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ace and Time

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by Hermann Minkowski, translated by Megh Nad Saha and Wikisource

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- Saha's translation: The Principle of Relativity (1920), Calcutta: University Press, pp. 70-88, Source (http://www.archive.org/details/principleofrelat00eins)
- In this Wikisource edition, Saha's notation was replaced by Minkowski's original notation. Also some passages were corrected, and the images and footnotes (omitted by Saha) were included and translated from the German original. (See Saha's original for comparison).

Space and Time

By HERMANN MINKOWSKI in Göttingen.

Gentlemen! The concepts about time and space, which I would like to develop before you today, have grown on experimental physical grounds. Herein lies their strength. Their tendency is radical. Henceforth, space for itself, and time for itself shall completely reduce to a mere shadow, and only some sort of union of the two shall preserve independence.

I

I would like to show you at first, how we can arrive – from mechanics as currently accepted – at the changed concepts about time and space, by purely mathematical considerations. The equations of NEWTONian mechanics show a twofold invariance. First, their form remains unaltered when we subject the underlying spatial coordinate system to any *change of position*, second, when we change the system in its state of motion, *i. e.*, when we impress upon it any *uniform motion of translation*; also the null-point of time plays no role. We are accustomed to look upon the axioms of geometry as settled, when we feel ready for the axioms of mechanics, and therefore the two invariants certainly are seldom mentioned in the same breath. Each one of these denotes a certain group of transformations in itself for the differential equations of mechanics. We look upon the existence of the first group as a fundamental characteristics of space. We always prefer to punish the second group with content, so as to get over the fact with a heart, that we can never decide from physical considerations whether the space, which is supposed to be at rest, may not finally be in uniform motion. So these two groups have quite separate existences besides each other. Their totally heterogeneous character may scare us away from the attempt to compound them. Yet it is the whole compounded group which as a whole gives us occasion for thought.

We wish to picture to ourselves the whole relation graphically. Let *x, y, z* be the rectangular coordinates of space, and *t* denote the time. Subjects of our perception are always places and times connected. No one has observed a place except at a particular time, or has observed a time except at a particular place. Yet I still respect the dogma that time and space have independent existences each. I will call a space-point at a time-point, *i.e.*, a system of values *x, y, z, t*, as a *world-point*. The manifold of all possible value systems of *x, y, z, t*, shall be denoted as the *world*. I boldly could draw four world-axes with a chalk upon a table. Even *one* axis drawn consists of nothing but quickly vibrating molecules, and besides, takes part in all the journeys of the earth in the universe; and therefore gives us plenty occasions for abstractions. The greater abstraction connected with the number of 4 does not cause the mathematician any trouble. In order not to allow any yawning gap to exist, we shall suppose that at every place and time, something perceptible exists. In order not to say either matter or electricity, we shall simply use the word substance for this something. We direct our attention to the substantial point located at world-point *x, y, z, t*, and suppose that we are in a position to recognize this substantial point at any other time. Let *dt* be the time element corresponding to the changes *dx, dy, dz* of space coordinates of this substantial point. Then we obtain (as a picture, so to speak, of the perennial life-career of the substantial point), a curve in the *world*, the *world-line*, whose points can unambiguously be connected to the parameter *t* from $+\infty$ to $-\infty$. The whole world appears to be resolved in such world-lines, and I may just anticipate, that according to my opinion the physical laws would find their most perfect expression as mutual relations among these world-lines.

By concepts of time and space, the *x, y, z* manifold $t = 0$ and its two sides $t < 0$ and $t > 0$ fall apart. If, for the sake of simplicity, we keep the null-point of time and space fixed, then the first mentioned group of mechanics signifies that at $t = 0$ we can give the *x, y, z*-axes an arbitrary rotation about the null-point, corresponding to the homogeneous linear transformation of the expression

$$x^2 + y^2 + z^2$$

in itself. Yet the second group denotes that – also without changing the expression for the mechanical laws – we can substitute

$$x, y, z, t \text{ by } x - \alpha t,\ y - \beta t,\ z - \gamma t,\ t$$

with any constants α, β, γ. According to this we can give the time-axis any possible direction in the upper half of the world $t > 0$. Now, what has the demand of orthogonality in space to do with this perfect freedom of the time-axis towards the upper direction?

To establish this connection, let us take a positive parameter *c*, and let us consider the figure

$$c^2t^2 - x^2 - y^2 - z^2 = 1.$$

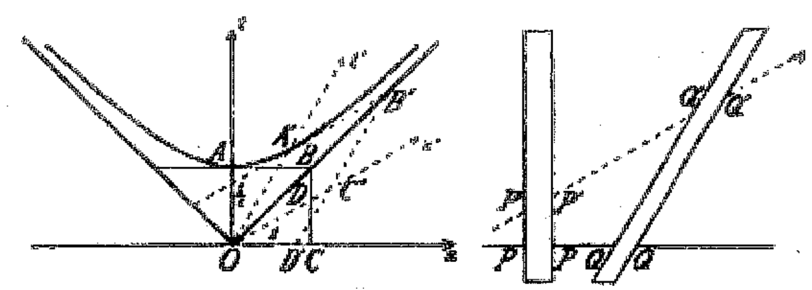

Fig. 1.

According to the analogy of the hyperboloid of two sheets, this consists of two sheets separated by $t = 0$. Let us consider the sheet in the region of $t > 0$, and let us now conceive the transformation of *x, y, z, t* into four new variables *x', y', z', t'*, and the expression of this sheet in the new variables will be equivalent. Clearly the rotations of space round the null-point belongs to this group of transformations. We can already have a complete idea of the transformations, when we look upon one of them, in which *y* and *z* remain unaltered. Let us draw the cross section of that sheet with the plane of the *x*-and *t*-axes, *i.e.*, the upper branch of the hyperbola $c^2 t^2 - x^2 = 1$, with its asymptotes (Fig. 1). Then let us draw an arbitrary radius vector OA' of that hyperbola branch from the origin *O*, the tangent in A' at the hyperbola to the cutting B' with the asymptote given at the right, and completing $OA'B'$ to the parallelogram $OA'B'C'$; at last for what follows, $B'C'$ is drawn to meet the *x*-axis at D'. Let us now take OC' and OA' as axes for the parallel coordinates x', t' with measuring rods $OC' = 1$, $OA' = 1/c$; then that hyperbola branch is again expressed in the form $c^2 t'^2 - x'^2 = 1, t' > 0$ and the transition from *x, y, z, t* to *x' y z t'* is one of the transitions in question. Let us add to those characteristic transformations an arbitrary displacement of the space- and time-nullpoints; by the we form a group of transformations still depending on the parameter *c*, which I may denote by G_c.

Now let us increase *c* to infinity, thus $1/c$ converges to zero, and it appears from the above figure, that the branch of the hyperbola gradually approaches the *x*-axis, the asymptotic angle extends becomes more obtuse, that the special transformation in the limit changes into one where the *t*-axis can have any direction upwards, and x' more and more approaches *x*. With respect to this it is clear that the group G_c in the limit for $c = \infty$, *i.e.* as group G_∞, exactly becomes the full group belonging to NEWTONian Mechanics. In this state of affairs, and since G_c is mathematically more intelligible than G_∞, a mathematician may, by a free play of imagination, hit upon the thought that natural phenomena actually possess an invariance, not for the group G_∞, but rather for a group G_c, where *c* is definitely finite, and only *exceedingly large* using the ordinary measuring units. Such a preconception would have been an extraordinary triumph for pure mathematics. Now, although mathematics only shows irony at this place, still the satisfaction remains for it, that thanks to its fortunate antecedents by its senses sharpened in free remote-view, it is instantly able to grasp the deep consequences of such a modification of our view of nature.

At the same time I shall remark about which value of *c* we eventually will arrive. For *c*, we shall substitute *the propagation velocity of light in free space*. In order to avoid speaking either of space or of vacuum, we again may take this quantity as the ratio between the electrostatic and electromagnetic units of the quantity of electricity.

We could form an idea of the invariant character of natural laws for the corresponding group G_c in the following manner:

Out of the totality of natural phenomena, we can, by successive higher approximations, deduce with increased precision a coordinate system *x, y, z,* and *t*, space and time, by means of which we can represent the phenomena according to definite laws. This system of reference, however, is by no means uniquely determined by the phenomena. *We can change the system of reference in any possible manner corresponding to the transformation of the above mentioned group* G_c, *but the expressions for natural laws will not be changed thereby.*

For example, corresponding to the figure described above, we also can denote t' as time, but in connection with this, we must necessarily define the space by the manifoldness of the three parameters *x' y z*. The physical laws are now expressed by means of *x', y, z, t',* — and the expressions are exactly the same as in the case of *x, y, z, t*. According to this, we shall have, not *one* space, but an infinite number of spaces in the world, — analogous to the case that the three-dimensional space consists of an infinite number of planes. The three-dimensional geometry becomes a chapter of four-dimensional physics. Now you perceive, why I said in the beginning that time and space shall reduce to mere shadows and we shall have only one world itself.

II.

Now the question may be asked, — what circumstances forces upon us these changed views about time and space, are they actually never in contradiction with observed phenomena, do they finally guarantee us advantages for the description of natural phenomena?

Before we enter into the discussion, a very important point must be noticed. Suppose we have individualized time and space in any manner; then a substantial point as a world-line corresponds to a line parallel to the *t*-axis; a uniformly moving substantial point corresponds to a world-line inclined to the *t*-axis; and non-uniformly moving substantial point will correspond to a somehow curved world-curve. Let us consider the world-line passing through any world point *x, y, z, t*; now if we find the world-line parallel to any radius vector OA' of the hyperboloidal sheet mentioned before, then we can introduce OA' as a new time-axis, and then according to the new conceptions of time and space the substance in the corresponding world point will appear to be at rest. We shall now introduce this fundamental axiom:

The substance existing at any world point can always be conceived to be at rest, if time and space are interpreted suitably.

The axiom means, that in a world-point the expression

$$c^2dt^2 - dx^2 - dy^2 - dz^2,$$

shall always be positive or what is equivalent to the same thing, every velocity *v* should always be smaller than *c*. *c* shall therefore be the upper limit for all substantial velocities and exactly herein lies a deep significance for the quantity *c*. At the first impression, the axiom in this different form seems to be rather unsatisfactory. However, it is to be remembered that a modified mechanics will hold now, in which the square root of this differential combination takes is included, so that cases in which the velocity is greater than *c* will only play a role in a similar way as figures with imaginary coordinates in geometry.

The *impulse* and real cause for accepting *the group* G_c, came from the fact that the differential equation for the propagation of light in vacant space possesses that group G_c.[1] On the other hand, the idea of rigid bodies has any sense only in a system mechanics with the group G_∞. Now if we have an optics with G_c, and on the other hand if there were rigid bodies, it is easy to see that *one* t-direction is preferred by the two hyperboloidal shells belonging to the groups G_∞, and G_c, which would have got the further consequence, that by means of suitable rigid instruments in the laboratory, we can perceive a change in natural phenomena, in case of different orientations with regard to the direction of progressive motion of the earth. But all efforts directed towards this goal, and even the celebrated interference-experiment of MICHELSON have given negative results. In order to supply an explanation for this result, H. A. LORENTZ formed a hypothesis whose success lies exactly in the invariance of optics for the group G_c. According to LORENTZ every body in motion, shall suffer a contraction in the direction of its motion, namely at velocity v in the ratio

$$1 : \sqrt{1 - \frac{v^2}{c^2}}$$

This hypothesis sounds rather fantastical. For the contraction is not to be thought of as a consequence of resistances in the ether, but purely as a gift from above, as a condition accompanying the state of motion.

I shall show in our figure, that LORENTZ's hypothesis is fully equivalent to the new conceptions about time and space, by which it becomes more intelligible. Let us now, for the sake of simplicity, neglect y and z and imagine a spatially two dimensional world, in which upright strips parallel to the t-axis represent a state of rest and another parallel strip inclined to the t-axis represent a state of uniform motion for a body (see fig. 1). If OA' is parallel to the second strip, we can take t' as time and x' as the space coordinate, then the second body will appear to be at rest, and the first body in uniform motion. We shall now assume that the first body supposed to be at rest, has the length l, *i.e.*, the cross section PP of the first strip upon the x-axis $= l \cdot OC$, where OC is the unit measuring rod upon the x-axis — and that on the other hand, the second body also, when *supposed to be at rest*, has the same length l, this means that the cross section is $Q'Q' = l \cdot OC'$ when measured *parallel to the* x'*-axis*. In these two bodies, we have now images of two equal LORENTZ-electrons, one of which is at rest and the other moves uniformly. Now if we stick to our original coordinates, then the extension of the second electron is given by the cross section QQ of the corresponding strip is to be given *parallel to the* x*-axis*. Now it is clear, since $Q'Q' = l \cdot OC'$, that $QQ = l \cdot OD'$. If $\frac{dx}{dt} = v$ for the second strip, a simple calculation gives $OD' = OC\sqrt{1 - \frac{v^2}{c^2}}$, therefore also $PP : QQ = 1 : \sqrt{1 - \frac{v^2}{c^2}}$. This, however, is the sense of LORENTZ's hypothesis about the contraction of electrons in motion. On the other hand, if we conceive the second electron to be at rest, and therefore adopt the system x', t', then the cross-section $P'P'$ of the strip of the electron parallel to OC' is to be regarded as its length and we shall find the first electron shortened with reference to the second in the same proportion, for it is in the figure

$$P'P' : Q'Q' = OD : OC' = OD' : OC = QQ : PP$$

LORENTZ denoted the combination t' of (t and x) as the *local time (Ortszeit)* of the uniformly moving electron, and used a physical construction of this idea for a better comprehension of the contraction-hypothesis. But to perceive clearly that the time of an electron is as good as the time of any other electron, *i.e.*, that t and t' are to be treated equivalently, has been the service of A. EINSTEIN.[2] By that, the concept of time was shown to be unambiguously established by natural phenomena. But the concept of space was not altered, either by EINSTEIN or LORENTZ, probably because in the case of the above-mentioned spatial transformations, where the x', t' plane coincides with the x, t plane, the interpretation is possible as if the x-axis of space somehow remains conserved in its position. To step over the concept of space in a corresponding manner, is certainly only to assess as the boldness of mathematical culture. After this inevitable step for the true understanding of the group G^c, however, the word "Relativity-Postulate" for the demands of invariance in the group G^c, seems to be rather weak to me. Because the sense of the postulate is that the four-dimensional world is given in space and time by phenomena only, but the projection in time and space can be handled with a certain freedom, and therefore I would rather like to give to this assertion the name "*The Postulate of the Absolute World*", (or shortly World-Postulate).

III.

By the world-postulate a similar treatment of the four determining parts x, y, z, t becomes possible. Thereby the forms under which the physical laws come forth, gain in intelligibility, as I shall presently show. Above all, the idea of *acceleration* becomes much more striking and clear.

I shall again use the geometrical method of expression, which presents itself by tacitly neglecting x from the triple x, y, z. Let us call any world-point O as a space-time-null-point. The *cone*

$$c^2t^2 - x^2 - y^2 - z^2 = 0$$

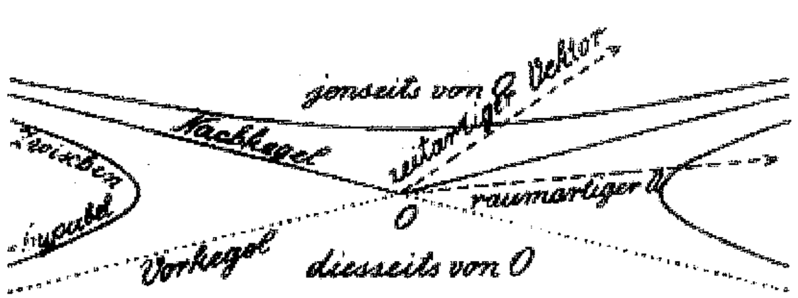

Fig. 2

consists of two parts with O as apex, one part having $t < 0$, the other having $t > 0$ (*Vide* fig. 2).. The first, the *fore-cone* consists of all those points which "send light towards O", the second, the *aft-cone*, consists of all those points which "receive their light from O". The region bounded by the fore-cone may be called the "fore-side of O", and the region bounded by the aft-cone may be called the "aft-side of O". On the aft-side of O we have the already considered hyperboloidal shell

$$F = c^2t^2 - x^2 - y^2 - z^2 = 1,\ t > 0.$$

The region *between the cones* will be occupied by the hyperboloid forms of one sheet

$$-F = x^2 + y^2 + z^2 - c^2t^2 = k^2,$$

to all constant positive values k^2. The hyperbolas which lie upon this figure with *O* as center, are important for us. For the sake of shortness the individual branches of this hyperbola will be called the *interhyperbola to center O*. Such a hyperbolic branch, when thought of as a world-line of a substantial point, would represent a motion which for $t = -\infty$ and $t = +\infty$ asymptotically approaches the velocity of light *c*.

If, by way of analogy to the idea of vectors in space, we call any directed length in the manifoldness *x, y, z, t* a *vector*, then we have to distinguish between a *time-like* vector directed from *O* towards the sheet $+F = 1, t > O$ and a *space-like* vector directed from *O* towards the sheet $-F = 1$. The time-axis can be parallel to any vector of the first kind. Any world-point between the *fore* and *aft cones* of *O*, may by means of the system of reference be regarded either as *synchronous* with *O*, as well as *later* or *earlier* than *O*. Every world-point on the fore-side of *O* is necessarily always earlier, every point on the aft side of *O*, necessarily later than *O*. The limit $c = \infty$ corresponds to a complete folding up of the wedge-shaped cross-section between the cones in the plane manifoldness $t = 0$. In the figure drawn, this cross-section has been intentionally drawn with a different breadth.

Let us decompose a vector drawn from *O* towards *x, y, z, t* into its four components *x, y, z, t*. If the directions of the two vectors are respectively the directions of the radius vector *OR* of *O* at one of the surfaces $\pm F = 1$, and additionally a tangent *RS* at the point *R* of the relevant surface, then the vectors shall be called *normal* to each other. Accordingly

$$c^2tt_1 - xx_1 - yy_1 - zz_1 = 0,$$

which is the condition that the vectors with the components *x, y, z, t* and $(x_1\ y_1\ z_1\ t_1)$ are normal to each other.

For the *sums* of vectors in different directions, the *unit measuring rods* are to be fixed in the following manner; — a space-like vector from to $-F = 1$ is always to have the sum 1, and a time-like vector from *O* to $+F = 1, t > 0$ is always to have the sum $1/c$.

Let us now fix our attention upon the world-line of a substantial point running through the world-point *P(x, y, z, t)*; then as we follow the progress of the line, the quantity

$$d\tau = \frac{1}{c}\sqrt{c^2dt^2 - dx^2 - dy^2 - dz^2},$$

corresponds to the time-like vector-element *dx, dy, dz, dt*.

The integral $\tau = \int d\tau$ of this sum, taken over the world-line from any fixed initial point P_0 to any variable endpoint *P*, may be called the "proper-time" of the substantial point in *P*. Upon the world-line, we may regard *x, y, z, t, i.e.*, the components of the vector *OP*, as functions of the "proper-time" τ; let $\left(\dot{x},\ \dot{y},\ \dot{z},\ \dot{t}\right)$ denote their first differential-quotients with respect to τ, and $\left(\ddot{x},\ \ddot{y},\ \ddot{z},\ \ddot{t}\right)$ their second differential quotients with respect to τ, and denote the corresponding vectors, *i.e.* the derivation of the vector *OP* with respect to τ the *motion-vector in P*, and the derivation of this motion-vector with respect to τ the *acceleration-vector in P*. There we have

$$\left.\begin{array}{l} c^2\dot{t}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2 = c^2 \\ c^2\dot{t}\ddot{t} - \dot{x}\ddot{x} - \dot{y}\ddot{y} - \dot{z}\ddot{z} = 0 \end{array}\right\}$$

i.e., the motion-vector is the time-like vector in the direction of the world-line at *P* of sum 1, the acceleration-vector at *P* is normal to the motion-vector at *P*, and is in any case a space-like vector.

Now there is, as can be easily seen, a certain hyperbola, which has three infinitely contiguous points in common with the world-line at *P*, and of which the asymptotes are the generators of a fore-cone and an aft-cone. This hyperbola may be called the *hyperbola of curvature* at *P* (*vide* fig. 3). If *M* be the center of this hyperbola, then we have to deal here with an inter-hyperbola with center *M*. Let ϱ be the sum of the vector *MP*, *then we perceive that the acceleration-vector at P is a vector of magnitude* c^2/ϱ *in the direction of MP*.

If $\ddot{x},\ \ddot{y},\ \ddot{z},\ \ddot{t}$ are nil, then the hyperbola of curvature at *P* reduces to the straight line touching the world-line at *P*, and $\varrho = \infty$.

IV

In order to demonstrate that the assumption of the group G_c for the physical laws does not possibly lead to any contradiction, it is inevitable to undertake a revision of the whole of physics on the basis of the assumptions of this group. The revision has, to a certain extent, already been successfully made in the case of thermodynamics and tadiation,"[3] for electromagnetic phenomena",[4] and finally for Mechanics with the maintenance of the idea of mass.

For the latter area, the question may be asked: if there is a force with the components *X, Y, Z* (with respect to the space-axes) at a world-point *P(x, y, z, t)*, where the motion-vector is $\left(\dot{x},\ \dot{y},\ \dot{z},\ \dot{t}\right)$, then how are we to regard this force when the system of reference is changed in any possible manner? Now, certain well-tested theorems about the ponderomotive force in electromagnetic fields exist, where the group G_c is undoubtedly permissible. These theorems lead us to the following simple rule; *if the system of reference be changed, then the supposed force is to be put as a force in the new space-coordinates in such a manner, that the corresponding vector with the components*

$$\dot{t}X,\ \dot{t}Y,\ \dot{t}Z,\ \dot{t}T,$$

where

$$T = \frac{1}{c^2}\left(\frac{\dot{x}}{\dot{t}}X + \frac{\dot{y}}{\dot{t}}Y + \frac{\dot{z}}{\dot{t}}Z\right)$$

is the work of the force divided by c^2 *at the world-point, remains unaltered.* This vector is always normal to the motion-vector at *P*. Such a force-vector belonging to a force at *P*, may be called a *moving force-vector at P*.

Now the world-line passing through *P* will be described by a substantial point with the constant *mechanical mass m*. Let us call *m*-times the velocity-vector at *P* as the *momentum-vector*, and *m*-times the acceleration-vector at *P* as the *force-vector of motion at P*. According to these definitions, the law telling us how the motion of a point-mass takes place under a given moving force-vector[5]:

The force-vector of motion is equal to the moving force-vector.

This enunciation comprises four equations for the components in the four directions, of which the fourth can be deduced from the first three, because both of the above-mentioned vectors are perpendicular to the velocity-vector from the outset. From the above-mentioned definition of *T*, we see that the fourth certainly expresses the energy-law. Accordingly, c^2*-times the component of the momentum-vector in the direction of the t-axis* is to be defined as *the kinetic-energy* of the point-mass. The expression for this is

$$mc^2\frac{dt}{d\tau} = mc^2/\sqrt{1-\frac{v^2}{c^2}}$$

i.e., if we deduct from this the additive constant mc^2, we obtain the expression $\frac{1}{2}mv^2$ of NEWTONian-mechanics up to magnitudes of the order of $1/c^2$. Hence it illustratively appears that *the energy depends upon the system of reference*. But since the *t*-axis can be laid in the direction of any time-like axis, however, the energy-law formed for any possible system of reference, therefore comprises already the whole system of equations of motion. This fact retains its significance for the axiomatic construction of Newtonian mechanics, even in the considered passage to the limit $c = \infty$, as has already been recognized out by J. R. SCHÜTZ.[6]

From the very beginning, we can establish the ratio between the units of length and time in such a manner, that the natural limiting velocity becomes $c = 1$. If we now write $\sqrt{-1}t = s$, in the place of *t*, then the quadratic differential expression

$$d\tau^2 = -\left(dx^2 + dy^2 + dz^2 + ds^2\right),$$

becomes symmetrical in *x, y, z, s*; this symmetry then enters into each law, which does not contradict the world-postulate. We can clothe the essential nature of this postulate in the mystical, but mathematically significant formula

$$3\cdot 10^5 \text{ km} = \sqrt{-1} \text{ Sec.}$$

V.

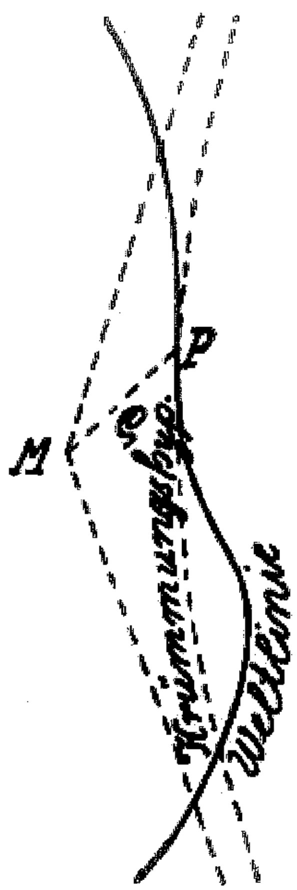

Fig. 3.

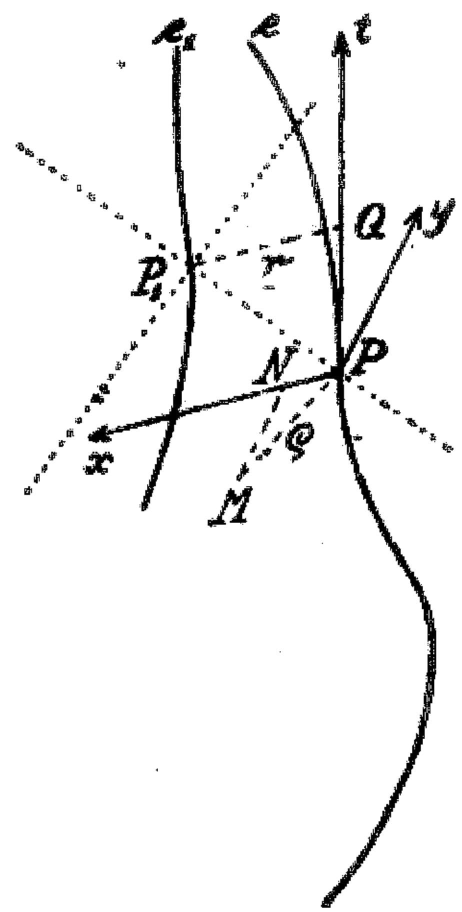

Fig. 4.

The advantages arising from the formulation of the world-postulate are illustrated by nothing so strikingly as by giving the expressions for the reactions exerted by a *point-charge moving in any manner* according to the MAXWELL-LORENTZ theory. Let us conceive the world-line of such a point-like electron with the charge *e*, and let us introduce upon it the proper-time τ reckoned from any initial point. In order to obtain the field caused by the electron at any world-point P_1, let us construct the fore-cone belonging to P_1 (*vide* fig. 4). Evidently this cuts the unlimited world-line of the electron at a single point *P*, because these directions are all time-like vectors. At *P*, let us draw the tangent to the world-line, and let us draw from P_1 the normal P_1Q to this tangent. Let *r* be the sum of P_1Q. According to the definition of a fore-cone, r/c is to be reckoned as the sum of *PQ*. Now at the world-point P_1, the vector with respect to *PQ* of magnitude e/r in its components along the *x*-, *y*-, *z*-axes, is represented by the vector-potential of the field multiplied by *c*; the component along the *t*-axis is represented by the scalar-potential of the field excited by *e*. This is the elementary law found out by A. LIÉNARD, and E. WIECHERT.[7]

In the description of the field caused by the electron itself, then it will appear that the division of the field into electric and magnetic forces is a relative one with respect to the time-axis assumed; the two forces considered together can most vividly be described by a certain analogy to the force-screw in mechanics; the analogy is, however, imperfect.

I shall now describe *the ponderomotive force which is exerted by a point-charge moving in an arbitrary way, to another point-charge moving in an arbitrary way.* Let us suppose that the world-line of a second point-electron passes through the world-point P_1. Let us determine *P*, *Q*, *r* as before, construct the middle-point of the hyperbola of curvature at *P*, and finally the normal *MN* from *M* upon a line through *P* which is parallel to QP_1. With *P* as the initial point, we shall establish a system of reference in the following way: the *t*-axis will be laid along *PQ*, the *x*-axis in the direction of QP_1; the *y*-axis in the direction of *MN*, then the direction of the *z*-axis is automatically determined as normal to the *t*-,*x*-,*y*--axes. Let $\ddot{x}, \ddot{y}, \ddot{z}, \ddot{t}$ be the acceleration-vector at P, $\dot{x}_1, \dot{y}_1, \dot{z}_1, \dot{t}_1$ be the motion-vector at P_1. Then the force-vector exerted by the first electron *e* (moving in any possible manner) upon the second electron e_1 (likewise moving in any possible manner) at P_1 is represented by

$$-ee_1\left(\dot{t}_1 - \frac{\dot{x}_1}{c}\right)\mathfrak{K}$$

For the components $\mathfrak{K}_x, \mathfrak{K}_y, \mathfrak{K}_z, \mathfrak{K}_t$ *of the vector* $\mathfrak{K}$ *the three relations hold:* —

$$c\mathfrak{K}_t - \mathfrak{K}_x = \frac{1}{r^2}, \quad \mathfrak{K}_y = \frac{\ddot{y}}{c^2 r}, \quad \mathfrak{K}_z = 0$$

and fourthly this vector $\mathfrak{K}$ *is normal to the motion-vector* P_1, *and through this circumstance alone, its dependence on this latter motion-vector arises.*

If we compare with this expression the previous formulations[8] giving the same elementary law about the ponderomotive action of moving electric point-charges upon each other, then we cannot but admit, that the relations which occur here only reveal the inner essence of full simplicity first in four dimensions; but upon a space of three dimensions that is forced upon them from the outset, they cast very complicated projections.

In the mechanics reformed according to the world-postulate, the disharmonies which have disturbed the relations between NEWTONian mechanics and modern electrodynamics automatically disappear. I still shall consider the position of the *NEWTONian law of attraction* to this postulate. I will assume that when two point-masses *m* and m_1 describe their world-lines, a moving force-vector is exerted by *m* upon m_1, and the expression is just the same as in the case of the electron previously discussed; we only have to write $+mm_1$ instead of $-ee_1$. We shall consider now the special case in which the acceleration-vector of *m* is constantly zero; then *t* may be introduced in such a manner that *m* may be regarded as fixed, the motion of m_1 is now subjected to the moving force-vector of *m* alone. If we now modify this given vector, first by writing $\dot{t}^{-1} = \sqrt{1 - \frac{v^2}{c^2}}$, which = 1 up to magnitudes of the order $1/c^2$, then it is shown[9] that KEPLER's laws hold good for the positions x_1, y_1, z_1 of m_1 at any time, except that in place of the times t_1 we have to write the proper times τ_1 of m_1. On the basis of this simple remark, it can be seen that the proposed law of attraction in combination with new mechanics is not less suited for the explanation of astronomical phenomena than the NEWTONian law of attraction in combination with NEWTONian mechanics.

Also the fundamental equations for electromagnetic processes in ponderable bodies are in accordance with the world-postulate throughout. I shall also show on a later occasion that even the deduction of these equations, as taught by LORENTZ on the basis of the concepts of electron theory, are by no means to be given up.

The fact that the world-postulate holds without exception is, I would like to believe, the true core of an electromagnetic picture of the world; this core was first hit to LORENTZ, it was further carved out by EINSTEIN, and finally it is fully brought to daylight. In course of further developing the mathematical consequences, enough suggestions will be forthcoming for the experimental verification of the postulate; in this way even those, who find it unsympathetic or even painful to give up the old, time-honoured concepts, will be reconciled by idea of a pre-established harmony between pure mathematics and physics.

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1. ↑ An essential application of this fact can already be found in W. VOIGT. Göttinger Nachr. 1887, p.41
2. ↑ A. EINSTEIN, Ann. d. Phys. 17, 1905, p. 891; Jahrb. d. Radioaktivität u. Elektronik 4, 1907, p. 411.
3. ↑ M. PLANCK, Zur Dynamik bewegter Systeme, Berliner Ber. 1907, p. 542 (also Ann. d. Phys. 26, 1908, p. 1).
4. ↑ H. MINKOWSKI, Die Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körpern, Göttinger Nachr. 1908, p. 53.
5. ↑ H. MINKOWSKI, l.c. p. 107. — see also M. PLANCK, Verh. d. Physik. Ges. 4, p. 136, 1906.
6. ↑ J. R. SCHÜTZ, Das Prinzip der absoluten Erhaltung der Energie. Göttinger Nachr. 1897, p. 110.
7. ↑ A. LIÉNARD, Champ électrique et magnétique produit par une charge concentrie en un point et animée d'un mouvement quelconque, L'Éclairage électrique 16 (1898), p. 5, 53, 106; WIECHERT, Elektrodynamische Elementargesetze, Arch. néerl. (2), 5 (1900), p. 549.
8. ↑ K. SCHWARZSCHILD, Göttinger Nachr. 1903, p. 132. — H. A. LORENTZ, Enzykl. d. math. Wissensch., Art. V, 14, p. 199.
9. ↑ H. MINKOWSKI, l.c., p. 110.

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