Abstract: We consider a problem related to Hadwiger’s Conjecture. Let $D = (d_1, d_2, \ldots, d_n)$ be a graphic sequence with $0 \leq d_1 \leq d_2 \leq \cdots \leq d_n \leq n-1$. Any simple graph $G$ with $D$ its degree sequence is called a realization of $D$. Let $R[D]$ denote the set of all realizations of $D$. Define $h(D) = \max\{h(G) : G \in R[D]\}$ and $\chi(D) = \max\{\chi(G) : G \in R[D]\}$, where $h(G)$ and $\chi(G)$ are Hadwiger number and chromatic number of a graph $G$, respectively. Hadwiger’s Conjecture implies that $h(D) \geq \chi(D)$. In this paper, we establish the above inequality for near regular degree sequences. © 2009 Wiley Periodicals, Inc.


1. INTRODUCTION

All graphs in this paper are finite and have no loops or multiple edges. A graph $H$ is a minor of a graph $G$ if $H$ can be obtained from a subgraph of $G$ by contracting edges. An $H$ minor is a minor isomorphic to $H$. If $H$ is a complete graph, we also say that $G$ contains a clique minor of size $|H|$. For a graph $G$, the Hadwiger number $h(G)$ of $G$ is the maximum integer $k$ such that $G$ contains a clique minor of size $k$. As usual we denote by $\chi(G)$ the chromatic number of $G$ and by $\chi'(G)$ the edge chromatic number of $G$.

Our research is motivated by Hadwiger’s Conjecture from 1943 which suggests a far-reaching generalization of the Four Color Theorem [1, 2, 7] and is considered by many as one of the deepest open problems in graph theory. Hadwiger’s Conjecture states the following.

Conjecture 1.1. For every integer $k \geq 1$, every $k$-chromatic graph has a $K_k$ minor.

Conjecture 1.1 is trivially true for $k \leq 3$, and reasonably easy for $k=4$, as shown by Dirac [5] and Hadwiger himself [6]. However, for $k \geq 5$, Conjecture 1.1 implies the Four Color Theorem. In 1937, Wagner [10] proved that the case $k=5$ of Conjecture 1.1 is, in fact, equivalent to the Four Color Theorem. In 1993, Robertson, Seymour, and Thomas [8] proved that a minimal counterexample to the case $k=6$ is a graph $G$ which has a vertex $v$ such that $G-v$ is planar. By the Four Color Theorem, this implies Conjecture 1.1 for $k=6$. Hence the cases $k=5,6$ are each equivalent to the Four Color Theorem [1, 2, 7]. Conjecture 1.1 is open for $k \geq 7$. Note that Conjecture 1.1 also states that for every graph $G$, $h(G) \geq \chi(G)$.

In this paper, we consider a weaker version of Hadwiger’s Conjecture. The goal is to establish more evidence for Conjecture 1.1. Let $D=(d_1,d_2,\ldots,d_n)$ be an integer sequence with $0 \leq d_1 \leq d_2 \leq \cdots \leq d_n \leq n-1$. We say that $D$ is graphic if there is a graph $G$ with $V(G)={u_1,u_2,\ldots,u_n}$ such that $d_G(u_i)=d_i$. In those circumstances, we say that $G$ is a realization of $D$. If a sequence $D$ consists of the terms $d_1,\ldots,d_t$ with multiplicities $m_1,\ldots,m_t$, we may write $D=(d_1^{m_1},\ldots,d_t^{m_t})$. For a graphic degree sequence $D$, let $R[D]$ denote the set of all realizations of $D$. Define $h(D)=\max\{h(G):G \in R[D]\}$ and $\chi(D)=\max\{\chi(G):G \in R[D]\}$. Hadwiger’s Conjecture implies the following conjecture.

Hadwiger’s Conjecture for Degree Sequences: For any graphic degree sequence $D$, $h(D) \geq \chi(D)$.

For a graphic sequence $D=(d_1,d_2,\ldots,d_n)$, we say that $D$ is near regular if $D=((k-1)p,k^n-p)$ for some integers $k \geq 1$ and $p$ satisfying $0 \leq p \leq n-1$. The purpose of this paper is to prove that Hadwiger’s Conjecture for Degree Sequences is true for near

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regular degree sequences. Our proof technique is to construct a graph \( G \) in \( R[D] \) with a clique minor of desired size.

We need to introduce more notation. Let \( G \) be a graph. The \textit{complement} of \( G \) is denoted by \( \overline{G} \). If \( X \subseteq V(G) \), we denote the subgraph of \( G \) induced on \( X \) by \( G[X] \). We use \( G \setminus X \) denote the subgraph of \( G \) induced on \( V(G) \setminus X \). If \( A, B \subseteq V(G) \) are disjoint, we say that \( A \) is \textit{complete} to \( B \) if every vertex in \( A \) is adjacent to every vertex in \( B \), and \( A \) is \textit{anticomplete} to \( B \) if no vertex in \( A \) is adjacent to a vertex in \( B \). If \( A = \{x\} \), we simply say \( x \) is complete to \( B \) or \( x \) is anticomplete to \( B \). We use \( G[A,B] \) to denote the bipartite graph obtained from \( G[A \cup B] \) by deleting all edges with both ends in \( A \) or in \( B \). If \( F \subseteq E(G) \) and \( M \subseteq E(\overline{G}) \), then \( G - F \) (resp. \( G + M \)) denotes the graph obtained from \( G \) by deleting the edges in \( F \) from \( G \) (resp. adding the edges in \( M \) to \( G \)). For two disjoint graphs \( G \) and \( H \), \( G + H \) is the join of \( G \) and \( H \) with vertex set \( V(G) \cup V(H) \) and edge set \( E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\} \). As usual we denote by \( K_n \) and \( K_{n,m} \), respectively, the complete graph on \( n \) vertices and the complete bipartite graph such that one partite set has \( n \) vertices and the other partite set has \( m \) vertices.

The following well-known results [3, 9] will be used later in the proof of the main results. We list them below. A proof of Hall’s Theorem and other notation not introduced here can be found in [4].

**Theorem 1.2.** Let \( G \) be a connected graph with maximum degree \( \Delta \). Suppose \( G \) is neither a complete graph nor an odd cycle. Then \( \chi(G) \leq \Delta \).

**Theorem 1.3.** If \( G \) is a \( r \)-regular bipartite graph with \( r \geq 1 \), then \( G \) has \( r \) pairwise disjoint perfect matchings.

**Theorem 1.4.** If \( G \) is simple graph with maximum degree \( r \), then \( \chi'(G) = r + 1 \) or \( r \).

### 2. NEAR REGULAR DEGREE SEQUENCES

In this section, we prove that Hadwiger’s Conjecture for Degree Sequences is true for near regular degree sequences. We first prove some preliminary results.

**Lemma 2.1.** For integers \( r \geq 2 \) and \( n \geq r + 1 \), if \( nr \) is even, then there exists an \( r \)-regular graph \( G \) of order \( n \) such that \( G \) has at least two pairwise disjoint perfect matchings if \( n \) is even and one near perfect matching if \( n \) is odd.

**Proof.** If \( r = n - 1 \), then \( G = K_n \) has the desired property. So we may assume that \( r \leq n - 2 \). Assume that \( n \) is even. If \( r \leq \frac{n}{2} \), let \( G \) be an \( r \)-regular bipartite graph with each partite of size \( \frac{n}{2} \). If \( r \geq \frac{n}{2} + 1 \), let \( G \) be a graph with \( V(G) \) partitioned into \( A \) and \( B \) such that \( G[A] = \overline{G[B]} = K_{n/2} \) and \( G[A,B] \) is \( s \)-regular, where \( s = r - (\frac{n}{2} - 1) \geq 2 \). In either case, by Theorem 1.3, \( G \) has at least two pairwise disjoint perfect matchings. If \( n \) is odd, then \( r \) and \( n - 1 \) must be even. Let \( H \) be the \( r \)-regular graph of order \( n - 1 \), as constructed above, with two disjoint perfect matchings, say \( M_1, M_2 \). Let \( F \subseteq M_1 \) with \( |F| = \frac{n}{2} \). We may assume that \( F = \{x_1, x_2, x_3, x_4, \ldots, x_{r-1} x_r\} \). Let \( G \) be obtained from \( H - F \).
by joining a new vertex \( w \) to \( x_i, \ i = 1, 2, \ldots, r \). Clearly \( G \) is \( r \)-regular and \( M_2 \) is a near perfect matching of \( G \).

This completes the proof of Lemma 2.1.

Lemma 2.2. Let \( D = ((k-1)p, k^{n-p}) \) be the degree sequence of a near \( k \)-regular graph on \( n \) vertices, where \( 0 \leq p \leq n-1 \). If \( 3n \geq 4k+4 \), then \( h(D) \geq k+1 \) if \( p = 0 \) and \( h(D) \geq k \) if \( p > 0 \).

Proof. The statement is trivially true if \( k \leq 2 \). So we may assume that \( k \geq 3 \). We consider the following two cases.

Case 1. \( n \) is even.

Since \( \sigma(D) = nk - p \) is even, \( p \) must be an even integer. Suppose that \( n \geq 2k+2 \). If \( n-p \geq k+1 \), by Lemma 2.1, let \( H \) be a \( k \)-regular graph of order \( n-k-1 \) and let \( F \) be a matching of size \( \frac{p}{2} \) in \( H \). Then \( K_{k+1} \cup G \in \mathcal{R}(D) \), where \( G = H - F \). Thus \( h(D) \geq h(K_{k+1} \cup G) \geq k+1 \). If \( n-p \leq k \), by Lemma 2.1, let \( H \) be a \((k-1)\)-regular graph of order \( n-k \) and let \( F \) be a matching of size \( \frac{n-p}{2} \) in \( \overline{H} \). Then \( K_k \cup G \in \mathcal{R}(D) \), where \( G = H + F \). Hence \( h(D) \geq h(K_k \cup G) \geq k \). So we may assume that \( n \leq 2k \). For integers \( r, s, t \) satisfying \( n = 2s + 2t \) and \( r = s + 2t - 1 \), define \( G(r,s,t) \) to be an \( r \)-regular graph of order \( n \) with its vertex set partitioned into \( A, B, C, D \) such that \( G(r,s,t)[A] = G(r,s,t)[B] = K_r \), \( G(r,s,t)[C] = G(r,s,t)[D] = K_t \), \( G(r,s,t)[A,B] \) is a \( t \)-regular bipartite graph. Moreover, \( C \) is complete to \( A \cup D \), \( D \) is complete to \( B \), \( A \) is anticomplete to \( D \), \( C \) is anticomplete to \( B \).

If \( p \leq 2(n-k-1) \), let \( s = n - k - 1 \) and \( t = k + 1 - \frac{n}{2} \). Then \( \frac{k}{4} \leq s \leq k - 1 \), \( 1 \leq t \leq \frac{k}{4} \), and \( s - \frac{p}{2} \geq 0 \). Observe that \( 2s + 2t = n \), \( s + 2t = k + 1 \) and \( t \leq s \) because \( 3n \geq 4k + 4 \). Let \( G = G(k,s,t) \). Then \( G \) is a \( k \)-regular graph of order \( n \). If \( t = 1 \), then \( n = 2k \) and so \( s + t = k \). Let \( M \) be a perfect matching of \( G[A,B] \) and \( F \subseteq M \) with \( |F| = \frac{p}{2} \). Let \( H = G - F \). Then \( H \in \mathcal{R}(D) \). By contracting each of the edges of \( M - F \) into single vertices, we have \( h(D) \geq h(H) \geq |A| + |C| + |D| = s + 2t = k + 1 \) if \( p = 0 \), and \( h(D) \geq h(H) \geq |A| + |C| = s + t = k \) if \( p > 0 \). So we may assume that \( t \geq 2 \). By Theorem 1.3, \( G[A,B] \) contains two disjoint perfect matchings \( M_1 \) and \( M_2 \). Let \( F \subseteq M_1 \) with \( |F| = \frac{p}{2} \) and let \( H = G - F \). Then \( H \in \mathcal{R}(D) \). By contracting each of the edges of \( M_2 \) into single vertices, we see that \( h(D) \geq h(H) \geq s + 2t = k + 1 \).

So we may assume that \( p \geq 2(n-k) \). Then \( n - p \leq n - 2(n-k) = 2k - n \leq 2(n-k) \) because \( 3n \geq 4k + 4 \). Note that \( n - p \) is even. Let \( s = n - k \) and \( t = k - \frac{n}{2} \). Then \( \frac{k}{3} \leq s \leq k \), \( 0 \leq t \leq \frac{k}{3} \). Observe that \( 2s + 2t = n \), \( s + 2t = k \) and \( t < s \) because \( 3n \geq 4k + 4 \). Let \( G = G(k - 1, s, t) \). Then \( G \) is a \((k-1)\)-regular graph of order \( n \). Since \( t < s \), by Theorem 1.3, \( G[A,B] \) contains an anti-matching \( F \) of size \( \frac{n-p}{2} \). Let \( H = G + F \), then \( H \in \mathcal{R}(D) \). If \( t = 0 \), then \( n = 2k \) and so \( h(D) \geq h(H) \geq |A| = s = \frac{k}{2} \). If \( t \geq 1 \), let \( M \) be a perfect matching of \( G[A,B] \). Clearly, \( M \) is also a matching of \( H \). By contracting each of the edges of \( M \) into single vertices, we see that \( h(D) \geq h(H) \geq s + 2t = k \).

Case 2. \( n \) is odd.
In this case, since \( \sigma(D)=nk-p \) is even, we have
(a) \( p \) and \( k \) are either both even or both odd.

By (a), both \( p-k \) and \( n-p-k-1 \) are even. Suppose that \( n \geq 2k+1 \). If \( n-p \leq k \), then \( p-k \leq 2(n-k) \). By Lemma 2.1, let \( H \) be a \( k \)-regular graph of order \( n-k \) and let \( F \) be a matching of size \( \frac{n-k}{2} \) in \( H \). Then \( K_k \cup G \in \mathcal{R}[D] \), where \( G=H-F \). Hence \( h(D) \geq h(K_k \cup G) \geq k \). We may assume that \( n-p \geq k+1 \). If \( n \geq 2k+3 \), by Lemma 2.1, let \( H \) be a \((k-1)\)-regular graph of order \( n-k-1 \geq k+2 \) and let \( F \) be a matching of size \( n-p-k-1 \) of \( H \). Then \( K_{k+1} \cup G \in \mathcal{R}[D] \), where \( G=H+F \). Thus \( h(D) \geq h(K_{k+1} \cup G) \geq k+1 \). We may assume that \( n=2k+1 \). Then \( p \leq n-k-1=k \). Let \( F=\{x_1x_2,\ldots,x_{2q-1}x_{2q}\} \) be a matching of size \( q \) of \( K_{k+1} \), where \( q=\frac{n-p-k-1}{2}=\frac{k-p}{2} \). Let \( y_1,y_2,\ldots,y_{2q} \) be \( 2q \) distinct vertices in \( K_k \). Let \( G \) be the graph obtained from \( K_{k+1} \cup F \) and \( K_k \) by joining \( y_i \) to \( x_i \), \( i=1,2,\ldots,2q \). Then \( G \in \mathcal{R}[D] \). By contracting \( G[\{y_{2i-1},y_{2i}\}] \) onto \( x_{2i-1}, i=1,2,\ldots,q \), we see that \( h(D) \geq h(G) \geq k+1 \).

So we may assume that \( n \leq 2k-1 \). For integers \( r,s,t \) satisfying \( n=2s+2t+1 \) and \( r=s+2t \), define \( I(r,s,t) \) to be a graph of order \( n \) with its vertex set partitioned into \( A, B, C, D \) such that \( I(r,s,t)[A]=K_k \), \( I(r,s,t)[B]=K_k-E \), \( I(r,s,t)[C]=K_t \), \( I(r,s,t)[D]=K_{k+1} \). \( I(r,s,t)[A,B] \) is a \((r+1)\)-regular bipartite graph, where \( E \) is matching of size \( \frac{n}{2} \) of \( K_k \). Moreover, \( C \) is complete to \( A \cup D \), \( D \) is complete to \( B \), \( A \) is anticomplete to \( D \), \( C \) is anticomplete to \( B \). By (a), we consider the following two subcases.

**Case 2.1.** \( p \) and \( k \) are even.

If \( p \leq 2(n-k-1) \), let \( s=n-k-1 \) and \( t=k+1-\frac{n+1}{2} \). Then \( \frac{k}{2} \leq t \leq k+1-\frac{n+1}{2} \). Observe that \( s \) is even, \( s+2t=k \) and \( t \leq s \). Let \( G=I(k,s,t) \). Then \( G \) is a \( k \)-regular graph of order \( n \). Since \( t \geq 1 \), by Theorem 1.3, let \( M_1 \) and \( M_2 \) be two perfect matchings of \( G[A,B] \). Let \( F \subseteq M_1 \) with \( |F|=\frac{t}{2} \). Let \( H=G-F \). Then \( H \in \mathcal{R}[D] \). By contracting each of the edges of \( M_2 \) into single vertices, we see that \( h(H) \geq s+2t+1=k+1 \). Hence \( h(D) \geq k+1 \).

So we may assume that \( p \geq 2(n-k) \). Then \( n-p \leq n-2(n-k)=2k-n<2(n-k) \) and \( n-p \) is odd. Let \( s=n-k \) and \( t=k-\frac{n+1}{2} \). Then \( \frac{k}{2} \leq t \leq k-1 \). Observe that \( s \) is odd, \( s+2t=k-1 \) and \( t < s \) because \( 3n \geq 4k+4 \). Let \( G=I(k-1,s,t) \). Then \( DS(G)=(k^1,(k-1)^{n-1}) \). Let \( w \) be the vertex of degree \( s-1 \) in \( G[B] \). Since \( n-p \leq 2s \), by Theorem 1.3, \( G[A,B \backslash w] \) contains an anti-matching, say \( F \), of size \( \frac{n-p-1}{2} \). Let \( H=G+F \). Then \( H \in \mathcal{R}[D] \). Let \( M \) be a perfect matching of \( G[A,B] \). By contracting each of the edges of \( M \) into single vertices, we see that \( h(D) \geq h(H) \geq s+2t+1=k \).

**Case 2.2.** \( p \) and \( k \) are odd.

In this case, \( n-p \) is even. If \( n-p \leq 2(n-k) \), let \( s=n-k \) and \( t=k-\frac{n+1}{2} \). Then \( \frac{k}{2} \leq s \leq k-1 \). Observe that \( s \) is even, \( s+2t+1=k \) and \( t \leq s \). Let \( G=I(k-1,s,t) \). Then \( G \) is a \((k-1)\)-regular graph of order \( n \). By Theorem 1.3, let \( M \) be a perfect matching and \( F \) be an anti-matching of size \( \frac{n-p}{2} \) of \( G[A,B] \), respectively. Let \( H=G+F \). Then \( H \in \mathcal{R}[D] \). By contracting each of the edges of \( M \) into single vertices, we see that \( h(H) \geq s+2t+1=k \). Hence \( h(D) \geq k \).
So we may assume that \( n - p \geq 2(n - k) + 2 \). Then \( p \leq n - 2(n - k) - 2 = 2k - n - 2 < 2(n - k) \). Let \( s = n - k - 1 \) and \( t = k + 1 - \frac{n + 1}{2} \). Then \( \frac{k}{2} \leq s \leq k - 2, \ 1 \leq t < \frac{k}{2} \). Observe that \( s \) is odd, \( s + 2t + 1 = k + 1 \) and \( t < s \) because \( 3n \geq 4k + 4 \). Let \( J \) be an Hamiltonian cycle of \( K_s \) and let \( E \) be a matching of size \( \frac{s - 1}{2} \) in \( J \). Now let \( G \) be a graph of order \( n \) with \( V(G) \) partitioned into \( A, B, C, D \) such that \( G[A] = K_s, \ G[B] = K_s - E(J) + E, \ G[C] = K_t, \ G[D] = K_{t+1}, \ G[A, B] \) is a \((t+1)\)-regular bipartite graph. Moreover, \( C \) is complete to \( A \cup D \), \( D \) is complete to \( B \), \( A \) is anticomplete to \( D \), \( C \) is anticomplete to \( B \). Let \( w \) be the vertex of degree \( s - 3 \) in \( G[B] \). Since \( t \geq 1 \), by Theorem 1.3, \( G[A, B] \) contains two perfect matchings \( M_1 \) and \( M_2 \). Since \( p < 2s \), there exists \( F \subseteq M_1 \) with \( |F| = \frac{p - 1}{2} \) and \( w \notin V(F) \). Let \( H = G - F \). Then \( H \in R[D] \). By contracting each of the edges of \( M_2 \) into single vertices, we see that \( h(D) \geq h(H) \geq s + 2t + 1 = k + 1 \).

This completes the proof of Lemma 2.2.

**Theorem 2.3.** Let \( D = ((k-1)^p, k^{n-p}) \) be the degree sequence of a near \( k \)-regular graph on \( n \) vertices, where \( 0 \leq p \leq n - 1 \). Then

\[
h(D) \begin{cases} 
  k + 1 & \text{if } p = 0 \text{ and } n = k + 1 \\
  k + 1 & \text{if } p = 0 \text{ and } n \geq \frac{4k + 4}{3} \\
  k & \text{if } p > 0 \text{ and } n \geq \frac{4k + 4}{3} \\
  k + 1 - \left\lceil \frac{p}{4} \right\rceil & \text{if } p > 0 \text{ and } n = k + 1 \\
  \left\lceil \frac{3n}{4} \right\rceil & \text{if } k + 2 \leq n < \frac{4k + 4}{3} 
\end{cases}
\]

**Proof.** If \( n = k + 1 \) and \( p = 0 \), then \( R[D] = \{K_{k+1}\} \) and \( h(D) = k + 1 \). If \( n = k + 1 \) and \( p > 0 \), then \( p \) must be even and \( D \) has a unique realization \( K_{k+1} - M \), where \( M \) is a matching of size \( \frac{k}{2} \) of \( K_{k+1} \). Clearly, \( h(D) \geq (n - p) + \frac{k}{2} + \left\lceil \frac{p}{4} \right\rceil \geq k + 1 - \left\lceil \frac{p}{4} \right\rceil \). So we may assume that \( n \geq k + 2 \). If \( n = k + 2 \) and \( p = 0 \), then \( k \) must be even and \( D \) has a unique realization \( K_n - M \), where \( M \) is a perfect matching of \( K_n \). It can be easily checked that \( h(D) = \left\lceil \frac{3n}{4} \right\rceil \). Suppose \( n = k + 2 \) and \( p > 0 \). If \( k \) is odd, then \( n \) and \( p \) must be odd. Let \( J \) be a Hamilton cycle of \( K_n \) and let \( F \) be a matching of size \( \frac{n - p}{2} \) of \( J \). Let \( G = K_n - E(J) + F \). Then \( G \in R[D] \). Let \( \{x_1, x_2, \ldots, x_n\} \) be the vertices of \( J \) in order. Let \( M = \{x_3x_{n+1}, x_5x_{n+2}, \ldots, x_{3i+1}x_{n+1+2i}\} \) if \( \frac{n - p}{2} \) is even; and \( M = \{x_1x_{n+1+2i}, x_3x_{n+2}, x_5x_{n+2}, \ldots, x_{3i+1}x_{n+1+2i}\} \) if \( \frac{n - 1}{2} \) is odd. By contracting each of the edges of \( M \) into single vertices, we see that \( h(D) \geq h(G) \geq \left\lceil \frac{3n}{4} \right\rceil \). If \( k \) is even, then \( n \) and \( p \) must be even. Let \( x_1, x_2, \ldots, x_{\frac{n}{2}}, y_1, y_2, \ldots, y_{\frac{n}{2}} \) be the \( n \) vertices of \( K_n \). Let \( G = K_n - E \), where \( E = \{x_1y_1, x_2y_2, \ldots, x_{\frac{n}{2}}y_{\frac{n}{2}}, y_1x_2, y_2x_3, \ldots, y_{\frac{n}{2}}x_{\frac{n}{2}+1}\} \). By contracting each

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of the edges \( y_i v_{i+j}, \) where \( i=1,2,\ldots,\lfloor \frac{n}{4} \rfloor, \) into single vertices, we see that \( h(D) \geq h(G) \geq \lceil \frac{3n}{4} \rceil. \) Thus we may assume that \( n \geq k+3. \)

By Lemma 2.2, we may assume that \( k+3 \leq n < 4k+4. \) We next show that \( h(D) \geq \lceil \frac{3n}{4} \rceil. \)

Note that \( \overline{D} \) is the degree sequence of a near \( r \)-regular graph on \( n \) vertices, where \( 2 \leq r = n−1−k < \frac{k+1}{3}. \) Since \( 3n < 4k + 4, \) we have \( 4r = 4(n−k−1) < n. \) Let \( m, s \) be nonnegative integers so that \( n = 4m + s, \) where \( 0 \leq s \leq 3. \) Clearly, \( m \geq r \) because \( n > 4r. \) Assume that \( p = 0. \) For an integer \( t > 0, \) we denote by \( B_{t,t}^r \) an \( r \)-regular bipartite graph with each partite of size \( t, \) and let \( M \) be a matching of \( B_{t,t}^r \) of size \( \lfloor \frac{n}{2} \rfloor. \) Let \( B^* \) be obtained from \( B_{m,m}^r − M \) by adding a new vertex \( v \) joining to each vertex of \( V(M) \). Note that if \( n \) is odd, then \( r \) is even. Let

\[
\overline{G} = \begin{cases} 
K_{m,m}^r \cup K_{m,m}^r & \text{if } s = 0 \\
K_{m,m}^r \cup B^* & \text{if } s = 1 \\
K_{m,m}^r \cup K_{m+1,m+1}^r & \text{if } s = 2 \\
K_{m+1,m+1}^r \cup B^* & \text{if } s = 3 
\end{cases}
\]

It can be easily checked that \( G \in \mathcal{R}[D] \) and \( G \) contains a clique minor of size at least \( \lfloor \frac{3n}{4} \rfloor. \) Thus \( h(D) \geq \lceil \frac{3n}{4} \rceil, \) as desired. So we may assume that \( p > 0. \) We consider the following two cases.

**Case 1.** \( n \) is even.

Then \( n = 4m \) or \( 4m + 2, \) and \( p \) must be even. Since \( p \leq n−1, \) we have \( p \leq n−2. \) Let \( H \) be a \( k \)-regular graph of order \( n \) with \( V(H) \) partitioned into \( A, B, C, D \) such that \( H[A] = K_m, \) \( H[B] = K_{n−2m}, \) \( H[C] = K_{n−2m}, \) \( H[D] = K_m, \) \( A \cup D \) is complete to \( B \cup C, \) \( H[A,D] \) and \( H[B,C] \) are \( (m−r) \)-regular and \( (m+1−r) \)-regular bipartite graphs, respectively. If \( r < m, \) let \( M \) and \( M' \) be perfect matchings of \( H[A,D] \) and \( H[B,C], \) respectively. Let \( F \subseteq M \cup M' \) with \( |F| = \frac{p}{2}. \) If \( r = m \) and \( p \leq 2m + 2, \) then \( n = 4m + 2. \) Thus \( |B| = |C| = m + 1. \) Let \( M \) be a perfect matching of \( G[B,C]. \) Let \( F \subseteq M \) with \( |F| = \frac{p}{2}. \) In both cases, let \( H^* = H − F. \) Then \( H^* \in \mathcal{R}[D]. \) By contracting each of the edges of a matching of size \( m \) in \( H[A,B] \) into single vertices, we see that \( h(D) \geq h(H^*) \geq \lfloor \frac{3n}{4} \rfloor. \) So we may assume that \( r = m \) and \( p \geq 2m + 4. \) Since \( r = m, \) we have \( n = 4m + 2 \) and so \( k = 3m + 1. \) Note that \( n−p \leq (4m + 2)−(2m + 4) = 2m − 2. \) Let \( H = G(3m, m + 1, m), \) as defined in the proof of Lemma 2.2 (see Case 1). Let \( A \) and \( B \) be as given in the definition of \( G(3m, m + 1, m). \) Now let \( M \) and \( M' \) be a perfect matching and an anti-matching of size \( \frac{n−p}{2} \) of \( G(3m, m + 1, m)[A,B], \) respectively. Let \( H^* = H + M'. \) Then \( H^* \in \mathcal{R}[D]. \) By contracting each of the edges of \( M \) into single vertices, we have \( h(D) \geq h(H^*) \geq \lfloor \frac{3n}{4} \rfloor. \)

**Case 2.** \( n \) is odd.

In this case, \( n = 4m + 1 \) or \( 4m + 3. \) Since \( \sigma(D) = nk−p \) is even, we have \( (b) \) \( p \) and \( k \) are either both even or both odd.

Let \( s = k \) if \( k \) is even and \( s = k − 1 \) if \( k \) is odd. Then \( n−1−s \) is even and let \( r' = (n−1)−1−s. \) Then \( r' = r−1 \) if \( s = k \) and \( r' = r \) if \( s = k − 1. \) Define \( H(s) \) to be an \( (s−1) \)-regular graph of order \( n−1 \) with \( V(H) \) partitioned into \( A, B, C, D \) such that \( H(s)[A] = K_m, \)
Theorem 2.4. Let $D=((k-1)^p, kn^{p})$ be the degree sequence of a near $k$-regular graph on $n$ vertices, where $0 \leq p \leq n-1$. Then $h(D) \geq \chi(D)$.

Proof. By Theorem 1.2 and Theorem 2.3, we may assume that $n=k+1$ and $h(D) \geq k+1-\lceil \frac{n}{4} \rceil$ or $k+2 \leq n < \frac{4k+4}{5}$ and $h(D) \geq \lceil \frac{3n}{4} \rceil$. In the first case, $p$ is even and $D$ has a unique realization $K_{k+1}-M$, where $M$ is a matching of size $\frac{n}{4}$. Clearly, $\chi(D)=k+1-\frac{n}{2} \leq h(D)$. It remains to show that $\chi(D) \leq \lfloor \frac{3n}{4} \rfloor$ when $k+2 \leq n < \frac{4k+4}{5}$. Let $G \in R[D]$ and let $M$ be a maximum anti-matching of $G$. Then $G[V(M)]$ is a complete subgraph. Thus $\chi(G) \leq |G[V(M)]|+|M|=(n-2|M|)+|M|=n-|M|$. It suffices to show that $|M| \geq \frac{n}{4}$. If $p=0$, then $G$ is $k$-regular and $\overline{G}$ is $r$-regular, where $r=n-1-k \geq 1$. By Theorem 1.4, $\chi'(<G>) \leq r+1$, we have $\frac{rn}{2}=|E(<G>)| \leq (r+1)|M|$, which yields $|M| \geq \frac{n}{4}$. Assume that $p>0$. Then $\overline{G}$ contains $n-p$ vertices of degree $r=n-1-(k-1)=n-k$ and $p$ vertices of degree $n-1-k$. Let $r=n-k$. Then $r \geq 2$. If $r \geq 3$, by Theorem 1.4 again, $\chi'(<G>) \leq r+1$, we have $(r-1)n \frac{2}{2} < |E(<G>)| \leq (r+1)|M|$. It follows that $|M| \geq \frac{n}{4}$. So we may assume that $r=2$. Then $\overline{G}$ consists of disjoint unions of cycles and paths. One can easily check that $\overline{G}$ contains a matching of size at least $\frac{n}{2}$. Thus $|M| \geq \frac{n}{4}$ and so $\chi(G) \leq n-|M| \leq \frac{3n}{4}$, as desired. Consequently, $\chi(D) \leq \lfloor \frac{3n}{4} \rfloor$ by the arbitrary choice of $G \in R[D]$.

This completes the proof of Theorem 2.4.

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REFERENCES