitz [Sti88] for the cases $(2,4)$, $(3,4)$ and $(3,5)$ of Conjecture 1.1. We say that two proper vertex-colorings c_1 and c_2 of a graph G are *equivalent* if, for all $x, y \in V(G)$, $c_1(x) = c_1(y)$ iff $c_2(x) = c_2(y)$. A graph G with $\chi(G) \leq k$ is *uniquely k-colorable* if every two proper k-colorings of G are equivalent. It is not hard to see that the complete graph K_n is uniquely k-colorable for all $k \geq n$.

Lemma 3.1 (Brown and Jung [BJ69]). *Let H be an induced subgraph of a graph G. If $G \setminus V(H)$ has at most $\chi(G \setminus V(H)) - 1$ vertices each adjacent to at least $\chi(H)$ vertices of H, then*

$$\chi(G \setminus V(H)) \geq \chi(G) - \chi(H) + 1.$$

Proof. Let $s := \chi(H)$ and $t := \chi(G \backslash V(H))$. Let $V_1, \ldots, V_t$ be the color classes of a proper t-coloring of $G \setminus V(H)$ and $V_{t+1}, \ldots, V_{s+t}$ be the color classes of a proper s-coloring of H. Since $G \setminus V(H)$ has at most $t-1$ vertices each adjacent to at least s vertices of H, we may assume that no vertex in V_1 is adjacent to at least s vertices of H. Similar to the proof of Lemma 2.1, each vertex in V_1 can be placed into one of $V_{t+1}, \ldots, V_{s+t}$ and this yields a proper $(s+t-1)$-coloring of G. It follows that

$$\chi(G) \leq s + t - 1 = \chi(H) + \chi(G \setminus V(H)) - 1.$$

Therefore, $\chi(G \setminus V(H)) \geq \chi(G) - \chi(H) + 1$, as desired. □

Corollary 3.2 (Brown and Jung [BJ69])**.** *Let K be a clique of G. Then either*

$$\chi(G \setminus K) \geq \chi(G) - |K| + 1$$

or $G \setminus K$ has at least $\chi(G) - |K|$ vertices each adjacent to all vertices of K in G.

Lemma 3.3 (Brown and Jung [BJ69])**.** *Let G be a K_4-free graph and C be a shortest odd cycle in G. Then either $G = C$, or no vertex in $G \setminus V(C)$ is adjacent to at least three vertices on C.*

Proof. Since C is a shortest odd cycle in G, we see that C is an induced cycle, say with vertices $v_1, \ldots, v_{2\ell+1}$ in order for some positive integer ℓ. Suppose $G \neq C$ and there exists a vertex $v \in V(G) \setminus V(C)$ such that v is adjacent to at least three vertices on C, say v_i, v_j, v_k with $i < j < k$. Then $C \neq K_3$ because G is K_4-free. Let $Q_{i,j}$ be the path on C with vertices $v_i, \ldots, v_j$ in order; $Q_{j,k}$ be the path on C with vertices $v_j, \ldots, v_k$ in order, and $Q_{k,i} := C \setminus \{v_{i+1}, \ldots, v_{k-1}\}$. It follows that each of the paths $Q_{i,j}, Q_{j,k}, Q_{k,i}$ has length at least two, in particular, one of them has an odd length, say $Q_{i,j}$. But then we obtain a shorter odd cycle with vertices $v, v_i, \ldots, v_j$ in order, contrary to the choice of C. □

Lemma 3.4 (Stiebitz [Sti88])**.** *Every $(3,t)$-graph has a K_4 subgraph, where $t \geq 2$.*

Proof. Let G be a $(3,t)$-graph, where $t \geq 2$. Then $\chi(G) = t + 2 \geq 4$. Suppose G is K_4-free. Let C be a shortest odd cycle in G. Then $C \neq G$. By Lemma 3.3, no vertex in $G \setminus V(C)$ is adjacent to at least three vertices on C. By Lemma 3.1, $\chi(G \setminus V(C)) \geq \chi(G) - 2 = t$. Thus G is $(3,t)$-splittable, a contradiction. □

We are now ready to prove the known cases of Conjecture 1.1.

Proof. Let G be a graph with $\omega(G) < \chi(G) = s + t - 1$, where $s \leq t$ and

$$(s,t) \in \{(2,2), (2,3), (2,4), (3,3), (3,4), (3,5)\}.$$

Suppose G is not (s,t)-splittable. We choose G with $|G|$ minimum. Then G is connected and so G is an (s,t)-graph. Suppose $(s,t) \in \{(2,2), (2,3)\}$. Then $\omega(G) \geq \chi(G)$ by Theorem 2.2, a contradiction. Thus $(s,t) \in \{(2,4), (3,3), (3,4), (3,5)\}$. We claim that $\omega(G) = s + 1$ and $(s,t) \in \{(2,4), (3,5)\}$. Suppose $(s,t) = (2,4)$. Then $\omega(G) = 3$ by Theorem 2.2. Next, suppose $s = 3$. Then $\chi(G) \geq 5$. By Lemma 3.4, $\omega(G) \geq 4$. By Theorem 2.2, $(s,t) = (3,5)$ and $\omega(G) = 4$. It follows that $\omega(G) = s + 1$ and $(s,t) \in \{(2,4), (3,5)\}$, as claimed. Thus $s \in \{2,4\}$ and $t = s + 2$.

Let $x_1, \ldots, x_r$ be a sequence of vertices of G satisfying

(i) for each $i \in [r]$, $H_i := G[\{x_1, \ldots, x_i\}]$ is uniquely $(s+1)$-colorable,

(ii) for each $i \in \{s, s+1, \ldots, r\}$, x_i is contained in at least one K_s in G_i,

(iii) subject to (i, ii), r is maximum, and

(iv) subject to (i-iii), the length of a shortest odd cycle of $G \setminus \{x_1, \ldots, x_r\}$ is minimum.

Then $r \geq s+1$ because $\omega(G) = s+1$ and $K_1, K_2, \ldots, K_{s+1}$ are uniquely $(s+1)$-colorable. By (i), H_r is uniquely $(s+1)$-colorable, and every proper $(s+1)$-coloring of H_i can be extended to a proper $(s+1)$-coloring of H_r for all $i \in [r-1]$. It is easy to see that $H_r \neq G$ because $\chi(G) = s+t-1 > s+1$. By (ii), x_r belongs to an s-clique of H_r. Let $j \in [r]$ be the largest such that $x_j \in K$, where K is an s-clique of H_r with $x_r \in K$. When $s = 3$, let $\ell \in [j-1]$ be the largest such that $K = \{x_\ell, x_j, x_r\}$. Since H_r is uniquely $(s+1)$-colorable, we see that x_r is adjacent to at least s vertices in H_{r-1}. It follows that $j \geq 2$. Let $X := \{x_1, \ldots, x_r\}$ and

$$U := \{v \in V(G) \mid v \text{ is complete to } K \text{ in } G\}.$$

Since G is an (s,t)-graph, we have $\chi(G \setminus K) = t-1$. Let c be any proper $(t-1)$-coloring of $G \setminus K$. Then all the vertices in $U \cap X$ are colored by at most one color under c, otherwise the restriction $c|_{H_{j-1}}$ cannot be extended to a proper 3-coloring of H_r when $s = 2$; the restriction $c|_{H_{\ell-1}}$ cannot be extended to a proper 4-coloring of H_r when $s = 3$. By Theorem 2.2(i), $|U \setminus X| \geq t-2$. Let $u_1, \ldots, u_{t-2} \in U \setminus X$. For each $i \in [t-2]$, by the maximality of r (see (iii)), it follows that $G[X \cup \{u_i\}]$ is not uniquely $(s+1)$-colorable, and so $\chi(G[X \cup \{u_i\}]) = s+2$.

Suppose $(s,t) = (2,4)$. Then both $V(G) \setminus (X \cup \{u_1\})$ and $V(G) \setminus (X \cup \{u_2\})$ are independent set in G because G is a $(2,4)$-graph. Thus $u_1u_2 \in E(G)$ because $\chi(G) = 5$. But then $G[\{x_r, x_j, u_1, u_2\}] = K_4$, contrary to the fact $\omega(G) = s+1 = 3$. Thus $(s,t) = (3,5)$. Then $K = \{x_\ell, x_j, x_r\}$ and $\chi(G) = 7$. Note that $\chi(G \setminus X) \geq 3$. Let C be a shortest odd cycle of $G \setminus X$. For each $i \in [3]$, since $\chi(G[X \cup \{u_i\}]) = s+2 = 5$ and G is a $(3,5)$-graph, we see that $\chi(G \setminus (X \cup \{u_1\})) \leq 2$; thus $u_i \in V(C)$. If $|C| = 3$, then $G[X \cup V(C)] = K_6$, contrary to the fact $\omega(G) = s+1 = 4$. Thus $|C| \geq 5$. Note that x_r is adjacent to three vertices u_1, u_2, u_3 on C. Similar to the proof of Lemma 3.3, there exists $i \in [3]$ such that $G \setminus X^*$ has an odd cycle that is shorter than C, where $X^* := \{x_1, \ldots, x_{r-1}, u_i\}$. Then $\chi(G \setminus X^*) = 3$ because $G \setminus X^*$ has an odd cycle and $\chi(G \setminus (X \cup \{u_i\})) \leq 2$. Thus $\chi(G[X^*]) = 4$ as G is a $(3,7)$-graph. This, together with the fact that H_{r-1} and H_r are uniquely 4-colorable and $\chi(G[X \cup \{u_i\}]) = 5$, implies that $G[X^*]$ is uniquely 4-colorable and u_i is contained in a triangle in $G[X^*]$. It follows that $x_1, \ldots, x_{r-1}, u_i$ is a sequences satisfying (i-iii) such that $G \setminus \{x_1, \ldots, x_{r-1}, u_i\}$ has an odd cycle that is shorter than C, contrary to the choice of $x_1, \ldots, x_r$ (see (iv)).

This completes the proof of Conjecture 1.1 for the values of (s,t), where $s \in \{2,3\}$ and $t \in \{s, s+1, s+2\}$. □

4 Double-Critical Graph Conjecture

In this section, we introduce an attractive special case of the the Erdős-Lovász Tihany Conjecture. If we restrict $s = 2$ in Conjecture 1.1, then the Erdős-Lovász Tihany Conjecture states that every graph G with $\chi(G) > \omega(G)$ contains an edge $xy \in E(G)$ such that $\chi(G \setminus \{x, y\}) \geq \chi(G) - 1$. To prove this special case of Conjecture 1.1, suppose for a contradiction that no such an edge exists. We choose a counterexample G with $|G|$ minimum. Then G is connected and $\chi(G \setminus \{x, y\}) = \chi(G) - 2$ for every edge $xy \in E(G)$. This motivates the definition of double-critical graphs. A connected graph G is *double-critical* if for every edge $xy \in E(G)$, $\chi(G \setminus \{x, y\}) = \chi(G) - 2$. The $(2, t)$ case of Conjecture 1.1 is then equivalent to the following conjecture, which is referred to as the *Double-Critical Graph Conjecture* of Erdős and Lovász from 1968.

Conjecture 4.1 (Double-Critical Graph Conjecture [Erd68]). *Let G be a double-critical, k-chromatic graph. Then $G = K_k$.*

Since Conjecture 4.1 is a special case of Conjecture 1.1, it has been settled in the affirmative for $k \leq 5$ [Moz87, Sti87], for line graphs [KS08]; quasi-line graphs [BKPS09]; graphs G with $\alpha(G) = 2$ [BKPS09]; graphs G with $\alpha(G) \geq 3$ and no hole of length between 4 and $2\alpha(G)-1$ [Son19]. It is hard to prove Conjecture 4.1: by Theorem 2.2(i), every edge of a non-complete, double-critical, k-chromatic graph belongs to at least $k - 2$ triangles; however, it remains unknown whether such a graph contains K_4 as a subgraph. We next list basic properties of non-complete double-critical k-chromatic graphs established in [KPT10].

Proposition 4.2 (Kawarabayashi, Pedersen and Toft [KPT10]). *If G is a non-complete double-critical k-chromatic graph, then the following hold.*

(a) G does not contain K_{k-1} as a subgraph.
(b) $\delta(G) \geq k + 1$.
(c) For any $x \in V(G)$, $\alpha(G[N(x)]) \leq d(x) - |N(x) \cap N(y)| - 1 \leq d(x) - k + 1$, where $y \in N(x)$ is any vertex contained in an maximum independent set of $G[N(x)]$.
(d) If H is a connected subgraph of G, then the graph G/H obtained by contracting H to a single vertex is $(k - 1)$-colorable.
(e) Every edge $xy \in E(G)$ belongs to at least $k - 2$ triangles.
(f) Every vertex $x \in V(G)$ has a neighbor y such that y is not complete to $N(x)$.
(g) There exists at least one edge $xy \in E(G)$ which is not a dominating edge of G.
(h) If $x \in V(G)$ has neighbors y, z say, such that $yz \notin E(G)$, then x has another neighbor, say w, such that $wz \in E(G)$ and $wy \notin E(G)$.
(i) Any vertex x with a non-neighbor in G satisfies $\chi(G[N(x)]) \leq k - 3$.
(j) If $x \in V(G)$ is a vertex of degree $k + 1$, then $\overline{G[N(x)]}$ consists only of isolated vertices and disjoint cycles of length at least five. Moreover, $\overline{G[N(x)]}$ contains at least one such cycle.
(k) If $x, y \in V(G)$ are both of degree $k + 1$, then $xy \notin E(G)$.
(l) G is 6-connected and no minimal separating set of G can be partitioned into two sets A and B such that $G[A]$ and $G[B]$ are edge-empty and complete, respectively.

Further properties were obtained by Rolek and the present author in [RS17, RS18]. Recall that two proper vertex-colorings c_1 and c_2 of a graph G are *equivalent* if, for all $x, y \in V(G)$, $c_1(x) = c_1(y)$ iff $c_2(x) = c_2(y)$. We say that two proper vertex-colorings c_1 and c_2 of a graph G are *equivalent on a set* $A \subseteq V(G)$ if the restrictions $c_{1|A}$ and $c_{2|A}$ to A are equivalent on the subgraph $G[A]$. Let S be a separating set of G, and let G_1, G_2 be connected subgraphs of G such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = G[S]$. If c_1 is a proper k-coloring of G_1 and c_2 is a proper k-coloring of G_2 such that c_1 and c_2 are equivalent on S, then it is clear that c_1 and c_2 can be combined to a proper k-coloring of G by a suitable permutation of the color classes of, say c_1. The main technique in the proof of Proposition 4.2(ℓ) involves reassigning and permuting the colors on a separating set S of a non-complete double-critical k-chromatic graph G so that c_1 and c_2 are equivalent on S to obtain a contradiction, where c_1 is a proper $(k-1)$-coloring of G_1 and c_2 is a proper $(k-1)$-coloring of G_2. It seems hard to use this idea to prove that every non-complete double-critical k-chromatic graph is 7-connected, but one can use it to say a bit more about minimal separating sets of size 6 in non-complete double-critical graphs.

Lemma 4.3 (Rolek and Song [RS18]). *Suppose G is a non-complete double-critical k-chromatic graph. If S is a minimal separating set of G with $|S| = 6$, then either $G[S] \subseteq K_{3,3}$ or $G[S] \subseteq K_{2,2,2}$.*

Lemma 4.4 (Rolek and Song [RS17]). *If G is a non-complete, double-critical, k-chromatic graph, then for any $x \in V(G)$ with at least one non-neighbor in G, $\chi(G \setminus N[x]) \geq 3$. In particular, $G \setminus N[x]$ must contain an odd cycle, and so $d(x) \leq |V(G)| - 4$.*

Lemma 4.5 (Rolek and Song [RS17]). *Let G be a double-critical, k-chromatic graph. For any edge $xy \in E(G)$, let c be any $(k-2)$-coloring of $G \setminus \{x, y\}$ with color classes $V_1, V_2, \ldots, V_{k-2}$. Then the following two statements are true.*

(a) For any $i, j \in \{1, 2, \ldots, k-2\}$ with $i \neq j$, if $N(x) \cap N(y) \cap V_i$ is anti-complete to $N(x) \cap V_j$, then there exists at least one edge between $(N(y) \setminus N(x)) \cap V_i$ and $N(x) \cap V_j$ in G. In particular, $(N(y) \setminus N(x)) \cap V_i \neq \emptyset$.

(b) Assume that $d(x) = k + 1$ and y belongs to a cycle of length $\ell \geq 5$ in $\overline{G[N(x)]}$.

(b_1) If $\ell \geq 7$, then $d(y) \geq k + e(\overline{G[N(x)]}) - 4$;

(b_2) If $\ell = 6$, then $d(y) \geq \max\{k + 2, k + e(\overline{G[N(x)]}) - 5\}$; and

(b_3) If $\ell = 5$, then $d(y) \geq \max\{k + 2, k + e(\overline{G[N(x)]}) - 6\}$.

Theorem 4.6 (Rolek and Song [RS17]). *If G is a non-complete, double-critical, k-chromatic graph with $k \geq 6$, then for any vertex $x \in V(G)$ with $d(x) = k + 1$, the following hold:*

(a) $e(\overline{G[N(x)]}) \geq 8$; and

(b) for any vertex $y \in N(x)$, $d(y) \geq k+4$. Furthermore, if $d(y) = k+4$, then $|N(x) \cap N(y)| = k-2$ and $\overline{G[N(x)]}$ contains either only one cycle, which is isomorphic to C_8, or exactly two cycles, each of which is isomorphic to C_5.

Corollary 4.7 below follows immediately from Theorem 4.6.

Corollary 4.7 (Rolek and Song [RS17]). *If G is a non-complete, double-critical, k-chromatic graph with $k \geq 6$, then no vertex of degree $k+1$ is adjacent to a vertex of degree $k+1$, $k+2$, or $k+3$ in G.*

We end this section with the progress towards Conjecture 4.1 on claw-free graphs. Huang and Yu [HY16] proved that the only double-critical, 6-chromatic, claw-free graph is K_6. Applying properties of non-complete double-critical graphs stated in Proposition 4.2, Theorem 4.6 and Corollary 4.7, Rolek and the present author [RS17] then proved the next two cases.

Theorem 4.8 (Rolek and Song [RS17]). *Let G be a double-critical, k-chromatic graph with $k \in \{6, 7, 8\}$. If G is claw-free, then $G = K_k$.*

It would be interesting to know if Conjecture 4.1 holds for all claw-free graphs. Rolek and Yu [RY] announced a solution to Conjecture 4.1 for all claw-free graphs G with $\alpha(G) \geq 4$ at the 31st Cumberland Conference on Combinatorics, Graph Theory and Computing, May 18-19, 2019. It would also be interesting to know if Conjecture 4.1 holds for all even-hole-free graphs.

5 Minor version of the Erdős-Lovász Tihany Conjecture

Before we state a weakening conjecture of Conjecture 1.1, we begin this section with a brief introduction on the celebrated Hadwiger's Conjecture [Had43].

5.1 Hadwiger's Conjecture

A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. We write $G \succcurlyeq H$ if H is a minor of G. In those circumstances we also say that G has an H minor. The celebrated result of Wagner [Wag37] states that if G is an edge-maximal graph without a K_5 minor and $|G| \geq 4$, then G can be constructed recursively, by pasting along triangles and K_2's from plane triangulations and copies of the graph W, where W is the Wagner graph depicted below. Motivated by what was then the Four Color Conjecture and inspired by Wagner's reuslt, Swiss mathematician Hugo Hadwiger posed his famous conjecture at a colloquium at Eidgenössiche Technische Hochschule on December 15, 1942 [Tof96].

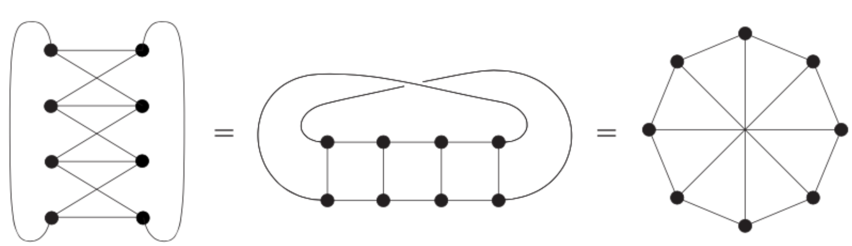

Figure 1: The Wagner graph W.

Conjecture 5.1 (Hadwiger's conjecture [Had43]). *For every integer $k \geq 1$, every graph with no K_k minor is $(k-1)$-colorable.*

Hadwiger's original presentation [Had43] of Conjecture 5.1 contains proofs for the cases $k \leq 4$. Dirac [Dir52] also independently supplied a proof for these cases in 1952. However, for $k \geq 5$, Hadwiger's conjecture implies the Four Color Theorem [AH77, AHK77]. (To see that, let H be a planar graph, and let G be obtained from H by adding $k-4$ vertices, each joined to every other vertex of the graph. Then G has no K_{k+1} minor, and hence is k-colorable by Hadwiger's conjecture, and hence H is 4-colorable.) The aforementioned celebrated result of Wagner [Wag37] shows that the case $k = 5$ of Hadwiger's conjecture is, in fact, equivalent to the Four Color Theorem, and the same was shown for $k = 6$ by Robertson, Seymour and Thomas [RST93]. Further historical explanation of the development of Conjecture 5.1 can be found in [Tof96]. Despite receiving considerable attention over the years, Hadwiger's conjecture remains wide open for all $k \geq 7$ and is widely considered among the most important problems in graph theory and has motivated numerous developments in graph coloring and graph minor theory. The best known upper bound on the chromatic number of graphs with no K_k minor is $O(k(\log\log k)^6)$ due to Postle [Pos20], improving a recent breakthrough of Norin, Postle, and the present author [NPS20] who improved a long-standing bound obtained independently by Kostochka [Kos82, Kos84] and Thomason [Tho84]. We refer the reader to a recent survey by Seymour [Sey16] for further background on Conjecture 5.1.

5.2 The extremal function for K_p minors

The extremal function for K_p minors when p is small is instrumental in proving Hadwiger's Conjecture for graphs G with $\chi(G) \leq 6$, and also settling the weakening conjectures of Conjecture 1.1. We list the known results here.

Theorem 5.2 (Mader [Mad68]). *For every integer $p \leq 7$, every graph on $n \geq p$ vertices and at least $(p-2)n - \binom{p-1}{2} + 1$ edges has a K_p minor.*

Jørgensen [Jør94] and later the present author and Thomas [ST06] generalized Theorem 5.2 to $p = 8$ and $p = 9$, respectively, as follows.

Theorem 5.3 (Jørgensen [Jør94]). *Every graph on $n \geq 8$ vertices with at least $6n - 20$ edges either contains a K_8-minor or is isomorphic to a $(K_{2,2,2,2,2}, 5)$-cockade.*

Theorem 5.4 (Song and Thomas [ST06]). *Every graph on $n \geq 9$ vertices with at least $7n - 27$ edges either contains a K_9-minor, or is isomorphic to $K_{2,2,2,3,3}$, or is isomorphic to a $(K_{1,2,2,2,2,2}, 6)$-cockade.*

Theorem 5.2 is such a nice result that it raises the question of whether it can be generalized to all values of p. Seymour and Thomas [ST06] conjecture the following which is open for all $p \geq 10$.

Conjecture 5.5 (Seymour and Thomas [ST06]). *For every $p \geq 1$ there exists a constant $N = N(p)$ such that every $(p-2)$-connected graph on $n \geq N$ vertices and at least $(p-2)n - \binom{p-1}{2} + 1$ edges has a K_p minor.*

5.3 Minor version of the Erdős-Lovász Tihany Conjecture

Kawarabayashi, Pedersen and Toft [KPT11] observed that if Hadwiger's Conjecture holds, then the following conjecture might be easier to settle than the Erdős-Lovász Tihany conjecture.

Conjecture 5.6 (Kawarabayashi, Pedersen, Toft [KPT11]). *Every graph G satisfying $\omega(G) < \chi(G) = s + t - 1$ has two vertex-disjoint subgraphs G_1 and G_2 such that $G_1 \succcurlyeq K_s$ and $G_2 \succcurlyeq K_t$, where $t \geq s \geq 2$ are integers.*

Since Hadwiger's Conjecture holds for graphs G with $\chi(G) \leq 6$, we see that Conjecture 5.6 holds for all the values of $(s,t) \in \{(2,2),(2,3),(2,4),(3,3),(3,4),(3,5)\}$. In the same paper [KPT11], Kawarabayashi, Pedersen and Toft settled Conjecture 5.6 for a few additional values of $(s,t) \in \{(2,6),(3,6),(4,4),(4,5)\}$. Let $K_m \cup K_n$ denotes the disjoint union of K_m and K_n.

Theorem 5.7 (Kawarabayashi, Pedersen and Toft [KPT11]). *Every 7-chromatic graph G with $\omega(G) \leq 6$ has a $K_2 \cup K_6$ minor and a $K_4 \cup K_4$ minor.*

Theorem 5.8 (Kawarabayashi, Pedersen and Toft [KPT11]). *Every 8-chromatic graph G with $\omega(G) \leq 7$ has a $K_3 \cup K_6$ minor and a $K_4 \cup K_5$ minor.*

Very recently, the present author [Son21] settled the $(4,6)$ case for Conjecture 5.6, that is, we prove that every graph G with $\chi(G) = 9 > \omega(G)$ has a $K_4 \cup K_6$ minor. We recall the proof here which utilizes the celebrated result of Stiebitz [Sti96].

Theorem 5.9 (Stiebitz [Sti96]). *Every graph G satisfying $\delta(G) \geq s + t + 1$ has two vertex-disjoint subgraphs G_1 and G_2 such that $\delta(G_1) \geq s$ and $\delta(G_2) \geq t$.*

Theorem 5.10 (Song [Son21]). *Every 9-chromatic graph G with $\omega(G) \leq 8$ has a $K_4 \cup K_6$ minor.*

Proof. Suppose for a contradiction that G is a counterexample to the statement with minimum number of vertices. Then G is vertex-critical, and so $\delta(G) \geq 8$ and G is connected. Suppose G contains two vertex-disjoint subgraphs G_1 and G_2 such that $\chi(G_1) \geq 4$ and $\chi(G_2) \geq 6$. Since Hadwiger's conjecture holds for k-chromatic graphs with $k \leq 6$, we see that $G_1 \succcurlyeq K_4$ and $G_2 \succcurlyeq K_6$, a contradiction. Thus G is a $(4,6)$-graph, and so $\omega(G) \leq 5$ by Theorem 2.2. Note that G is not necessarily contraction-critical, as a proper minor of G may have clique number 9. We claim that

Claim 1. $2 \leq \alpha(G[N(x)]) \leq d(x) - 7$ for each $x \in V(G)$.

Proof. Let $x \in V(G)$. Since $\omega(G) \leq 5$ and $\delta(G) \geq 8$, we see that $\alpha(G[N(x)]) \geq 2$. Suppose $\alpha(G[N(x)]) \geq d(x) - 6$. Let A be a maximum independent set of $G[N(x)]$. Let G^* be obtained from G by contracting $G[A \cup \{x\}]$ into a single vertex, say w. Note that $\omega(G^*) < 8$ and G^* has no $K_4 \cup K_6$ minor. By the minimality of G, $\chi(G^*) \leq 8$. Let $c : V(G^*) \to [8]$ be a proper 8-coloring of G^*. Since $|N(x) \setminus A| = d(x) - |A| \leq 6$, we may assume that $c(N(x) \setminus A) \subseteq [6]$ and $c(w) = 7$. But then we obtain a proper 8-coloring of G from c by coloring all the vertices in A with color 7 and the vertex x with color 8, a contradiction. Thus $2 \leq \alpha(G[N(x)]) \leq d(x) - 7$, as claimed. $\square$

By Claim 1, $\delta(G) \geq 9$. Suppose $\delta(G) \geq 13$. By Theorem 5.9, G contains two vertex-disjoint subgraphs G_1 and G_2 such that $\delta(G_1) \geq 4$ and $\delta(G_2) \geq 8$. By Theorem 5.2, we see that $G_1 \succcurlyeq K_4$ and $G_2 \succcurlyeq K_6$, a contradiction. Thus $9 \leq \delta(G) \leq 12$. We next claim that

Claim 2. $G[N(x)]$ is even-hole-free and $\chi(G[N(x)]) \leq 2\omega(G[N(x)]) - 1$ for each $x \in V(G)$.

Proof. Let $x \in V(G)$. Suppose $G[N(x)]$ contains an even hole C. Then $\chi(G[V(C) \cup \{x\}]) = 3$ and so $\chi(G \setminus (V(C) \cup \{x\})) \geq \chi(G) - 3 = 6$. It is easy to see that $G[V(C) \cup \{x\}] \succcurlyeq K_4$. Since Hadwiger's conjecture holds for 6-chromatic graphs, we see that $G \setminus (V(C) \cup \{x\})$ has a K_6 minor, and so G has a $K_4 \cup K_6$ minor, a contradiction. Thus $G[N(x)]$ is even-hole-free. By Theorem 1.7, $\chi(G[N(x)]) \leq 2\omega(G[N(x)]) - 1$. □

Let $v \in V(G)$ with $d(v) = \delta(G)$, and let $p := \chi(G[N(v)])$. Since $9 \leq d(v) \leq 12$, we see that $p \geq 3$ by Claim 1. Suppose $G[N(v)]$ is K_3-free. By Claim 2, $p \leq 2\omega(G[N(v)]) - 1 = 3$. Thus $\chi(G[N[v]]) = 4$ and $\chi(G \setminus N[v]) = \chi(G \setminus N(v)) \geq 9 - 3 = 6$, contrary to the fact that G is a $(4, 6)$-graph. Thus $\omega(G[N(v)]) \geq 3$. Let $v_1, v_2, v_3 \in N(v)$ be pairwise adjacent in G and let $H := G \setminus \{v, v_1, v_2, v_3\}$. Then $G[\{v, v_1, v_2, v_3\}] = K_4$ and

$$\begin{aligned} 2e(H) &\geq (d(v) - 3)(|G \setminus N[v]|) + (d(v) - 4) \cdot |N(v) \setminus \{v_1, v_2, v_3\}| \\ &= (d(v) - 3)(|G| - d(v) - 1) + (d(v) - 4)(d(v) - 3) \\ &= (d(v) - 3)(|H| - 1). \end{aligned}$$

Suppose $d(v) \in \{11, 12\}$. Then $2e(H) \geq 8(|H| - 1)$. By Theorem 5.2, $H \succcurlyeq K_6$, and so G has a $K_4 \cup K_6$ minor, a contradiction. This proves that $9 \leq d(v) \leq 10$. Then $p \geq 4$ by Claim 1. Since $\omega(G[N(v)]) \leq 4$, we see that $G[N(v)]$ has an anti-matching of size at least three. It follows that $4 \leq p \leq d(v) - 3$. Let $V_1, \ldots, V_p$ be the color classes of a proper p-coloring of $G[N(v)]$ with $|V_1| \geq \cdots \geq |V_p| \geq 1$. If $p \in \{4, 5\}$, then $|V_4| \leq 2$ because $d(v) \leq 10$. Thus $|V_4 \cup \cdots \cup V_p| \leq 4 = \chi(G) - 4 - 1$. By Lemma 1.8 applied to G and v with $r = 4$, we see that G is $(4, 6)$-splittable, contrary to the fact that G is a $(4, 6)$-graph. It remains to consider the case $6 \leq p \leq d(v) - 3$. Since $d(v) \leq 10$, we see that $|V_5| = 1$. Thus $|V_6 \cup \cdots \cup V_p| = p - 5 \leq (d(v) - 3) - 5 \leq 2 = \chi(G) - 6 - 1$. By Lemma 1.8 applied to G and v with $r = 6$, we see that G is $(4, 6)$-splittable, a contradiction.

This completes the proof of Theorem 5.10. □

6 Minor version of the Double-Critical Graph Conjecture

Given the difficulty in settling the Double-Critical Graph Conjecture and motivated by Hadwiger's Conjecture, Kawarabayashi, Pedersen and Toft [KPT10] proposed the following weaker conjecture.

Conjecture 6.1 (Kawarabayashi, Pedersen and Toft [KPT10]). *For every integer $k \geq 1$, every double-critical k-chromatic graph contains a K_k minor.*

Conjecture 6.1 is true for $k \leq 6$ because Conjecture 4.1 holds for graphs G with $\chi(G) \leq 6$. In the same paper [KPT10], Kawarabayashi, Pedersen and Toft verified Conjecture 6.1 for $k \in \{6, 7\}$.

Theorem 6.2 (Kawarabayashi, Pedersen and Toft [KPT10]). *For every integer $k \leq 7$, every double-critical k-chromatic graph contains a K_k minor.*

Pedersen [Ped11] proved a weakening of Conjecture 6.1 for the case $k = 8$ that every double-critical 8-chromatic graph has a K_8^- minor, where K_8^- is obtained from K_8 by deleting an edge. Albar and Gonçalves [AG18] later settled Conjecture 6.1 for the case $k = 8$.

Theorem 6.3 (Albar and Gonçalves [AG18]). *Every double-critical 8-chromatic graph has a K_8 minor.*

The proof of Theorem 6.3 is computer-assisted. Rolek and the present author [RS18] gave a computer-free proof of the same result and further showed that any double-critical, k-chromatic graph contains a K_9 minor for all $k \geq 9$.

Theorem 6.4 (Rolek and Song [RS18]). *For integers k, t with $1 \leq k \leq 9$ and $t \geq k$, every double-critical t-chromatic graph contains a K_k minor.*

Rolek and the present author [RS18] actually proved a much stronger result, the following.

Theorem 6.5 (Rolek and Song [RS18]). *For $k \in \{6, 7, 8, 9\}$, let G be a $(k-3)$-connected graph with $k + 1 \leq \delta(G) \leq 2k - 5$. If every edge of G is contained in at least $k - 2$ triangles and for any minimal separating set S of G and any $x \in S$, $G[S \setminus \{x\}]$ is not a clique, then $G \geq K_k$.*

The proofs of Theorem 6.2, Theorem 6.4 and Theorem 6.4 rely closely on the properties of non-complete double-critical graphs stated in Section 4. In particular, Theorem 6.4 follows directly from Proposition 4.2(e,ℓ) and Theorem 6.5. The proof of Theorem 6.5 closely follows the proof of the extremal function for K_9 minors by the present author and Thomas [ST06]. Note that the proof of Theorem 6.2 for $k = 7$ is about ten pages long and the proof of Theorem 6.3 is computer-assisted. The proof of Theorem 6.4 is much shorter and computer-free for $k \leq 8$. For $k = 9$, the proof is computer-assisted as it applies a computer-assisted lemma from [ST06, Lemma 3.7]. Note that a computer-assisted proof of Theorem 6.5 for all $k \leq 8$ (and hence computer-assisted proofs of Theorem 6.2 and Theorem 6.3) follows directly from Theorem 6.5 for $k = 9$. (To see that, let G and $k \leq 8$ be as in Theorem 6.2 or Theorem 6.3, and let H be obtained from G by adding $9 - k$ vertices, each adjacent to every other vertex of the graph. Then H is 6-connected and satisfies all the other conditions as stated in Theorem 6.5. Thus $H \succcurlyeq K_9$ and so $G \succcurlyeq K_k$.) Conjecture 6.1 remains open for all $k \geq 10$. It seems hard to generalize Theorem 6.4. It is worth noting that Conjecture 6.1 remains open for claw-free graphs and even-hole-free graphs.

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