The Smallest Degree Sum that Yields Potentially $P_k$-Graphical Sequences*

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Abstract: A simple graph $G$ is said to have property $P_k$ if it contains a complete subgraph of order $k + 1$, and a sequence $\pi$ is potentially $P_k$-graphical if it has a realization having property $P_k$. Let $\sigma(k, n)$ denote the smallest degree sum such that every $n$-term graphical sequence $\pi$ without zero terms and with degree sum $\sigma(\pi) \geq \sigma(k, n)$ is potentially $P_k$-graphical. Erdős, Jacobson, and Lehel [Graph Theory, 1991, 439–449] conjectured that $\sigma(k, n) = (k - 1)(2n - k) + 2$. In this article, we prove that the conjecture is true for $k = 4$ and $n \geq 10$.

1. INTRODUCTION

Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphical sequence. For each $m = 1, 2, \ldots, n$, denote $\sigma_m(\pi) = d_1 + d_2 + \cdots + d_m$ and $\sigma_n(\pi) = \sigma(\pi)$. $\sigma(\pi)$ is called the degree sum of $\pi$.

A simple graph $G$ is said to have property $P_k$ if it has a complete subgraph of order $k + 1$ and a graphical sequence $\pi = (d_1, d_2, \ldots, d_n)$ with $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$

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is said to be potentially $P_k$-graphical if it has a realization having property $P_k$. In [2], Erdős, Jacobson, and Lehel raised the following problem: Determine the smallest degree sum $\sigma(k,n)$ such that every graphical sequence $\pi = (d_1, d_2, \ldots, d_n)$ with $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$ and $\sigma(\pi) \geq \sigma(k,n)$ is potentially $P_k$-graphical. Using an example, they pointed out that $\sigma(k,n) \geq (k-1)(2n-k) + 2$ and conjectured that $\sigma(k,n) = (k-1)(2n-k) + 2$. They also proved that the conjecture was true for $k = 2$. The following is one of their results.

**Theorem 1.1.** Let $n \geq 6$ and $\pi = (d_1, d_2, \ldots, d_n)$ be a graphical sequence, where $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$. If $\sigma(\pi) \geq 2n$, then $\pi$ is potentially $P_2$-graphical. In other words, $\sigma(2,n) = 2n$ for $n \geq 6$.

Recently, J. S. Li and Z. X. Song [6] and Gould, Jacobson, and Lehel [3] proved, respectively, the following.

**Theorem 1.2.** If $n \geq 8$, then $\sigma(3,n) = 4n - 4$.

Moreover, [6] also determined the value of $\sigma(k,n)$ for $n = 2k + 1$.

**Theorem 1.3.** $\sigma(k, 2k + 1) = 2k(2k - 1)$.

In this article, we will determine the value $\sigma(4, n)$. The following theorems are repeatedly used in our proofs.

For a nonincreasing sequence $\pi = (d_1, d_2, \ldots, d_n)$ of nonnegative integers, $d_i \leq n - 1, i = 1, 2, \ldots, n$, define an $n$-by-$n$ matrix $\bar{A} = (a_{ij})$ as follows: If $d_i \geq i$, then

$$a_{ij} = \begin{cases} 1 & \text{if } 1 \leq j \leq d_i + 1 \text{ and } j \neq i \\ 0 & \text{otherwise,} \end{cases}$$

and if $d_i < i$, then

$$a_{ij} = \begin{cases} 1 & \text{if } 1 \leq j \leq d_i \\ 0 & \text{otherwise.} \end{cases}$$

The matrix $\bar{A}$ is called the off-diagonal matrix of $\pi$. The column sum vector of $\bar{A}$ is denoted by $\bar{\pi}$, and called the corrected conjugate vector of $\pi$. Clearly, the row sum vector of $\bar{A}$ is $\pi$.

**Theorem 1.4** (Berge [1]). Let $\pi = (d_1, d_2, \ldots, d_n)$ be a nonincreasing sequence of nonnegative integers, where $d_i \leq n - 1, i = 1, 2, \ldots, n$ and $\sigma(\pi)$ is even. Then $\pi$ is graphical if and only if $\sigma_i(\pi) \leq \sigma_i(\bar{\pi})$ for $i = 1, 2, \ldots, n - 1$, and $\sigma(\pi) = \sigma(\bar{\pi})$.

For a nonincreasing sequence $\pi = (d_1, d_2, \ldots, d_n)$ of nonnegative integers, let

$$\pi' = \begin{cases} (d_1 - 1, \ldots, d_{k-1} - 1, d_{k+1} - 1 \cdots, d_{d_k+1} - 1, d_{d_k+2}, \ldots, d_n) & \text{if } d_k \geq k \\ (d_1 - 1, \ldots, d_{d_k} - 1, d_{d_k+1}, \ldots, d_{d_k-1}, d_{k+1}, \ldots, d_n) & \text{if } d_k < k. \end{cases}$$

Then $\pi'$ is called the residual sequence after laying off $d_k$ from $\pi$.

**Theorem 1.5** (Kleitman and D. L. Wang [5]). A nonincreasing sequence $\pi$ of nonnegative integers is graphical if and only if $\pi'$ is graphical.
In [7], A. R. Rao introduced the following concept: Let $G$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let $\pi = (d_1, d_2, \ldots, d_n)$ be the degree sequence of $G$, where $d_i$ is the degree of $v_i$. Then $G$ is said to have property $A_k$ if the subgraph induced by $V = \{v_1, v_2, \ldots, v_{k+1}\}$ is complete. A graphical sequence $\pi$ is said to be potentially $A_k$-graphical if it has a realization having property $A_k$. A. R. Rao proved the following.

**Theorem 1.6.** A graphical sequence $\pi$ is potentially $P_k$-graphical if and only if $\pi$ is potentially $A_k$-graphical.

A. R. Rao [8] also gave a criterion for a sequence $\pi$ being potentially $A_k$-graphical.

**Theorem 1.7.** Let $\pi = (d_1, d_2, \ldots, d_n)$ be a sequence of nonnegative integers in which $d_1 \geq d_2 \geq \cdots \geq d_{k+1}$ and $d_{k+2} \geq d_{k+3} \geq \cdots \geq d_n$. Then $\pi$ is potentially $A_k$-graphical if and only if the following conditions hold:

1. $d_{k+1} \geq k$,
2. $\sigma(\pi)$ is even,
3. For any $s$ and $t$, $0 \leq s \leq k+1$ and $0 \leq t \leq n-k-1$,

\[
\sum_{i=1}^{s} d_i + \sum_{j=1}^{t} d_{k+1+j} \leq (s+t)(s+t-1) + \sum_{i=s+1}^{k+1} \min\{s+t, d_i-k+s\} + \sum_{j=k+2+t}^{n} \min\{s+t, d_j\}.
\]

Although the proof of Theorem 1.7 is unpublished until now, Kézdy and Lehel [4] gave a proof of Theorem 1.7.

### 2. MAIN RESULTS

Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphical sequence, where $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$ and $n \geq 2k+2$. For given $s$ and $t$, $0 \leq s \leq k+1$ and $0 \leq t \leq n-k-1$, let

\[
L(s, t) = \sum_{i=1}^{s} d_i + \sum_{j=1}^{t} d_{k+1+j}
\]

and

\[
R(s, t) = (s+t)(s+t-1) + \sum_{i=s+1}^{k+1} \min\{s+t, d_i-k+s\} + \sum_{j=k+2+t}^{n} \min\{s+t, d_j\}.
\]

We first prove the following.
Theorem 2.1. Let \( n \geq 2k + 2 \) and let \( \pi = (d_1, d_2, \ldots, d_n) \) be a graphical sequence. If
\[
n - 2 \geq d_1 \geq \cdots \geq d_k = d_{k+1} = \cdots = d_{d_1+2} \geq d_{d_1+3} \geq \cdots \geq d_n \geq k,\]
then \( \pi \) is potentially \( P_k \) graphical.

Proof. By Theorem 1.7, we need to verify only that
\[
R(s, t) \geq L(s, t) \tag{1}
\]
for any \( s \) and \( t \), \( 0 \leq s < k + 1 \) and \( 0 \leq t \leq n - k - 1 \). We consider the following cases.

Case 1. \( d_k \leq s + t - 1 \). If \( s \geq k \), then \( d_i - k + s \geq (d_k - k) + s \geq s \geq k \) for \( 1 \leq i \leq k + 1 \). Hence, \( \min\{s + t, d_i - k + s\} \geq s \geq k \) for \( s + 1 \leq i \leq k + 1 \). Observe that \( s + t > d_k \geq \cdots \geq d_n \geq k \). So we have
\[
R(s, t) \geq (s + t)(s + t - 1) + (k + 1 - s)k + (n - k - 1 - t)k
= (k - 1)(n - 2) + (s - k + 1 + t)(s + t - 1) + (n + k - 1 - s - t)
\geq (k - 1)(n - 2) + (s - k + 1)(s + t - 1) + (s + t - 1)
\geq (k - 1)d + (s - k + 1)d_k + td_k \geq L(s, t).
\]
If \( s \leq k - 1 \), then \( d_i - k + s \geq (d_k - k) + s \geq s \) for \( 1 \leq i \leq k + 1 \). Moreover, \( s + t > d_k \geq \cdots \geq k > s \). Hence,
\[
R(s, t) \geq (s + t)(s + t - 1) + (k + 1 - s)s + (n - k - 1 - t)s
= s(n - 1) + t(s + t - 1) > sd_1 + td_k \geq L(s, t).
\]
Thus, (1) holds for \( d_k \leq s + t - 1 \).

Case 2. \( d_k \geq s + t \). Clearly,
\[
d_k = d_{k+1} = \cdots = d_{t+k+2} = \cdots = d_{d_1+2} \geq s + t.
\]
If \( d_k \geq t + k \), then \( d_i - k + s \geq d_k - k + s \geq s + t \) for \( s + 1 \leq i \leq k + 1 \). Hence,
\[
R(s, t) \geq (s + t)(s + t - 1) + \sum_{i=s+1}^{k+1} \min\{s + t, d_i - k + s\}
+ \sum_{j=k+2+t}^{d_1+2} \min\{s + t, d_j\}
= (s + t)(s + t - 1) + (k + 1 - s)(s + t) + (d_1 + 1 - k - t)(s + t)
= (s + t)(d_1 + 1) \geq (s + t)d_1 \geq L(s, t).
\]
Now assume that \( s + t \leq d_k \leq t + k - 1 \). Then \( s \leq k - 1 \) and \( d_k - k + s \leq (t + k - 1) - k + s \leq s + t - 1 \). Moreover \( d_k - k + s \leq d_i - k + s \) for \( 1 \leq i \leq k + 1 \). Hence, \( \min\{s + t, d_i - k + s\} \geq d_k - k + s \) for \( 1 \leq i \leq k + 1 \). Denote
\[ d_k = t + m, \text{ where } 0 \leq s \leq m \leq k - 1. \text{ Then } s + t = d_k - (m - s) \geq k - (m - s). \]

In addition, \( d_n \geq k \geq k - (m - s). \) Therefore,

\[
R(s, t) \geq (s + t)(s + t - 1) + (k + 1 - s)(d_k - k + s)
+ (k - m + s)(n - t - k - 1)
= s(n - 2) + td_k + (k - m)(n - 2k - 2) + s(k + 1 - m)
\geq s(n - 2) + td_k \geq sd_1 + td_k = L(s, t).
\]

This shows that (1) holds for \( d_k \geq s + t \).}

We now will determine the values of \( \sigma(4, n) \) for \( n \geq 10 \). The main tools are induction, the laying off technique due to Kleitman and D. L. Wang, and A. R. Rao’s

Theorem 1.6 concerning equivalence on potentially \( P_k \)-graphical sequences and potentially \( A_k \)-graphical sequences. It is easy to see that, if the residual sequence

\[ \pi' = (d_1', d_2', \ldots, d_{n-1}') \]

obtained laying off \( d_1 \) from a graphical sequence \( \pi = (d_1, d_2, \ldots, d_n) \) is potentially \( A_{k-1} \)-graphical and \( d_2 - 1, \ldots, d_{k+1} - 1 \) are the \( k \)

largest numbers in \( \pi' \), then \( \pi \) is potentially \( P_k \)-graphical. We begin with the value

\[ \sigma(4, 10) = 50. \]

\textbf{Theorem 2.2.} \( \sigma(4, 10) = 50. \)

\textbf{Proof.} It is enough to prove that, if \( \pi = (d_1, d_2, \ldots, d_{10}) \) is a graphical sequence without zero terms and the degree sum \( \sigma(\pi) \) is at least 50, then \( \pi \) is potentially \( P_4 \)-graphical.

First, \( d_1 \geq 5 \), since \( 50 \leq \sigma(\pi) \leq 10d_1 \). Next, \( d_5 \geq 4 \), otherwise by the off-
diagonal matrix \( A \) of \( \pi \), we have \( \overline{d_i} \leq 9 \) for \( 1 \leq i \leq 3 \) and \( \overline{d_i} \leq 3 \). Hence, \( \sigma_4(\pi) \leq 30 \). On the other hand, we have \( 50 \leq \sigma(\pi) \leq \sigma_4(\pi) + 6d_5 \leq \sigma_4(\pi) + 18 \). Hence, \( \sigma_4(\pi) \geq 32 > 30 \geq \sigma_4(\pi) \). By Theorem 1.4, \( \pi \) is not graphical, contradiction.

\( d_8 \geq 2 \), otherwise by the off-diagonal matrix \( A \) of \( \pi \), we have \( \overline{d_i} = 9 \) and \( \overline{d_i} \leq 6 \) for \( 2 \leq i \leq 7 \). Hence, \( \sigma_7(\pi) \leq 9 + 6 \times 6 = 45 \). On the other hand, we have \( 50 \leq \sigma(\pi) \leq \sigma_7(\pi) + 3 \). Hence, \( \sigma_7(\pi) \geq 47 > 45 \geq \sigma_7(\pi) \). By Theorem 1.4, \( \pi \) is not graphical, contradiction. In other words, \( \pi \) contains at most

two 1’s.

The residual sequence obtained by laying off \( d_1 \) from \( \pi \) is denoted by \( \pi' = (d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_{10}) \). In realizing \( \pi' \), we delete the zero terms; this leaves 9, 8, or 7 vertices. By Theorems 1.2 and 1.3, we have \( \sigma(3, 9) = 32, \sigma(3, 8) = 28, \text{ and } \sigma(3, 7) = 30 \). Since \( \sigma(\pi') = \sigma(\pi) - 2d_1 \geq 50 - 18 \geq 32 \), \( \sigma(\pi') = 32 > 30 \geq \sigma_4(\pi) \) is potentially \( P_3 \)-graphical. By Theorem 1.6, \( \pi' \) is potentially \( A_3 \)-graphical. If \( \pi' \) has zero terms, then \( d_8 \geq 2 \) implies that \( d_1 \geq 8 \) and \( d_{d_1+2} \leq 1 \). Since \( d_5 \geq 4 \), \( d_2 - 1, \ldots, d_5 - 1 \) are the four largest numbers in \( \pi' \).

Thus, \( \pi \) is potentially \( A_4 \)-graphical.

Now suppose that \( \pi' \) has no zero terms. If there exists an integer \( t, 5 \leq t \leq d_1 + 1 \) such that \( d_1 > d_{t+1} \), then \( d_2 - 1, \ldots, d_5 - 1 \) are the four largest numbers in \( \pi' \). Since \( \pi' \) is potentially \( A_3 \)-graphical, \( \pi \) is potentially \( A_4 \)-graphical. So we may assume that

\[ d_1 \geq \cdots \geq d_4 \geq d_5 = \cdots = d_{d_1+2} \geq d_{d_1+3} \geq \cdots \geq d_{10}. \]
If \( d_1 > d_5 \), then by laying off \( d_{d_1+1} = l \) from \( \pi \), we obtain a sequence \( \pi'' = (d_1 - 1, \ldots, d_l - 1, d_{l+1}, \ldots, d_{d_1}, d_{d_1+2}, \ldots, d_{10}) \) without zero terms. Clearly, 
\[
\sigma(\pi'') = 50 - 2d_{d_1+1} \geq 32 = \sigma(3,9) \text{. Hence } \pi'' \text{ is potentially } A_3\text{-graphical.}
\]
Since \( d_1 - 1, d_2 - 1, d_3 - 1 \) and \( d_4 - 1 \) are the four largest numbers in \( \pi'' \), \( \pi \) is potentially \( A_4\)-graphical. So we may further assume that
\[
d_1 \geq \cdots \geq d_4 = d_5 = \cdots = d_{d_1+2} \geq d_{d_1+3} \geq \cdots \geq d_{10}.
\]
If \( d_1 = 9 \), then
\[
d_1 \geq \cdots \geq d_4 = d_5 = \cdots = d_{10} \geq 4.
\]
Since \( \pi' = (d_2 - 1, d_3 - 1, \ldots, d_{10} - 1) \) is potentially \( A_3\)-graphical, \( \pi \) is potentially \( P_4\)-graphical. So we assume that \( d_1 \leq 8 \). If \( d_{10} \geq 4 \), then by Theorem 2.1, \( \pi \) is potentially \( P_4\)-graphical. Now assume \( d_{10} \leq 3 \). Since \( d_{1+2} = d_5 \geq 4 \), we have \( d_1 + 2 < 9 \), i.e., \( d_1 \leq 7 \). Observe that \( 50 \leq \sigma(\pi) \leq 3d_1 + 6d_4 + d_{10} \). Hence, \( 6 \leq d_1 \leq 7 \) and \( 5 \leq d_4 \leq 7 \). We distinguish three cases according to the number of zeros in \( \pi' \) to prove that \( \pi \) is one of the following sequences:

\[
\begin{align*}
\pi_1 &= (7^9, 3^1), & \pi_2 &= (7^3, 6^6, 3^1), & \pi_3 &= (7^1, 6^8, 3^1), \\
\pi_4 &= (7^3, 5^6, 3^1), & \pi_5 &= (7^2, 5^7, 3^1), & \pi_6 &= (7^1, 6^2, 5^6, 3^1), \\
\pi_7 &= (7^1, 5^6, 3^1), & \pi_8 &= (6^8, 5^1, 3^1), & \pi_9 &= (6^8, 3^2), \\
\pi_{10} &= (6^3, 5^5, 4^1, 3^1), & \pi_{11} &= (6^2, 5^7, 3^1), & \pi_{12} &= (7^2, 6^7, 2^1), \\
\pi_{13} &= (7^2, 6^1, 5^6, 2^1), & \pi_{14} &= (7^1, 6^1, 5^7, 2^1), & \pi_{15} &= (6^9, 2^1), \\
\pi_{16} &= (6^8, 4^1, 2^1), & \pi_{17} &= (6^8, 2^2), & \pi_{18} &= (6^3, 5^6, 2^1), \\
\pi_{19} &= (7^9, 1^1), & \pi_{20} &= (7^3, 6^6, 1^1), & \pi_{21} &= (7^1, 6^8, 1^1), \\
\pi_{22} &= (7^1, 6^2, 5^6, 1^1), & \pi_{23} &= (7^3, 5^6, 1^1), & \pi_{24} &= (6^8, 1^2), \\
\pi_{25} &= (6^8, 3^1, 1^1), & \pi_{26} &= (6^8, 5^1, 1^1),
\end{align*}
\]

**Case 1.** \( d_{10} = 3 \). If \( d_1 = 7 \), then \( d_4 = \cdots = d_9 \). If \( d_4 = 7 \), then \( \pi = \pi_1 \). If \( d_4 = 6 \), then \( \pi = (7^1, d_2, d_3, 6^6, 3^1) \), where \( 12 \leq d_2 + d_3 \leq 14 \). Note that \( \sigma(\pi) \) is even. So \( \pi = \pi_2 \) or \( \pi_3 \). If \( d_4 = 5 \), then \( \pi = (7^1, d_2, d_3, 5^6, 3^1) \), where \( 10 \leq d_2 + d_3 \leq 14 \). Hence, \( \pi \) is one of \( \pi_4, \pi_5, \pi_6, \) and \( \pi_7 \). Therefore, we assume \( d_4 = 6 \). If \( d_4 = 6 \), then \( \pi = (6^8, d_9, 3^1) \). Hence, \( \pi = \pi_8 \) or \( \pi_9 \), because \( \sigma(\pi) \) is even. If \( d_4 = 5 \), then \( \pi = (6^1, d_2, d_3, 5^5, d_9, 3^1) \). Since \( 50 \leq \sigma(\pi) \leq 18 + 25 + d_9 + 3 \), we have \( 4 \leq d_9 \leq 5 \). Thus, \( \pi = \pi_{10} \) or \( \pi_{11} \).

**Case 2.** \( d_{10} = 2 \). If \( d_1 = 7 \), then \( d_4 = \cdots = d_9 \). If \( d_4 = 7 \), then \( \pi = (7^1, 2^1) \) is not graphical. If \( d_4 = 6 \), then \( 12 \leq d_2 + d_3 \leq 14 \). Since \( \sigma(\pi) \) is even, we have \( \pi = \pi_{12} \). If \( d_4 = 5 \), then \( \pi = \pi_{13} \) or \( \pi_{14} \). If \( d_4 = 6 \), then \( d_4 = \cdots = d_8 \). If \( d_4 = 6 \), then \( \pi = (6^8, d_9, 2^1) \) and \( \pi \) is one of \( \pi_{15}, \pi_{16}, \) and \( \pi_{17} \), since \( \sigma(\pi) \) is even. If \( d_4 = 5 \), then \( 50 \leq \sigma(\pi) \leq 18 + 25 + d_9 + 2 \). Hence, \( d_9 = 5 \) and \( \pi = \pi_{18} \).

**Case 3.** \( d_{10} = 1 \). If \( d_1 = 7 \), then \( d_4 = \cdots = d_9 \). If \( d_4 = 7 \), then \( \pi = \pi_{19} \). If \( d_4 = 6 \), then \( \pi = (7^1, d_2, d_3, 6^6, 1^1) \), where \( 12 \leq d_2 + d_3 \leq 14 \). Since \( \sigma(\pi) \) is even, \( \pi = \pi_{20} \) or \( \pi_{21} \). If \( d_4 = 5 \), then \( \pi = (7^1, d_2, d_3, 5^6, 1^1) \), where \( 12 \leq d_2 + d_3 \leq 14 \) and \( d_9 + d_9 \) is even. Hence, \( \pi = \pi_{22} \) or \( \pi_{23} \). If \( d_1 = 6 \), then \( d_4 = \cdots = d_8 \). Note that \( 50 \leq \sigma(\pi) \leq 18 + 6d_4 + 1 \). Hence, \( d_4 = 6 \), and \( \pi(6^8, d_9, 1^1) \), where \( d_9 \) is odd. Thus, \( \pi = \pi_{24}, \pi_{25}, \) or \( \pi_{26} \).
We now will check that all sequences \( \pi_1, \pi_2, \ldots, \pi_{26} \) are potentially \( A_k \)-graphical. First \( \sigma(4, 9) = 56 \) by Theorem 1.3. Hence, \( \pi_1, \pi_{19}, \) and \( \pi_{20} \) are potentially \( A_4 \)-graphical. Next for every remained sequence \( \pi = (d_1, d_2, \ldots, d_{10}) \), we may construct its realization \( G \) having property \( A_4 \) in the following way: Place a clique on the set \( S \) consisted of the 5 vertices of largest degree. The number \( r = \sum_{i=1}^{5} (d_i - 4) \) is the number of edges that must go from \( \mathcal{S} \) to \( V(G) \) as a bipartite subgraph. Hence, \( V(G) - \mathcal{S} \) induces a subgraph with \( ((\sum_{i=0}^{10} d_i - r)/2 \) edges. After forming such a subgraph, it is generally easy to insert the \( r \) edges between the sets \( \mathcal{S} \) and \( V(G) - \mathcal{S} \) to obtain the realization \( G \).

Figure 1 in the Appendix shows that all these remained sequences are potentially \( A_4 \)-graphical.

**Theorem 2.3.** If \( n \geq 10 \), then \( \sigma(4, n) = 6n - 10 \).

**Proof.** We only need to prove that if \( \pi = (d_1, d_2, \ldots, d_n) \) is a graphical sequence without zero terms and the degree sum \( \sigma(\pi) \) is at least \( 6n - 10 \), then \( \pi \) is potentially \( P_4 \)-graphical. We use induction on \( n \). By Theorem 2.2, the theorem holds for \( n = 10 \). Now suppose that \( n > 10 \). If \( d_n \leq 3 \), then \( \pi'' = (d_1 - 1, \ldots, d_{n-1}, d_{n+1}, \ldots, d_{n-1}) \) is graphical and has no zero terms. Clearly \( \sigma(\pi'') = \sigma(\pi) - 2d_n \geq 6(n - 1) - 10 \). By induction hypothesis, \( \pi'' \) is potentially \( P_3 \)-graphical, and, therefore, \( \pi \) is potentially \( P_4 \)-graphical. Hence, we may assume that \( d_n \geq 4 \). If \( d_1 = n - 1 \), then \( \pi' = (d_1 - 1, d_2 - 1, \ldots, d_n - 1) \) is graphical and has no zero terms. Clearly, \( \sigma(\pi') = \sigma(\pi) - 2(n - 1) \geq 4(n - 1) - 4 = \sigma(3, n - 1) \), where \( n - 1 \geq 9 \). By Theorem 1.2, \( \pi' \) is potentially \( P_3 \)-graphical, so \( \pi \) is potentially \( P_4 \)-graphical. Hence, we may further assume that \( d_1 \leq n - 2 \). If there exists an integer \( t, 5 \leq t \leq d_1 + 1 \) such that \( d_t > d_{t+1} \), then \( d_2 - 1, \ldots, d_5 - 1 \) are the four largest numbers in \( \pi' \). Since \( \pi' \) is potentially \( A_3 \)-graphical, \( \pi \) is potentially \( A_4 \)-graphical. Consequently, we may assume that

\[
n - 2 \geq d_1 \geq \cdots \geq d_4 \geq d_5 = \cdots = d_{d_1 + 2} \geq d_{d_1 + 3} \geq \cdots \geq d_n \geq 4.
\]

If \( d_4 > d_5 \), then by laying off \( d_{d_1 + 2} = l \) from \( \pi' \), the sequence \( \pi''' = (d_1 - 1, \ldots, d_l - 1, d_{l+1}, \ldots, d_{d_1 + 1}, d_{d_1 + 3}, \ldots, d_n) \) has no zero terms and \( \sigma(\pi''') = \sigma(\pi) - 2l \geq 4(n - 1) - 4 \). By Theorem 1.2, \( \pi''' \) is potentially \( A_3 \)-graphical. Since \( d_1 - 1, d_2 - 1, d_3 - 1, \) and \( d_4 - 1 \) are the four largest numbers in \( \pi''' \), \( \pi \) is potentially \( A_4 \)-graphical. Hence, we may assume that

\[
n - 2 \geq d_1 \geq \cdots \geq d_4 = d_5 = \cdots = d_{d_1 + 2} \geq d_{d_1 + 3} \geq \cdots \geq d_n \geq 4.
\]

By Theorem 2.1, \( \pi \) is potentially \( P_4 \)-graphical.

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Appendix

\[ \pi_2: \quad \pi_3: \quad \pi_4: \]

\[ \pi_5: \quad \pi_6: \quad \pi_7: \]

\[ \pi_8: \quad \pi_9: \quad \pi_{10}: \]

\[ \pi_{11}: \quad \pi_{12}: \quad \pi_{13}: \]

FIGURE 1. Potentially $\lambda_4$-graphical sequences.
FIGURE 1. (continued)
References