A Quick Introduction to Fourier Analysis

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1 Introduction

Welcome to the Fourier analysis notes! If you're reading this, you're probably either brand new to the subject and looking for an accessible introduction or interested in learning a little more than you already know. I have tried to include content which appeals to both groups of people. I have also tried to keep a reasonable limit on the size of the document.

This is not intended to be a comprehensive tome on the theory, but I have made an effort to keep it thorough. I believe that a close study of these notes will give you a more than solid foundation with the principles of the theory, and you will be well-prepared to use it in practical application after mastering the material. However, in order to keep a limit on the size, I had to omit the discussion of certain technical details, in particular working through a multitude of examples as most texts do. For instance, I never do the explicit calculation of a Fourier series, and I don't explain how to solve the whole zoo of differential equations with various boundary conditions.

There are two reasons for this. The first is that if you have a good grasp of what I have talked about, it should be well within your reach to work a great deal of this out yourself, and would be a good exercise to do so! The second is that there are already many places you can go to learn about this subject that talk about this material in great detail (as it is rather standard). I have linked a couple of places in the references section. While I have omitted some of this more standard stuff, I think the new material I included in some of the examples (and in section 5) is a nice complement to it and displays some unique and exciting applications of the ideas, in some cases to real problems in current-research modern physics (of course I am exaggerating; you need to know a lot more than Fourier analysis to really understand the depth of these problems, but it is surprising how far you can get with just Fourier theory).

I will briefly outline the structure of the notes and my philosophy of presentation. I have chosen to start with the discussion of a very useful tool, the Dirac delta function, then develop the Fourier transform, and finally the series. The reason for going in this order is because in this way we can develop the subject directly as an application of calculus (maybe with some more i's than some of you are used to). This will, I believe, make it more accessible and easier to learn for beginners. I work at the level of mathematical rigor acceptable to physicists. While the distinction between rigor in math and physics is usually negligible at the undergraduate level, there have been serious tomes written on the mathematical theory of Fourier analysis, and I want to be clear that this is not the place to go for those kind of niceties. If you're interested in L^p spaces and Lebesgue measures, this is definitely not the place for you.

As far as omissions, perhaps the biggest omission was a more complete discussion of the convergence issues of Fourier series. I left this out because at the practical level, it's not too big of a deal, and a proper discussion of it belongs to pure mathematics. I have left out some other interesting stuff related to the series, like the Gibbs phenomenon, but this is inessential. I also left out the discrete Fourier transform, so I sincerely apologize to all of the discrete math and lattice enthusiasts.

Finally, I have tried to show that the subject is really of great utility and can be used to understand a lot of interesting and exciting physics. So my final comment for the introduction is to just have fun!

2 The Dirac Delta Function

The heart of Fourier analysis is an object known as the Dirac delta function, denoted as $\delta(x)$. Informally, it is simply a function which is zero for $x \neq 0$ and infinite at $x = 0$, such that

$$\int_{-\infty}^{\infty} dx\delta(x) = 1$$

2.1 Definition and Properties

To define this more formally, consider the function $d_\epsilon(x)$ defined by the following integral:

$$d_\epsilon(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-\epsilon|k|+ikx} \tag{2.1.1}$$

Here, $i = \sqrt{-1}$. The integral is clearly convergent for all $\epsilon > 0$. Furthermore, it can be evaluated explicitly:

$$d_\epsilon(x) = \frac{1}{2\pi}\Big(\frac{1}{\epsilon - ix}\Big) + \frac{1}{2\pi}\Big(\frac{1}{\epsilon + ix}\Big)$$

$$d_\epsilon(x) = \frac{1}{\pi}\frac{\epsilon}{x^2 + \epsilon^2} \tag{2.1.2}$$

For ϵ small, we see that the function becomes sharply peaked at $x = 0$ with value $\frac{1}{\pi\epsilon}$, and rapidly drops to zero for $x \neq 0$. Furthermore,

$$\int_{-\infty}^{\infty} dx d_\epsilon(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} dx \frac{\epsilon}{x^2 + \epsilon^2} = \frac{1}{\pi}\int_{-\infty}^{\infty} dy \frac{1}{y^2 + 1} = 1 \tag{2.1.3}$$

In the second equality, we substitute $x = \epsilon y$. The final step is a simple arctan integral. We then see that the area under $d_\epsilon(x)$ is 1, independent of ϵ.

The Dirac delta function is defined as $\delta(x) = \lim_{\epsilon\to 0^+} d_\epsilon(x)$. Observe that $\delta(x)$ is even. By equations 2.1.1, 2.1.2, and 2.1.3, we see that $\delta(x)$ satisfies all of the properties we required out of it. It should be clear from the limit definition given that graphically, $\delta(x)$ is just a huge spike peaked at the origin. Then, we can see that

$$\int_{-\varepsilon}^{\varepsilon} dx\delta(x) = 1$$

for any positive number ε. The idea is that the part of the integral from ε to ∞ (and $-\varepsilon$ to $-\infty$) does not contribute, since $\delta(x)$ is zero in that entire region.

At this point, it will be of use to us to develop various properties of $\delta(x)$. First, consider the integral

$$\int_{-\infty}^{\infty} dx\delta(x - a)f(x)$$

for any real number a, and any nonsingular function $f(x)$. We see that

$$\int_{-\infty}^{\infty} dx\delta(x-a)f(x) = f(a)\int_{-\infty}^{\infty} dx\delta(x-a) = f(a) \tag{2.1.4}$$

In the first equality, we use the fact that $\delta(x-a) = 0$ for all $x \neq a$, so that we can replace $f(x)$ by $f(a)$ in the integral. But $f(a)$ is just a constant which pulls out of the integral over x, and the remaining integral is just 1 by the established properties of $\delta(x)$. So, the delta function makes all integrals incredibly easy! By the same argument used above, we see that

$$\int_{a-\varepsilon}^{a+\varepsilon} dx\delta(x-a)f(x) = f(a)$$

for any $\varepsilon > 0$.

Another property of interest of $\delta(x)$ is its behavior under scaling. That is, consider $\delta(bx)$ for $b \neq 0$. To see what is going on, simply use the definition:

$$\delta(bx) = \lim_{\epsilon\to 0^+} \frac{1}{\pi}\frac{\epsilon}{(bx)^2+\epsilon^2} = \lim_{\epsilon\to 0^+} \frac{1}{\pi}\frac{|b|\epsilon}{(bx)^2+b^2\epsilon^2} = \lim_{\epsilon\to 0^+} \frac{1}{\pi}\frac{|b|}{b^2}\frac{\epsilon}{x^2+\epsilon^2} = \frac{1}{|b|}\delta(x) \tag{2.1.5}$$

In the second equality, we recognize that the limit $\epsilon \to 0^+$ of ϵ is the same as $\epsilon \to 0^+$ of $|b|\epsilon$ for any fixed (finite) b. Observe that the absolute value is necessary to keep the limit coming from the right.

Now, consider $\delta(f(x))$, with $f(x)$ a smooth function with a set of isolated zeroes x_i such that $f'(x_i) \neq 0$. $\delta(x)$ will only be nonzero at x_i, and furthermore will only be sensitive to $f(x)$ in the immediate vicinity of x_i (for the purists, take an ϵ-neighborhood and make ϵ arbitrarily small). Thus, it is legal to replace $f(x)$ by its linear approximation about x_i inside $\delta(x)$, since $\delta(x)$ is insensitive to the errors due to the fact that it decays infinitely rapidly. Putting these words into equations, we find

$$\delta(f(x)) = \sum_i \delta(f'(x_i)(x-x_i))$$

But, using the scaling property, we then find that

$$\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|} \tag{2.1.6}$$

Our final property, which is the most fundamental equation in Fourier analysis, simply comes from taking $\lim_{\epsilon\to 0^+} d_\epsilon(x)$ directly from the integral definition. This gives

$$\delta(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tag{2.1.7}$$

At this point, it is instructive to pause to give a physical interpretation. The real and imaginary parts of e^{ikx} are simply waveforms with definite wavelength $\lambda = \frac{2\pi}{k}$ and unit amplitude. The integral over k then represents the superposition of such waveforms over all wavelengths. The above equation is then simply the statement that in a superposition of a large number of waves of equal amplitudes but differing wavelengths, the interference averages out to be destructive in such a manner that produces the characteristic delta function spike.

From here forward, I will use all established properties of $\delta(x)$ without alerting you to the fact that I am doing so, so it is well worth making sure you understand eqs. 2.1.3-7 thoroughly.

2.2 Alternative Definitions

$\delta(x)$ appears in many different guises. Here, we will give two additional commonly used representations.

The starting point for all definitions is a limit of equation 2.1.6.

The first is the most naive, namely to just define

$$d_K(x) = \int_{-\frac{K}{2}}^{\frac{K}{2}} \frac{dk}{2\pi} e^{ikx}$$

and then define $\delta(x) = \lim_{K\to\infty} d_K(x)$. This gives

$$\delta(x) = \lim_{K\to\infty} \frac{1}{\pi x} \sin \frac{Kx}{2} \tag{2.2.1}$$

This obviously shares the usual properties: sharply peaked at zero, integral is 1 independently of K, drops off rapidly. However, due to the trigonometric oscillation, this definition is not very convenient to work with. This is because the $K \to \infty$ limit is much more difficult to control than, by contrast, the $\epsilon \to 0^+$ limit of the previous section, precisely because it is oscillatory rather than damped.

By contrast, the most controlled limit definition is a Gaussian dampening. Define

$$\Delta_\epsilon(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-\frac{\epsilon}{2}k^2 + ikx}$$

This integral converges for $\epsilon > 0$, and its convergence properties are even better than our original d_ϵ (since Gaussians damp faster than exponentials). Obviously, $\delta(x) = \lim_{\epsilon\to 0^+} \Delta_\epsilon(x)$. We can explicitly carry out the Gaussian integral[1], giving the representation

$$\delta(x) = \lim_{\epsilon\to 0^+} \frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{x^2}{2\epsilon}} \tag{2.2.2}$$

Observe that it clearly satisfies all of the usual properties, by virtue of the properties of Gaussian integrals. This representation is quite useful when dealing with heat equations.

3 The Fourier Transform

We will begin with the proof of Fourier's theorem, and then comment on the assumptions. Then, we will discuss applications of the Fourier transform.

3.1 Fourier's Theorem

Now that we have established the essential properties of $\delta(x)$, defining the Fourier transform is quite simple. Take any suitably well-behaved function $f(x)$ defined on the domain

[1] If unfamiliar, check appendix now

$(-\infty, \infty)$. We will define what we mean by "suitably well-behaved" shortly. Define the function

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx e^{-ikx} f(x) \tag{3.1.1}$$

$\tilde{f}(k)$ is known as the *Fourier transform* of $f(x)$. So far, we haven't really done anything useful. All we have given is a definition. The key property is the following: consider the integral

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{f}(k) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \int_{-\infty}^{\infty} dy e^{-iky} f(y) = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-y)} f(y) \\ = \int_{-\infty}^{\infty} dy \delta(x-y) f(y) = f(x)\end{aligned} \tag{3.1.2}$$

What we have just established is that

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{f}(k)$$

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx e^{-ikx} f(x)$$

This result is *Fourier's theorem*, also known as the Fourier inversion theorem. $f(x)$ and $\tilde{f}(k)$ are known as a *Fourier transform pair*. This shows that the operation of Fourier transform is invertible, and in a sense squares to one, as the Fourier transform of the Fourier transform is the original function (up to a 2π). Observe that $\tilde{f}(k)$ will, in general, be a complex-valued function of a real variable. We see that the Fourier transform remains well-defined for complex-valued $f(x)$. If $f(x)$ is real, $(\tilde{f}(k))^* = \tilde{f}(-k)$. This follows immediately from the definition.

Now, we should comment on the assumptions of Fourier's theorem. I am of the school of thought that "too much rigor leads to rigor mortis", but here it pays to be careful. Since real-valued is a special case of complex-valued, we will take $f(x)$ to be complex valued. What we have assumed is that the Fourier transform integral exists, and that $f(x)$ is sufficiently well-behaved that we can exchange orders of integration so that the transform is invertible.

For $\tilde{f}(k)$ to be bounded, we need

$$|\tilde{f}(k)| = \left| \int_{-\infty}^{\infty} dx e^{-ikx} f(x) \right| \leq \int_{-\infty}^{\infty} dx |e^{-ikx} f(x)| = \int_{-\infty}^{\infty} dx |f(x)| < \infty$$

Therefore, if $|f(x)|$ is integrable, then $\tilde{f}(k)$ will exist. Obviously, for the inversion formula to make sense, we require that $|\tilde{f}(k)|$ must be integrable as well. An important result known as the *Riemann-Lebesgue lemma* ensures that $\tilde{f}(k) \to 0$ as $|k| \to \infty$ if $|f(x)|$ is integrable. I will not give a rigorous proof, but here is a "physicist proof". Let the function $\chi_{a.b}(x) = 1$ if $x \in [a, b]$ and 0 otherwise. It is essentially a step function over an interval. The Fourier transform of χ is

$$\frac{e^{ibk} - e^{ika}}{ik}$$

which obviously goes to zero as $|k| \to \infty$. The idea is that now, any function can be approximated arbitrarily well by writing

$$f_\epsilon(x) = \sum_i f(x_i)\chi_i(x)$$

for each $\chi_i(x)$ nonzero over some interval $[x_i, x_{i+1}]$ of length ϵ. Observe that this is precisely the same limit process used to define the Riemann integral. The point is that then, $\lim_{|k|\to\infty} \tilde{f}_\epsilon(k) = 0$ for all $\epsilon > 0$. Take $\epsilon \to 0$, and you have the result. Integrability of $f(x)$ is required because the limiting process used here is precisely the same as that of the Riemann integral, so the Riemann integral must make sense in order for the limit $\epsilon \to 0$ to be well-defined. Notice that this does not ensure that $\tilde{f}(k)$ is integrable, but for any reasonable f it will be.

The moral of the story is that the functions we are dealing with have to die at infinity fast enough and be sufficiently nonsingular for the Fourier transform to make sense. Observe that they do not even have to be continuous. In fact, they can have an arbitrarily large number of discontinuities, provided they are still integrable.

Finally, we comment on the physical interpretation. Going back to the wave picture, we see that the Fourier transform is simply a superposition, but now with a k-dependent amplitude. It is simply the statement that any arbitrary waveform can be written as a superposition of definite-wavelength waves, with suitably chosen amplitudes.

3.2 Some Common Fourier Transforms

Let us do some examples. For our first example, I will choose $f(x) = e^{-a|x|}$ for $a > 0$. The Fourier transform is given by

$$\tilde{f}(k) = \frac{2a}{k^2 + a^2}$$

Note that this is essentially the same computation that we did in defining the delta function. Observe also that if $a \to \infty$, rendering $f(x)$ sharply peaked, $\tilde{f}(k)$ disperses. This is a general property of the Fourier transform: sharp peaks in one representation are converted to wide spreads in the other.

The equation

$$\delta(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx}$$

simply states that the Fourier transform of $\delta(x)$ is 1. Observe the pattern again: 1 is essentially the most spread out you can get, but $\delta(x)$ is the most sharply peaked you can get. Observe also that 1 is not integrable. There is no contradiction here, because $\delta(x)$ is not bounded: it is singular at zero. This seems to conflict with our assumptions in proving Fourier's theorem, but the point is that the only reason the assumption is there is to ensure that the integrals converge. However, we have discussed the behavior of this integral extensively in the first section, and should be convinced that it is a sensible object. Making these considerations rigorous is rather subtle[2], but unimportant.

[2]For those who must know, in mathematical circles, it's called "distribution theory". Technically, due to its poor behavior at the origin, $\delta(x)$ is a distribution, not a function. This means that $\delta(x)$ really only makes sense when used under the integral sign, corresponding to the mathematical definition of distributions as linear functionals on the space of functions.

Using the fact that the Fourier transform of $\delta(x)$ is just 1, we can derive yet another representation of $\delta(x)$. Consider the integral

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{K^2}{k^2+K^2} e^{ikx}$$

We will take the limit $K \to \infty$. We already know that this will produce $\delta(x)$. However, we know from Fourier's theorem and the integral done above that $K^2/(k^2+K^2)$ is the Fourier transform of $\frac{K}{2}e^{-K|x|}$. This gives us

$$\delta(x) = \lim_{K\to\infty} \frac{K}{2} e^{-K|x|} \tag{3.2.1}$$

Another common Fourier transform is of the Gaussian $f(x) = e^{-\frac{a}{2}x^2}$. The integral is a Gaussian, yielding

$$\tilde{f}(k) = \sqrt{\frac{2\pi}{a}} e^{-\frac{k^2}{2a}}$$

Observe that this reflects the general spread/peak pattern discussed previously.

An instructive exercise is to see how Fourier inversion works explicitly. In the delta function case, it is obvious. In the Gaussian case, one has

$$f(x) = \int_{-\infty}^{\infty} dk \frac{1}{\sqrt{2\pi a}} e^{-\frac{k^2}{2a}+ikx} = \frac{1}{\sqrt{2\pi a}} \sqrt{2\pi a} e^{-\frac{a}{2}x^2} = e^{-\frac{a}{2}x^2}$$

Finally, we are brought to the exponential case. We need the integral

$$f(x) = \frac{a}{\pi} \int_{-\infty}^{\infty} dk \frac{e^{ikx}}{k^2+a^2} \tag{3.2.2}$$

The most elegant way to evaluate this is using contour integrals. However, since this may be unfamiliar to some readers, we opt for a different approach. Observe that we have, for any $A > 0$,

$$\frac{1}{A} = \int_0^{\infty} ds e^{-sA}$$

(if this is unfamiliar, check it by doing the s integral). Then, in the integral, we may write

$$f(x) = \frac{a}{\pi} \int_0^{\infty} ds \int_{-\infty}^{\infty} dk e^{-s(k^2+a^2)+ikx}$$

The interchange of order is legal because since $k^2 + a^2 > 0$ for all k, all integrals are convergent. But now, the integral over k is a simple Gaussian[3] yielding

$$f(x) = \frac{a}{\pi} \int_0^{\infty} ds \sqrt{\frac{\pi}{s}} e^{-\frac{x^2}{4s}-sa^2}$$

To make the integrand more symmetric, it is useful to rescale $s \to s/2$, giving

$$f(x) = \frac{a}{\sqrt{2\pi}} \int_0^{\infty} \frac{ds}{\sqrt{s}} \exp\left\{-\frac{1}{2}\left(\frac{x^2}{s}+sa^2\right)\right\}$$

[3] I have heard some people call this trick of introducing s "integrating under the integral sign". I think a better name is "using identities".

Making the substitution $t = \sqrt{s}$, one has

$$f(x) = a\sqrt{\frac{2}{\pi}} \int_0^\infty dt \exp\left\{ -\frac{1}{2}\left(\frac{x}{t}\right)^2 - \frac{1}{2}(at)^2 \right\}$$

Observe that this may be written as

$$f(x) = a\sqrt{\frac{2}{\pi}} \int_0^\infty dt \exp\left\{ -\frac{1}{2}\left(at - \frac{|x|}{t}\right)^2 - a|x| \right\} = a\sqrt{\frac{2}{\pi}} e^{-a|x|} I(x)$$

We have used the absolute value to keep it manifest that $f(x)$ is even, and we have defined the function

$$I(x) = \int_0^\infty dt \exp\left\{ -\frac{1}{2}\left(at - \frac{|x|}{t}\right)^2 \right\} \tag{3.2.3}$$

To evaluate this the trick is to exploit the symmetry between t and $1/t$. First observe that rescaling $t \to t/a$ (which flips no signs since $a > 0$), we have

$$I(x) = \int_0^\infty \frac{dt}{a} \exp\left\{ -\frac{1}{2}\left(t - \frac{a|x|}{t}\right)^2 \right\}$$

Now, substitute $t = a|x|/u$, under which

$$I(x) = \int_0^\infty \frac{du}{a}\frac{a|x|}{u^2} \exp\left\{ -\frac{1}{2}\left(u - \frac{a|x|}{u}\right)^2 \right\} = \int_0^\infty \frac{dt}{a}\frac{a|x|}{t^2} \exp\left\{ -\frac{1}{2}\left(t - \frac{a|x|}{t}\right)^2 \right\} \tag{3.2.4}$$

In the last step, I have simply relabeled the integration variable $u \to t$. Now, add 3.2.3 and 3.2.4 together to obtain

$$2I(x) = \int_0^\infty \frac{dt}{a}\left(1 + \frac{a|x|}{t^2}\right) \exp\left\{ -\frac{1}{2}\left(t - \frac{a|x|}{t}\right)^2 \right\}$$

Now, observe that we may simply substitute $v = t - \frac{a|x|}{t}$, so that

$$2I(x) = \int_{-\infty}^\infty \frac{dv}{a} \exp\left\{ -\frac{1}{2}v^2 \right\} = \frac{\sqrt{2\pi}}{a}$$

Leaving

$$I(x) = \frac{1}{a}\sqrt{\frac{\pi}{2}}$$

Amusingly, $I(x)$ does not depend on x at all. Substituting into our above formula, we find

$$f(x) = e^{-a|x|}$$

as expected.

3.3 Properties of the Fourier Transform

The most important property of the Fourier transform is that it is linear in f. This makes it very useful in solving linear differential equations, both ordinary and partial, as we will see in our examples.

A rather obvious property of the transform is that a phase factor in x space, namely $f(x) \to e^{-ipx} f(x)$ leads to a translation in k space, $\tilde{f}(k) \to \tilde{f}(k+p)$, since it follows directly from the definition. Here, I have introduced some useful jargon. We refer to manipulations of $f(x)$ as "working in x space", and manipulations with $\tilde{f}(k)$ as "working in k space". This terminology is particularly illuminating in quantum mechanics, but for us it's just convenient nomenclature.

The most useful aspect of the Fourier transform for problem solving is how it behaves under differentiation and multiplication. Consider the Fourier transform of $\frac{df}{dx}$. Here, we will assume all functions vanish rapidly at infinity (as they must for the transform to even make sense). We clearly have

$$\int_{-\infty}^{\infty} dx e^{-ikx} \frac{df}{dx} = -\int_{-\infty}^{\infty} dx \frac{d}{dx}(e^{-ikx}) f(x) = \int_{-\infty}^{\infty} dx e^{-ikx} ik f(x) = ik\tilde{f}(k)$$

Obviously, this extends to multiple derivatives by repeatedly integrating by parts, so the rule is that under Fourier transform,

$$\frac{d^n}{dx^n} \to (ik)^n \tag{3.3.1}$$

This is obviously invaluable in solving differential equations, because a differential equation in x-space becomes an algebraic equation in k-space.

Now, consider the Fourier transform of $xf(x)$. One has

$$\int_{-\infty}^{\infty} dx e^{-ikx} x f(x) = i\frac{d}{dk} \int_{-\infty}^{\infty} dx e^{-ikx} f(x) = i\frac{d}{dk}\tilde{f}(k)$$

This obviously generalizes to multiplication by x^n by repeatedly differentiating, which yields

$$x^n \to \left(i\frac{d}{dk}\right)^n \tag{3.3.2}$$

Note that this relation is "conjugate" to the one involving $\frac{d}{dx}$.

A further useful property of the Fourier transform is how it behaves under convolution. Define the convolution product

$$(f * g)(x) = \int_{-\infty}^{\infty} dy f(y) g(x-y) \tag{3.3.3}$$

For simplicity, we take f and g to be real-valued functions. To see how the convolution behaves under Fourier transformation, simply substitute the Fourier transforms of f and g:

$$\begin{aligned}(f * g)(x) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dy \frac{dk}{2\pi}\frac{dp}{2\pi}\tilde{f}(k)\tilde{g}(p) e^{iky} e^{ip(x-y)} = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\tilde{f}(k)\int_{-\infty}^{\infty}\frac{dp}{2\pi}\tilde{g}(p)e^{ipx}\int_{-\infty}^{\infty} dy e^{i(k-p)y} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk}{2\pi} dp \tilde{f}(k)\tilde{g}(p)\delta(k-p)e^{ipx} = \int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ikx}\tilde{f}(k)\tilde{g}(k)\end{aligned} \tag{3.3.4}$$

From this we can read off that the Fourier transform of $(f * g)(x)$ is simply $\tilde{f}(k)\tilde{g}(k)$. The convolution product in x space, which is obviously a nonlocal operation (it requires summing contributions over the entire real line), becomes a local pointwise product in k-space! This is known as the *convolution theorem*. I cannot help but comment that in quantum field theory, the convolution theorem implies conservation of momentum.

3.4 Examples

Finally, after all of these abstract and general considerations, we apply the Fourier transform to the solution of some equations.

3.4.1 Green's Function for ODE

Suppose we are asked to solve the equation

$$\Big(-\frac{d^2}{dt^2}+\Omega^2\Big)G(t-t')=\delta(t-t') \tag{3.4.1}$$

for some constant Ω. $G(t-t')$ is known as a *Green's function.* After solving the equation, we will see an application of G. Since we are Fourier transforming, we are interested in G in the domain $t-t'\in(-\infty,\infty)$. To solve this equation, start by Fourier transforming both sides. Define

$$\tilde{G}(\omega)=\int_{-\infty}^{\infty}dte^{-i\omega t}G(t)$$

Note that I have changed notation from (x,k) to (t,ω). When dealing with time variables, it is conventional in physics to call the Fourier conjugate variable ω. Using our properties of the Fourier transform, and the known transforms for G and δ, we find

$$(\omega^2+\Omega^2)\tilde{G}(\omega)=1$$

But this is just an algebraic equation! We can immediately solve for $\tilde{G}(\omega)$:

$$\tilde{G}(\omega)=\frac{1}{\omega^2+\Omega^2}$$

Now, using Fourier's theorem to invert this, we find that

$$G(t-t')=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{e^{i\omega(t-t')}}{\omega^2+\Omega^2}$$

However, we have already done this integral in our "common Fourier transforms" section, so we immediately find

$$G(t-t')=\frac{1}{2\Omega}e^{-\Omega|t-t'|} \tag{3.4.2}$$

Observe that this satisfies the original differential equation! We see that it is not differentiable at $t=t'$, reflecting the singular nature of the delta function.

What is the use of the Green's function? Suppose we are asked to solve the equation

$$\Big(-\frac{d^2}{dt^2}+\Omega^2\Big)x(t)=J(t) \tag{3.4.3}$$

for arbitrary $J(t)$, over the domain $t\in(-\infty,\infty)$. This is a linear inhomogeneous differential equation. We can write down the most general solution as

$$x(t)=x_0(t)+x_p(t)$$

where $x_0(t)$ is a solution to the homogeneous ($J = 0$) equation such that x satisfies the right boundary conditions. For arbitrary J, it may seem like a very tricky task to find $x_p(t)$. However, observe that the function

$$x_p(t) = \int_{-\infty}^{\infty} dt' G(t - t')J(t')$$

satisfies the equation, by virtue of the differential equation for G! We can then immediately write down the most general solution to our problem:

$$x(t) = Ae^{\Omega t} + Be^{-\Omega t} + \frac{1}{2\Omega}\int_{-\infty}^{\infty} dt' e^{-\Omega|t-t'|}J(t') \tag{3.4.4}$$

The interpretation of this is that the Green's function G is the response of the system described by the differential equation to a momentary impulse (the delta function spike). The idea is then the same one we have been using for all of this: the response to an arbitrary profile $J(t)$ can be obtained as the superposition of the responses of impulses, appropriately weighted. Note that the Green's function is linear in J. This is because what we have done here is basically the same as inverting a matrix to solve a matrix equation. The Green's function acts as the inverse of the differential operator (compare $M^{-1}M = 1$ for matrix M to eq. 3.4.1).

Note that the method easily generalizes to more complicated differential operators. The only difference is that the ω integral will be harder. Furthermore, it is important that the differential operators only have constant coefficients, since variable coefficients render the Fourier transform less friendly (you will no longer get an algebraic equation). However, it still may be useful, provided the variable coefficients are not too complicated.

The final caveat is that the Fourier transform methods used here to solve differential equations make the implicit assumption that the solution to the equation will have the properties required for the Fourier transform and its inverse to exist. Namely, the solution must be defined on all of $(-\infty, \infty)$, and furthermore die off rapidly at infinity. In most physics problems, this is not an issue, since we assume all sources to be localized, meaning they cannot reach infinity. The only problem is if the equations are only defined on a finite interval with some boundary conditions (which is an often-studied scenario in introductory courses on differential equations). In this case, the Fourier transform does not apply, since the functions are not defined on all of the real line. Instead, one must use the Fourier series, about which more later.

3.4.2 The Airy Function

The Airy equation is given by

$$\frac{d^2y}{dx^2} = xy \tag{3.4.5}$$

There is no closed-form solution to this equation, and it is quite challenging to solve by conventional methods. We will use the Fourier transform to solve it, obtaining an integral representation for the solution.

Fourier transforming $y(x)$ to $\tilde{y}(k)$ and using our properties, we have

$$-k^2\tilde{y}(k) = i\tilde{y}'(k)$$

Observe that the variable coefficient x has become a derivative: we see that variable-coefficient differential equations are no longer algebraic in k space. However, we have converted the second order differential equation to a first order one, which is easily solved. The equation is

$$\tilde{y}'(k) = ik^2\tilde{y}(k)$$

The solution is

$$\tilde{y}(k) = e^{i\frac{k^3}{3}}$$

We have dropped the integration constant, because it is irrelevant for our purposes here.

Using Fourier's theorem, we find that the solution to the original differential equation is given by the integral

$$y(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp\left\{ i\left(\frac{k^3}{3} + kx\right)\right\} \tag{3.4.6}$$

The convergence of the integral is somewhat subtle, but it has to do with the fact that rapid oscillations tend to cancel. A more careful treatment involves complex analysis. However, here we will just take it to converge. We can make the integral representation manifestly real by noting

$$y(x) = \int_{-\infty}^{0} \frac{dk}{2\pi} \exp\left\{ i\left(\frac{k^3}{3} + kx\right)\right\} + \int_{0}^{\infty} \frac{dk}{2\pi} \exp\left\{ i\left(\frac{k^3}{3} + kx\right)\right\}$$

Sending $k \to -k$ in the first integral, we find

$$y(x) = \int_{0}^{\infty} \frac{dk}{2\pi} \exp\left\{ -i\left(\frac{k^3}{3} + kx\right)\right\} + \int_{0}^{\infty} \frac{dk}{2\pi} \exp\left\{ i\left(\frac{k^3}{3} + kx\right)\right\}$$

so that

$$y(x) = \mathrm{Ai}(x) = \frac{1}{\pi}\int_{0}^{\infty} dt \cos\left(\frac{t^3}{3} + tx\right) \tag{3.4.7}$$

This is the integral definition of a special function known as the *Airy function*. We have simply relabeled the integration variable k to t. It is not obvious at all that the integral describing $\mathrm{Ai}(x)$ solves the original differential equation, in the sense that one would not be able to easily guess a solution of that form. It is straightforward to verify that the integral does solve the equation.

While it may seem not very useful to have a solution to a differential equation in terms of an integral which can't be done in closed form, what is useful about the integral definition is that a great deal more can be understood about the properties of the solution than simply by staring at the equation. In particular, using some methods in integral asymptotics, one can easily find the behavior of $\mathrm{Ai}(x)$ as $x \to \infty$, which cannot easily be obtained from staring at the equation[4]. Discussing this is in detail is outside of our scope. Furthermore, the integral definition is quite useful for discussing analytic continuation of the Airy function to complex values of x. This information is obviously very difficult to obtain from the original Airy equation. There are many beautiful results in asymptotic analysis linked to the asymptotics of the analytically continued Airy function. Remarkably enough, current research in quantum field theory actually makes direct contact with

[4]Actually, the asymptotics for the leading behavior can be obtained from the equation by making a very clever ansatz. Getting the higher order terms becomes very computationally involved, but the integral makes it very easy.

these classical results, because the integral describing $\mathrm{Ai}(x)$ is a zero-dimensional toy model for what is known as a Chern-Simons gauge theory, a topological field theory related to knot invariants.

A final detail which some readers may have questioned is that we only encountered one integration constant in our solution to this equation. However, the Airy equation is second order, so there should be a 2-parameter family of solutions! In other words, there should be two different integration constants. The key lies in the assumptions: the Fourier transform only exists for $y(x)$ which die off at infinity. There is a second linearly independent solution to the Airy equation, known as $\mathrm{Bi}(x)$, but it grows at infinity, and therefore is invisible to Fourier transform methods.

3.4.3 The Heat Equation

In this section, we will apply the Fourier transform to a partial differential equation known as the heat equation. Take a function $u(t, x)$. It is required to obey

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \tag{3.4.8}$$

Suppose we are interested in finding the solution $u(t, x)$ which dies at infinity and satisfies the initial condition $u(t = 0, x) = f(x)$. We will use Fourier analysis to solve this problem.

First, consider a slightly different problem. Let us solve the heat equation with the initial condition $u(t = 0, x) = \delta(x-y)$ for some point y. We will see the utility shortly (it works similarly to a Green's function). First of all, at fixed t, we can simply Fourier transform the x-dependence of $u(t, x)$. This is

$$u(t, x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{u}(t, k)$$

Applying this to the heat equation, we find that

$$\frac{\partial \tilde{u}}{\partial t} = -\frac{k^2}{2}\tilde{u}(t, k)$$

This can easily be integrated at fixed k to find

$$\tilde{u}(t, k) = A(k) \exp\left\{-\frac{k^2}{2}t\right\}$$

Now, observe that the condition $u(0, x) = \delta(x-y)$ translates in Fourier space to $\tilde{u}(0, k) = e^{-iky}$, since this produces the integral definition of $\delta(x-y)$ from the Fourier transform. From our solution, we find

$$\tilde{u}(0, k) = A(k) = e^{-iky}$$

Therefore, the Fourier transform of the solution to the heat equation is

$$\tilde{u}(t, k) = \exp\left\{-\frac{k^2}{2}t - iky\right\}$$

Using Fourier's theorem, we find

$$u(t,x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp\left\{-\frac{k^2}{2}t + ik(x-y)\right\} \tag{3.4.9}$$

But this is a simple Gaussian integral in k, so we find

$$u(t,x) = \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{(x-y)^2}{2t}\right\} \tag{3.4.10}$$

This function is known as the *heat kernel* for the heat equation, and we will change notation and call it $K(x-y;t)$. Observe that our Gaussian representation of the delta function ensures that

$$\lim_{t\to 0} K(x-y;t) = \lim_{t\to 0} \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{(x-y)^2}{2t}\right\} = \delta(x-y)$$

Now, we may use this to solve our original problem. Suppose we want the solution $u(t,x)$ with initial condition $u(0,x) = f(x)$. Consider the integral

$$\int_{-\infty}^{\infty} dy K(x-y;t) f(y) \tag{3.4.11}$$

Due to the behavior of K as $t \to 0$, we find that at $t = 0$, 3.4.11 evaluates to $f(x)$. However, we know that since K solves the heat equation as a function of x, the integral must also, by linearity (check this). Therefore, the solution $u(t,x)$ satisfying $u(t=0,x) = f(x)$ is simply

$$u(t,x) = \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi t}} \exp\left\{-\frac{(x-y)^2}{2t}\right\} f(y)$$

Observe that this is simply the convolution of $f(x)$ with a Gaussian that has a t-dependent dispersion.

The heat equation has many applications beyond the flow of heat. For instance, in the theory of Brownian motion, the heat equation governs the time-dependent probability distribution for the position of the particle. However, the nontrivial issue here is not solving the heat equation, but showing that the probability distribution obeys the heat equation (in these contexts, it's called the Fokker-Planck equation for the distribution).

3.4.4 The Wave Equation

Consider a function $\psi(t,x)$ which satisfies

$$\frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial x^2} = 0 \tag{3.4.12}$$

This is a linear, homogeneous, second order partial differential equation known as the wave equation. We can write down its most general solution using Fourier transforms. Just as with the heat equation, Fourier transform in x:

$$\psi(t,x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{\psi}(t,k)$$

In Fourier space, the equation becomes

$$\frac{\partial^2 \tilde{\psi}}{\partial t^2} + k^2 \tilde{\psi}(t,k) = 0$$

This is the equation for a simple harmonic oscillator, and the general solution is

$$\tilde{\psi}(t,k) = A(k)e^{-ikt} + B(k)e^{ikt}$$

Notice that the coefficients A and B can depend on k. Using Fourier's theorem, we have

$$\psi(t,x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-t)} A(k) + \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x+t)} B(k)$$

But, these are just the Fourier transforms for arbitrary functions

$$f(x-t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-t)} A(k)$$

$$g(x+t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x+t)} B(k)$$

Although the Fourier transform requires that these die at infinity, we see that

$$\psi(t,x) = f(x-t) + g(x+t) \tag{3.4.13}$$

satisfies the wave equation for any twice differentiable f and g. This is the most general solution to the wave equation.

4 The Fourier Series

The Fourier series is the Fourier transform applied to periodic functions. The essential difference is that the spectrum of wave numbers k being superposed goes from a continuum to a discrete set. The integral then becomes a sum, and the resulting sum is called a Fourier series. It states that any waveform on an interval can be obtained from a superposition of interfering sines and cosines (or, equivalently, complex exponentials).

4.1 Derivation

The Fourier series can be obtained directly from Fourier's theorem, by simply taking f to be periodic. Obviously, we are sidestepping some of the issues related to integrability and the existence of the transform, but we will see that this is unimportant.

Consider the Fourier transform integral:

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx e^{-ikx} f(x)$$

We now suppose that f is periodic with period L: $f(x+L) = f(x)$. We then break up the integral as follows:

$$\tilde{f}(k) = \Big(... + \int_{-2L}^{-L} dx e^{-ikx} f(x) + \int_{-L}^{0} dx e^{-ikx} f(x) + \int_{0}^{L} e^{-ikx} f(x) + \int_{L}^{2L} dx e^{-ikx} f(x) + ... \Big)$$

Now, we will shift the integration variables independently in each integral. First, consider the integral from nL to $(n+1)L$. Shift $x \to x - nL$. We find

$$\int_{nL}^{(n+1)L} dx e^{-ikx} f(x) = \int_{0}^{L} dx e^{-ik(x+nL)} f(x+nL) = e^{-iknL} \int_{0}^{L} dx e^{-ikx} f(x)$$

where we have used the periodicity of f. Since we have

$$\tilde{f}(k) = \sum_{n=-\infty}^{\infty} \int_{nL}^{(n+1)L} dx e^{-ikx} f(x)$$

This leaves us with

$$\tilde{f}(k) = \sum_{n=-\infty}^{\infty} e^{iknL} \int_{0}^{L} dx e^{-ikx} f(x)$$

I have flipped $n \to -n$ for convenience. Observe that this simply reverses the order of the terms in the sum. This means that to understand the transform, we must understand the sum[5]

$$\sum_{n=-\infty}^{\infty} e^{iknL} \tag{4.1.1}$$

We can break this up as

$$1 + \sum_{n=1}^{\infty} e^{iknL} + \sum_{n=1}^{\infty} e^{-iknL}$$

The sums are wildly oscillatory, and their convergence depends on the mutual cancellation of the phase factors (similarly to the delta function, they interfere destructively on average). To improve convergence (just as with our original delta function integral) we add a factor r^n to each term, for some $r < 1$ (in our delta function integral, the role of r was played by $e^{-\epsilon|k|}$, which is indeed always less than 1). The sum is then a simple convergent geometric series, with result

$$1 + \frac{re^{ikL}}{1 - re^{ikL}} + \frac{re^{-ikL}}{1 - re^{-ikL}}$$

First, we combine denominators as

$$1 + \frac{2r\cos(kL) - 2r^2}{1 - 2r\cos(kL) + r^2} = \frac{1 - r^2}{1 - 2r\cos(kL) + r^2}$$

Now, observe that if $k \neq \frac{2\pi n}{L}$ for some integer n, then the sum is equal to zero in the $r \to 1$ limit. However, if $k = \frac{2\pi n}{L}$, then the denominator becomes $(1-r)^2$ and the limit is infinite. Walks like a delta function, quacks like a delta function! But we must show

[5] In the mathematical terminology, sums of this type (both finite and infinite) are called Dirichlet kernels.

this explicitly. Let us examine the nature of this divergence by writing $k = \frac{2\pi n}{L} + K$ and taking K small. The cosine behaves as

$$\cos(2\pi n + KL) = \cos(2\pi n)\cos(KL) - \sin(2\pi n)\sin(KL) = \cos(KL) \approx 1 - \frac{K^2L^2}{2}$$

In the last line, we have used the first two terms of the Taylor series for small K. We then find that the sum behaves as

$$\frac{1-r^2}{1-2r+rK^2L^2+r^2}$$

as $K \to 0$. Now, write $r = 1-\epsilon$, with $\epsilon \to 0^+$. This is equivalent to the $r \to 1$ limit. Assuming K to be of order ϵ, and expanding to order ϵ^2, we find

$$\frac{2\epsilon - \epsilon^2}{1-2+2\epsilon+K^2L^2+1-2\epsilon+\epsilon^2} = \frac{2\epsilon}{K^2L^2+\epsilon^2} - \frac{\epsilon^2}{K^2L^2+\epsilon^2} = \frac{2\epsilon}{K^2L^2+\epsilon^2} + \mathcal{O}(1) \quad (4.1.2)$$

We recognize the ϵ^2 term, which is finite as $K \to 0$, as subleading to the term kept in the limit $\epsilon, K \to 0$. We then recognize the remaining term as the definition of the delta function (times 2π), so we are left with

$$2\pi\delta(KL) = \frac{2\pi}{L}\delta(K) = \frac{2\pi}{L}\delta\Big(k - \frac{2\pi n}{L}\Big)$$

We have used the fact that L is positive without loss of generality for periodic functions.

The meaning of what we have shown is that the sum (eq. 4.1.1) is zero when $k \neq \frac{2\pi n}{L}$, and has a delta function spike when $k = \frac{2\pi n}{L}$. This means we may write

$$\sum_{n=-\infty}^{\infty} e^{iknL} = \frac{2\pi}{L}\sum_{n=-\infty}^{\infty}\delta\Big(k - \frac{2\pi n}{L}\Big) \quad (4.1.3)$$

We are allowed to superpose the delta functions since they are never coincident: only one term at a time in the series on the right hand side will be nonzero. If you are struggling to see this, it helps to think of what the graph of the right hand side looks like as a function of k (it is called the "Dirac comb", because it looks like a hair comb).

Now, we see that $\tilde{f}(k)$ is given by

$$\tilde{f}(k) = \frac{2\pi}{L}\sum_{n=-\infty}^{\infty}\delta\Big(k - \frac{2\pi n}{L}\Big)\int_0^L dx e^{-ikx} f(x) = 2\pi\sum_{n=-\infty}^{\infty}\delta\Big(k - \frac{2\pi n}{L}\Big)\frac{1}{L}\int_0^L dx e^{-2\pi i n x/L} f(x)$$

In the last equality, we have used the fact that in each term, e^{-ikx} is multiplied by a delta function in k, so we may set it equal to the value at which the delta function spikes. We now define

$$c_n = \frac{1}{L}\int_0^L dx e^{-2\pi i n x/L} f(x) \quad (4.1.4)$$

so that

$$\tilde{f}(k) = 2\pi\sum_{n=-\infty}^{\infty} c_n \delta\Big(k - \frac{2\pi n}{L}\Big)$$

Now, we use Fourier's theorem.

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{f}(k) = \sum_{n=-\infty}^{\infty} c_n \int_{-\infty}^{\infty} dk e^{ikx} \delta\Big(k - \frac{2\pi n}{L}\Big)$$

Using the delta function to do the integral, we find

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x/L} \tag{4.1.5}$$

with c_n given by eq. 4.1.4. This result is known as the *Fourier series.* While we sidestepped the issue of convergence, we observe that the only difficulties we encountered were delta function singularities, which are able to be treated, as should be familiar by now.

The physical interpretation of this is that an arbitrary waveform defined on the interval $[0, L]$ can be written as a superposition of waves with wavenumbers which are integer multiples of $\frac{2\pi n}{L}$.

4.2 Properties of Fourier Series

The derivation gave above holds for arbitrary complex-valued periodic functions $f(x)$. If $f(x)$ is real (equal to its own complex conjugate), application of this condition to the Fourier series gives

$$(c_n)^* = c_{-n}$$

In this case, it is useful to break up c_n into its real and imaginary parts:

$$c_n = \frac{a_n - ib_n}{2}$$

The factor of 2 and sign are conventional. We can write the Fourier expansion as

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{2\pi i n x/L} + \sum_{n=1}^{\infty} c_{-n} e^{-2\pi i n x/L}$$

This comes from breaking up the sum from $-\infty$ to ∞ as a sum from $-\infty$ to -1, the zero term, and 1 to ∞. In the last term, we have simply flipped $n \to -n$ in the $-\infty$ to -1 sum to make it another sum from 1 to ∞. This is a very useful trick and I will use it repeatedly. We can recognize the second and third terms as follows:

$$f(x) = c_0 + 2\,\mathrm{Re}\left\{ \sum_{n=1}^{\infty} c_n e^{2\pi i n x/L} \right\}$$

We then need

$$2\,\mathrm{Re}\left\{ \sum_{n=1}^{\infty} c_n e^{2\pi i n x/L} \right\} = \mathrm{Re}\left\{ \sum_{n=1}^{\infty} (a_n - ib_n)\Big(\cos\frac{2\pi n x}{L} + i \sin\frac{2\pi n x}{L} \Big) \right\}$$

This gives

$$f(x) = c_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{L}\right) + b_n \sin\left(\frac{2\pi n x}{L}\right) \tag{4.2.1}$$

By taking the real and imaginary part of eq. 4.1.4, we find

$$a_n = \frac{2}{L}\int_0^L dx f(x) \cos\left(\frac{2\pi n x}{L}\right)$$

$$b_n = \frac{2}{L}\int_0^L dx f(x) \sin\left(\frac{2\pi n x}{L}\right)$$

Furthermore, we see that c_0 can be written as $\frac{a_0}{2}$, so we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{L}\right) + b_n \sin\left(\frac{2\pi n x}{L}\right) \tag{4.2.2}$$

This is the Fourier sine and cosine series.

A key property of the Fourier series is that it uses *orthogonality relations* involving functions. To describe these conveniently, we introduce the symbol δ_{mn}, for two integers m and n. We define $\delta_{mn} = 1$ if $m = n$ and 0 otherwise. δ_{mn} is known as the Kronecker delta symbol, and in a sense is the discrete version of the Dirac delta. We see that

$$\frac{1}{L}\int_0^L dx e^{2\pi i (m-n)x/L} = \delta_{mn} \tag{4.2.3}$$

The easiest way to establish this is by directly computing the integral. When $m = n$, we are just integrating 1 and the result is 1. When $m \neq n$, periodicity causes the terms to cancel and we get 0.

By using the above relations for the complex exponentials, one can easily obtain the following orthogonality relations for sines and cosines:

$$\frac{2}{L}\int_0^L dx \cos\left(\frac{2\pi m x}{L}\right) \cos\left(\frac{2\pi n x}{L}\right) = \delta_{mn}$$

$$\frac{2}{L}\int_0^L dx \sin\left(\frac{2\pi m x}{L}\right) \sin\left(\frac{2\pi n x}{L}\right) = \delta_{mn}$$

$$\int_0^L dx \cos\left(\frac{2\pi m x}{L}\right) \sin\left(\frac{2\pi n x}{L}\right) = 0$$

Observe that the equation

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-y)} = \delta(x-y)$$

is the "continuum limit" of the orthogonality relations.

The resulting expressions for c_n, a_n and b_n can be derived by means of the orthogonality relations.

Further consequences can be derived if f is an even or an odd function. If $f(x)$ is even, then substituting $f(-x) = f(x)$ into the Fourier series gives

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nx}{L}\right) = 0$$

Multiplying by $\sin\frac{2\pi mx}{L}$ and integrating from 0 to L, we find

$$b_m = 0$$

for all m. This implies that, for $f(x)$ even, periodicity L, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nx}{L}\right)$$

Similar considerations show that for $f(x)$ odd, periodicity L one has

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nx}{L}\right)$$

This has the following consequence. Suppose $f(x)$ is defined on $[0, L]$ and satisfies $f(0) = f(L) = 0$. This is obviously a special case of periodic. We can extend it to an odd function on $[-L, L]$ as follows. For $x < 0$, define

$$f(x) = -f(x + L)$$

This defines an odd function with period $2L$. Therefore we can write

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi nx}{L}\right) \tag{4.2.4}$$

This is a *Fourier sine series.* It holds for any function satisfying $f(0) = f(L) = 0$.

4.3 Convergence of Fourier Series (Some Linear Algebra Required)

Just as with the existence of the transform, the convergence of the Fourier series is an issue which requires more care. To answer it in complete generality is a very challenging subject in pure mathematics; here, we will give a simple criterion which works in all cases relevant to practical application.

First, note that the orthogonality relations for the exponential functions $e^{2\pi ikx/L}$ match the orthogonality relations of basis vectors $\vec{e}_k$ in $\mathbb{R}^n$. From a course on linear algebra, mathematical methods, or quantum mechanics, you should be familiar with the idea that the set of all periodic functions on an interval of length L forms a vector space under addition and multiplication by constants. The additional structure relevant to the Fourier series is the introduction of an inner product on this vector space. One can verify that the map $\langle , \rangle : V \times V \to \mathbb{C}$ given by

$$\langle f, g \rangle = \int_0^L dx f^*(x) g(x)$$

satisfies the required axioms of an inner product. Obviously, for a function to make sense in this vector space, the norm

$$\langle f, f \rangle = \int_0^L dx |f(x)|^2$$

must exist. Now, observe that the linear operator

$$-\frac{d^2}{dx^2}$$

is self-adjoint with respect to this inner product. From familiar linear algebra, we should expect that its eigenvectors constitute a complete, orthogonal basis for V[6]. What are the eigenvectors? They are simply the solutions to

$$-\frac{d^2 f}{dx^2} = \lambda^2 f$$

for some complex λ. This is, of course, the easiest differential equation to solve: The solution is a linear combination of $e^{i\lambda x}$ and $e^{-i\lambda x}$. For L-periodicity, one must have

$$\lambda = \frac{2\pi n}{L}$$

for some $n \in \mathbb{Z}$. One now sees what is going on: the Fourier series is simply an eigenfunction expansion in the space of square integrable functions. Of course, a certain degree of smoothness is also required so that the action of the differential operator even makes sense.

We can then state a criterion for the convergence of the Fourier series: if f is square-integrable on L, then its Fourier series is a convergent basis expansion and equals f at every x.

For practical purposes, this is the only safety check that is necessary. For the more mathematically minded, some subtleties which arise include: square integrability does not imply square integrability of the derivative, so differentiation of a Fourier series is not always legal; sometimes, the Fourier series may converge, but not to the correct value of the function; the Fourier series may converge pointwise in x but not uniformly, and so on. These are all irrelevant at the practical level (unless term-by-term differentiation blows up in your face). Most of the time, one is not interested in actually summing up the Fourier series for some value of x; it is viewed as just an expression which represents the solution in some sense, and convergence takes a backseat.

4.4 Examples

We will discuss examples of the Fourier series. Since many of the applications of the Fourier series are identical in the key ideas to those of the Fourier transform, we will omit these (except for a brief discussion so that one may see how to problem solve) and opt for more interesting topics to maintain diversity. For example, to find a Green's function on a finite interval, one should use a Fourier series rather than a transform, but the manipulations are nearly identical, so we will not go through it again.

[6] Recall that hermitian matrices are a special case of self-adjoint operators. We know that hermitian matrices always generate a complete orthogonal basis of eigenvectors, so we expect that the same result holds here. Proving this result rigorously for function spaces is highly nontrivial and it goes under the name of the spectral theorem.

4.4.1 The Heat Equation

In this example, we will solve a partial differential equation with certain prescribed boundary conditions. Suppose we are interested in diffusion (heat flow) on a circle of unit radius. The appropriate differential equation is

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial \theta^2} \tag{4.4.1}$$

Here, $u(t,\theta)$ is the temperature at time t and position θ. Since we are on the unit circle, u must satisfy $u(t,\theta) = u(t,\theta+2\pi)$, since otherwise we would have a multivalued temperature function, which is physically nonsense (one point does not have two different temperatures). However, this is simply a 2π periodicity condition, so we may write a Fourier expansion:

$$u(t,\theta) = \sum_{n=-\infty}^{\infty} u_n(t)e^{in\theta}$$

Observe that our Fourier coefficients depend on t, since the Fourier expansion only accounts for the θ dependence. Now, we simply substitute the expansion into the heat equation to obtain

$$\sum_{n=-\infty}^{\infty} \dot{u}_n(t)e^{in\theta} = -\frac{1}{2}\sum_{n=-\infty}^{\infty} n^2 u_n(t)e^{in\theta}$$

The dot denotes differentiation with respect to t. Now, multiplying by $e^{-im\theta}$, integrating over θ, and using the orthogonality relations, one finds

$$\dot{u}_m(t) = -\frac{1}{2}m^2 u_m(t)$$

We have converted the partial differential equation into an infinite number of ordinary differential equations. Furthermore, these ordinary differential equations are very easy to solve! The solution is

$$u_m(t) = e^{-\frac{1}{2}m^2 t}c_m$$

The c_m can be constructed appropriately from an initial condition $u(0,\theta) = f(\theta)$. The most general solution with the appropriate initial and boundary conditions is then

$$u(t,\theta) = \sum_{n=-\infty}^{\infty} c_n e^{-\frac{1}{2}n^2 t + in\theta} \tag{4.4.2}$$

Compare with the Fourier transform solution to the heat equation on the infinite line. Note that this is very obviously convergent for any reasonable choice of c_n, its large-n behavior being controlled by the Gaussian $e^{-\frac{1}{2}n^2 t}$.

For a delta function initial condition, this series defines a Jacobi theta function for certain values of its complex parameters. This function appears in many places throughout the mathematical and physical literature. In the mathematical literature, the theta function is a modular form useful in analytic number theory. Perhaps its most noteworthy application here is its role in the derivation of Riemann's functional equation, which leads to his hypothesis. In the physics literature, theta functions and other modular forms appear when discussing one-loop Feynman amplitudes in string theory, with their complex parameters describing a space known as the *moduli space* of complex structures on the torus. The structure of moduli space is crucial in the search for consistent string theories.

4.4.2 Poisson Summation

Admittedly, Poisson summation uses the Fourier transform, not the series, but since it has to do with series more than integrals, I include it here. In my own experience, it has been a very useful tool for handling infinite sums.

Say you are given a sum of the form

$$\sum_{n=-\infty}^{\infty} f(n) \tag{4.4.3}$$

Here is the key idea: from Fourier's theorem, we can write

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{f}(k)$$

for any x. Substitute this into the series. We have

$$\sum_{n=-\infty}^{\infty} f(n) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{f}(k) \sum_{n=-\infty}^{\infty} e^{ikn}$$

But we already did this infinite sum earlier! The answer is

$$\sum_{n=-\infty}^{\infty} e^{ikn} = 2\pi \sum_{n=-\infty}^{\infty} \delta(k - 2\pi n)$$

Substituting this into the series, we have

$$\sum_{n=-\infty}^{\infty} f(n) = \int_{-\infty}^{\infty} dk \tilde{f}(k) \sum_{n=-\infty}^{\infty} \delta(k - 2\pi n)$$

But now, we can interchange order of summation and integration, and we know how to do the integrals with the delta functions. This leaves us with

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \tilde{f}(2\pi n) \tag{4.4.4}$$

This is the *Poisson summation formula.*

As an application of this, we will evaluate the infinite sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \tag{4.4.5}$$

We will handle this as follows. Consider the sum

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{1}{a^2} + 2\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2}$$

We will take the $a \to 0$ limit to extract the desired sum. Clearly, in this case we have

$$f(x) = \frac{1}{x^2 + a^2}$$

The Fourier transform is given by the integral

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx \frac{e^{-ikx}}{x^2 + a^2}$$

We already evaluated this integral in the "common Fourier transforms" section, so we have

$$\tilde{f}(k) = \frac{\pi}{a} e^{-a|k|}$$

Using the Poisson formula, we have

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a} \sum_{n=-\infty}^{\infty} e^{-2\pi a|n|} \tag{4.4.6}$$

But the right hand side is a simple geometric series! We have

$$\sum_{n=-\infty}^{\infty} e^{-2\pi a|n|} = 1 + 2\sum_{n=1}^{\infty} e^{-2\pi an} = 1 + \frac{2e^{-2\pi a}}{1 - e^{-2\pi a}} = 1 + \frac{2}{e^{2\pi a} - 1}$$

This leaves us with

$$\frac{1}{a^2} + 2\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a} + \frac{2\pi}{a} \frac{1}{e^{2\pi a} - 1}$$

So that

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} = \frac{1}{2a^2}\left(\pi a + \frac{2\pi a}{e^{2\pi a} - 1} - 1\right) \tag{4.4.7}$$

Now, in the limit $a \to 0$, we have to be very careful, since each term naively diverges. For the second term, observe that we have

$$\frac{2\pi a}{1 + 2\pi a + 2\pi^2 a^2 + \frac{4\pi^3 a^3}{3} + ... - 1} = \frac{1}{1 + \pi a + \frac{2\pi^2 a^2}{3} + ...}$$

We have simply used the Taylor expansion for $e^{2\pi a}$. Now, what is clear is that the limiting behavior that we need is the term in parenthesis of eq. 4.4.7 up to order a^2, since all higher terms when divided by a^2 will be a positive power of a, which vanishes as $a \to 0$. Taylor expanding, we have

$$\frac{1}{1 + \pi a + \frac{2\pi^2 a^2}{3} + ...} = 1 - \left(\pi a + \frac{2\pi^2 a^2}{3} + ...\right) + \pi^2 a^2 + ...$$

We have used the Taylor series for $\frac{1}{1+x}$ and consistently neglected terms of order a^3 and higher. We see that, to this order, the second term is

$$1 - \pi a + \frac{\pi^2 a^2}{3} + ...$$

We have combined the two a^2 terms, and the ... indicate the a^3 and higher terms. Conveniently, the first two terms exactly cancel the other terms appearing in parenthesis, so we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \lim_{a \to 0} \frac{1}{2a^2}\left(\pi a + 1 - \pi a + \frac{\pi^2 a^2}{3} - 1\right) = \frac{\pi^2}{6} \tag{4.4.8}$$

We then have that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

This result was first derived by Euler, and the challenge to find the sum is known as the Basel problem (after Basel, the hometown of Euler and the Bernoulli family). Euler also found a closed form solution for

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}}$$

for all positive integers k. It is actually quite easy to do this with our formula derived from Poisson summation: simply expand both sides in a^2 and compare terms. The general expression for the sum involves the Bernoulli numbers. This can seem somewhat mysterious, but the informed reader will note that our formula includes none other than the generating function for the Bernoulli numbers, so Poisson summation makes this nearly obvious.

4.4.3 Parseval's Identity

As a second derivation of the result for this infinite sum, we will use a result known as *Parseval's identity.* First, we must prove it. Consider an L-periodic square-integrable function $f(x)$. Now, consider the integral

$$\frac{1}{L}\int_0^L dx|f(x)|^2$$

Now, substitute the Fourier series for f into this formula. We find

$$\frac{1}{L}\int_0^L dx|f(x)|^2 = \frac{1}{L}\sum_{m,n=-\infty}^{\infty} c_m c_n^* \int_0^L dx e^{2\pi i(m-n)x/L}$$

Now, we use the orthogonality relation to find

$$\frac{1}{L}\int_0^L dx|f(x)|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2 \tag{4.4.9}$$

This result is known as Parseval's identity. It is obviously quite useful because integrals are much easier to evaluate than series.

To do the sum, take $f(x) = x$ on the interval $(-\pi, \pi)$, so that f is 2π periodic. Note that this is not of the same form as the Fourier series I have done so far, on an interval from $[0, L]$. Here, we have a function on an interval $[-L/2, L/2]$. The easiest way to see that it still works is that if we went back to our derivation of the Fourier series and broke up the integral in steps of $[-L/2, L/2]$ instead of $[0, L]$, the arguments would carry through exactly, the only difference being that the bounds of the integral in eq. 4.1.4 are now $-L/2$ and $L/2$. This principle holds for any interval, and is a consequence of translational invariance.

Observe that although $f(x)$ is discontinuous at every integer multiple of π (a saw wave), it is smooth and square integrable on the interval. Let us evaluate the coefficients. For c_0, we have

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx x = 0$$

because it is the integral of an odd function over a symmetric interval. For c_n, $n \neq 0$, note that

$$c_n = \frac{i}{2\pi}\frac{\partial}{\partial n}\int_{-\pi}^{\pi} dx e^{-inx} = \frac{\partial}{\partial n}\Big(\frac{1}{2\pi n}(e^{in\pi} - e^{-in\pi})\Big) = \frac{i}{n}(-1)^n$$

Clearly, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} dx x^2 = \frac{\pi^2}{3}$$

So Parseval's identity states that

$$\frac{\pi^2}{3} = \sum_{n=-\infty}^{\infty} |c_n|^2 = \sum_{n=-\infty}^{-1}\frac{1}{n^2} + 0 + \sum_{n=1}^{\infty}\frac{1}{n^2} = 2\sum_{n=1}^{\infty}\frac{1}{n^2}$$

so that

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Note that while the algebra was somewhat simpler, the result from Poisson summation is much more general.

5 The Fourier Transform in $d > 1$

In this section, we discuss the Fourier transform in more than one dimension. It is invaluable in the study of linear partial differential equations, particularly of the kind that appear in electrodynamics or quantum mechanics. We will first discuss the definitions and notation, being quite quick because it is a direct generalization of the one dimensional results. Then, we discuss three examples: causal Green's functions, Poisson's equation, and large extra dimensions. This section will be written at a more advanced level than those preceding it, and is intended for an audience which may already be exposed to some of the ideas mentioned earlier (particularly the differential equations in electrodynamics).

5.1 Definitions and Notation

First, we must define the Dirac delta function. In two dimensions, the definition is

$$\delta^{(2)}(\vec{x}) = \delta(x)\delta(y)$$

Note that while $(\delta(x))^2$ is an ill-defined object[7], this is not important here because x and y are allowed to vary independently. Furthermore, the "double singularity" at the origin is offset by the fact that we now do double integrals instead of single:

$$\int d^2x\delta^{(2)}(\vec{x}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdy\delta(x)\delta(y) = \int_{-\infty}^{\infty} dx\delta(x)\int_{-\infty}^{\infty} dy\delta(y) = 1$$

[7] I haven't discussed this point explicitly in these notes, but note that the integral of $\delta(x)^2$ is $\delta(0)$, which is infinite. This ruins any chance $\delta(x)^2$ has of being a friendly object.

In the first expression, I have introduced a convenient notation for integration in d dimensions. Integration over the region D is defined by the symbol

$$\int_D d^d x \equiv \int ... \int dx_1 ... dx_d$$

The limits of the various integrals are to be determined by the geometry of D. If no subscript is written on the integral, it is understood to mean integration over all of space.

In d dimensions, the delta function is given as

$$\delta^{(d)}(\vec{x}) = \prod_{n=1}^{d} \delta(x_n) = \delta(x_1)\delta(x_2)...\delta(x_d)$$

In the second equality, we have used the product symbol, and in the third we have written it out explicitly. Fourier's theorem states that, for scalar-valued $f(\vec{x})$,

$$f(\vec{x}) = \int \frac{d^d k}{(2\pi)^d} e^{i\vec{k}\cdot\vec{x}} \tilde{f}(\vec{k})$$

$$\tilde{f}(\vec{k}) = \int d^d x e^{-i\vec{k}\cdot\vec{x}} f(\vec{x})$$

Here, $\vec{k} \cdot \vec{x}$ is the usual dot product. I leave the proof as an exercise.

This is about all there is to it. To do Fourier analysis in more dimensions, you simply have to do more integrals. All properties of the Fourier transform which we exploited in solving problems generalize in the obvious way (prove this if you do not see it). There also exists a generalization of the Fourier series to higher dimensions, where periodic boundary conditions can be taken in some or all of the directions. Topologically, this corresponds to compactifying some or all of the directions into a higher-dimensional torus. The expression for the Fourier series is more complicated, though, and the sum from one dimension gets promoted to a sum over a lattice. Poisson summation becomes summation over a related lattice known as the dual lattice[8]. We will not need the Fourier series in more dimensions.

5.2 Examples

In this section, we discuss three examples of physical relevance.

5.2.1 Causal Green's Functions (Complex Analysis Required)

In this section, we will apply the Fourier transform in $d = 4$ to electrodynamics. Recall that in Lorenz gauge $\partial^\mu A_\mu = 0$, Maxwell's equations reduce to

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) A_\mu(t, \vec{x}) = J_\mu(t, \vec{x}) \tag{5.2.1}$$

[8]In passing, I note that in string theory, the Fourier series lattice one gets from compactifying on a torus is known as a Narain lattice, and requiring that the lattice and its dual satisfy certain physically motivated criteria leads to the construction of the $O(32)$ and $E_8 \times E_8$ heterotic theories. The lattice requirements actually constrain the only possible groups to $O(32)$ and $E_8 \times E_8$.

Here, A_μ is the vector potential and J_μ is the four-current. For the purposes of our discussion here, μ will just be an annoying extra index which creates notational baggage, so for simplicity we will study the scalar version of this equation[9]

$$\Big(\frac{\partial^2}{\partial t^2} - \nabla^2\Big)\phi(t,\vec{x}) = \rho(t,\vec{x})$$

This is essentially the wave equation with source. We will be interested in obtaining the Green's function $G(t,\vec{x})$ for the wave equation. This is given by the solution to

$$\Big(\frac{\partial^2}{\partial t^2} - \nabla^2\Big)G(t,\vec{x}) = \delta^{(3)}(\vec{x})\delta(t) \tag{5.2.2}$$

We define our Fourier transform by

$$G(t,\vec{x}) = \int \frac{d\omega}{2\pi}\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}-i\omega t}\tilde{G}(\omega,\vec{k})$$

I have trivially reversed my sign convention in the t transform. This can easily be absorbed by changing $\omega \to -\omega$ in the integral, but the convention I have taken is the most commonly used. Going through the usual machinery, we find

$$\tilde{G}(\omega,\vec{k}) = -\frac{1}{\omega^2 - \vec{k}^2}$$

Fourier's theorem implies

$$G(t,\vec{x}) = -\int \frac{d\omega}{2\pi}\frac{d^3k}{(2\pi)^3}\frac{e^{i\vec{k}\cdot\vec{x}-i\omega t}}{\omega^2 - \vec{k}^2} \tag{5.2.3}$$

But now we see the problem: this integral is not defined! In particular, if we try to do the integral $d\omega$ we run into poles at $\omega = \pm|\vec{k}|$. What is the origin of this difficulty? Well, observe that the equation defining $G(t,\vec{x})$ has no unique solution! This is because to any solution, we can add a solution to the homogenous (source-free) wave equation, of which we have an infinite family, as we saw when we applied the transform to solve the homogeneous wave equation. This means that we have a freedom in defining the integral, and we can choose a definition which is the most convenient. The only subtlety here is whether or not our choice of definition of the integral actually gives a solution to the equation. This can always be verified after-the-fact by simply checking explicitly.

For readers familiar with complex variables, the solution is clear: we should deform the integration contour in the complex plane so that it does not hit the poles. Equivalently, we can keep the integration contour along the real axis and shift the poles off of it. Since the integrand is a meromorphic function of ω, the results are equivalent. We consider shifting the poles along the imaginary axis by a small amount $i\epsilon$, with ϵ a positive infinitesimal. There are several ways to do this: we can move both poles above the real axis, both below the real axis, or put one in each half-plane. We will discuss the physical implications of these choices now. Also, provided we consider the limit $\epsilon \to 0$, it is clear that all of these procedures will lead to solutions of the equation.

[9] If you're a purist, make ϕ the scalar potential and ρ the charge density, i.e. take $\mu = 0$

First, consider the case of shifting both poles into the lower half plane. This can be accomplished by adding $i\epsilon\text{sgn}(\omega)$ to the denominator[10], and the resulting Green's function is called the *retarded* Green's function:

$$G_{\text{ret}}(t,\vec{x}) = -\int \frac{d\omega}{2\pi}\frac{d^3k}{(2\pi)^3}\frac{e^{i\vec{k}\cdot\vec{x}-i\omega t}}{\omega^2-\vec{k}^2+i\epsilon\text{sgn}(\omega)} \tag{5.2.4}$$

The poles are now at $\omega = \pm|\vec{k}| - i\epsilon$ (recall that ϵ is to be taken so small that any positive number multiplying it is again ϵ). Here is the key: if $t < 0$, we can close the contour by considering a large semicircle in the upper half plane, with its diameter along the real axis. The arc of the semicircle will not contribute, since for $t < 0$, $-i\omega t$ gives rise to exponential suppression in the upper half plane. This ensures that $G_{\text{ret}}(t,\vec{x}) = 0$ for $t < 0$, since the contour encloses no poles. This is a *causality* condition, because it ensures that the retarded Green's function only propagates signals forward in time. If $t > 0$, we can close the contour in the lower half plane and use the residue theorem on the poles. For the pole at $k - i\epsilon$ (for brevity, $k = |\vec{k}|$), the residue[11] is $\frac{1}{2k}e^{i\vec{k}\cdot\vec{x}-ikt}$. For the other pole, the residue is $-\frac{1}{2k}e^{i\vec{k}\cdot\vec{x}+ikt}$. Using the residue theorem for the $d\omega$ integral, we have

$$G_{\text{ret}}(t,\vec{x}) = i\int \frac{d^3k}{(2\pi)^3}\frac{1}{2k}\left(e^{i\vec{k}\cdot\vec{x}-ikt} - e^{i\vec{k}\cdot\vec{x}+ikt}\right)$$

Note the sign coming from the orientation of the contour. Now, to evaluate the $\vec{k}$ integral, we note that $\vec{x}$ is just some vector in space as far as the integral is concerned, so we can set up spherical coordinates (k,θ,ϕ) with $\vec{x}$ acting as the z-axis, so that $\vec{k}\cdot\vec{x} = kr\cos\theta$, where $r = |\vec{x}|$. As usual, $d^3k = k^2\sin\theta dk d\theta d\phi$. We find

$$G_{\text{ret}}(t,\vec{x}) = \frac{i}{4\pi^2}\int_0^\infty \frac{dk k^2}{2k}\int_{-1}^1 d(\cos\theta)\left(e^{ikr\cos\theta}(e^{-ikt} - e^{ikt})\right)$$

Doing the $\cos\theta$ integral and cleaning up, we find

$$G_{\text{ret}}(t,\vec{x}) = \frac{1}{2\pi^2 r}\int_0^\infty dk\sin(kr)\sin(kt)$$

Now, we see that we are integrating an even function, so that we may replace this with

$$\frac{1}{4\pi^2 r}\int_{-\infty}^\infty dk\sin(kr)\sin(kt)$$

Using the trigonometric formula $\sin(kr)\sin(kt) = \frac{1}{2}(\cos(k(r-t)) - \cos(k(r+t)))$, we have

$$\frac{1}{8\pi^2 r}\int_{-\infty}^\infty dk(\cos(k(r-t)) - \cos(k(r+t)))$$

We may now do the following trick: $e^{ikx} = \cos kx + i\sin kx$, but $\sin kx$ is an odd function of k, so in an integral over a symmetric interval of k we may replace $\cos(kx)$ with e^{ikx}. The integral then becomes $2\pi\delta(t-r) - 2\pi\delta(t+r)$, where we have recognized a delta

[10] In case this is new to you, $\text{sgn}(x) = 1$ if $x > 0$ and -1 if $x < 0$

[11] Recall that we take $\epsilon \to 0^+$

function definition. However, we know that $t > 0$ by assumption, so the delta function in $t + r$ is always zero. Therefore, the result is

$$G_{\rm ret}(t, \vec{x}) = \frac{1}{4\pi r}\delta(t - r) \tag{5.2.5}$$

for $t > 0$. Since it vanishes for $t < 0$, this can be neatly packaged as

$$G_{\rm ret}(t, \vec{x}) = \frac{1}{4\pi r}\Theta(t)\delta(t - r) \tag{5.2.6}$$

$\Theta(t)$ is the Heaviside step function. Thus, we have found a Green's function for the wave equation which propagates data forward in time. We see that our pole prescription leads to a notion of causality: the way in which we choose to define the integral in the complex plane dictates how the Green's function moves data through time.

Suppose we shifted both poles into the upper half plane, equivalent to adding $-i\epsilon\text{sgn}(\omega)$ to the denominator. This defines the *advanced* Green's function:

$$G_{\rm adv}(t, \vec{x}) = -\int \frac{d\omega}{2\pi}\frac{d^3k}{(2\pi)^3}\frac{e^{i\vec{k}\cdot\vec{x}-i\omega t}}{\omega^2 - \vec{k}^2 - i\epsilon\text{sgn}(\omega)} \tag{5.2.7}$$

I will not go through the whole manipulation again, but note that now if $t > 0$, we must close the contour in the lower half plane, in which there are no poles, so the integral vanishes. If $t < 0$, we pick up the residues and do the calculation as before, yielding

$$G_{\rm adv}(t, \vec{x}) = \frac{1}{4\pi r}\Theta(-t)\delta(t + r) \tag{5.2.8}$$

Note that the advanced Green's function can be obtained from the retarded one by a simple time reversal $t \to -t$. Thus, the advanced Green's function propagates information backwards in time. As such, advanced waves are better to use when you are given information about the present, and asked to infer what happened in the past[12].

The Green's functions we have described are known as *causal Green's functions* because they require an explicit choice to define how they move data through time. They act just like Green's functions in the usual sense, but now one must give a prescription as to which one to use to obtain the correct solution. The most intuitively accessible is the retarded Green's function, since we are accustomed to initial value problems. The particular solution is then

$$\phi(t, \vec{x}) = \int dt' d^3y G_{\rm ret}(t - t', \vec{x} - \vec{y})\rho(t', \vec{y})$$

We can actually go one step further and do the integral over t', which yields

$$\phi(t, \vec{x}) = \int d^3y \frac{\rho(t - |\vec{x} - \vec{y}|, \vec{y})}{4\pi|\vec{x} - \vec{y}|}$$

[12]Realistically, any solution to the inhomogeneous wave equation can be written using some choice of Green's function and homogeneous solution which gives us the right initial/boundary conditions. They should be viewed as mathematical objects which give us a machine to generate solutions. The philosophical niceties about cause, effect, present, and past are just useful to build some intuition for the objects, since the contour business is somewhat abstract.

Observe that at this point, it is exactly like an electrostatics problem, with some new $\vec{y}$-dependence induced by using the "retarded time" $t - |\vec{x} - \vec{y}|$. As such, the usual machinery, such as multipole expansion, can be developed.

We also note that the retarded Green's function simply reflects Huygen's principle applied to waves: $\frac{1}{4\pi r}$ is the source due to a point, $\Theta(t)$ is causality, and $\delta(t-r)$ says that the waves are spherical and travel at unit speed (recall we have set all the constants to one in our wave equation). Solving the equation using the Green's function then has the following interpretation: the effect of the theta function is to set the upper bound on the t' integral to t. This ensures that no data is taken from the future: the only profile of the source which contributes to the solution comes from the past. We then add up the contributions from every component of the source, which have the spherical wave profile $\frac{1}{r}f(t-r)$.

It appears I have let one detail slip: what about the pole prescription where we place one pole in the upper half plane and one pole in the lower? First of all, it is clear that this idea is more exotic than those preceding it: there is now a pole in both planes, so no matter which way we close the contour, the Green's function will be nonzero. Thus it sends data both forwards and backwards in time. At first glance, it seems that this is physically meaningless. How could there ever be a use for this? The answer is relativistic quantum mechanics[13].

I assume at some point you have been introduced to the de Broglie relations, that a quantum mechanical particle of energy E and momentum $\vec{p}$ is to be associated with a wave of definite wave number (more accurately, wave vector) $\vec{k}$ and frequency ω, such that $E = \omega$ and $\vec{p} = \vec{k}$ (we use $\hbar = c = 1$ units). This means we can package the quantum mechanical information by saying that the particle is described by the wave $e^{i\vec{k}\cdot\vec{x} - i\omega t}$. I also assume you have been introduced to the relativistic energy-momentum relation $E^2 = \vec{p}^2 + m^2$. Observe that there are actually two solutions for E: $E = \pm\sqrt{\vec{p}^2 + m^2}$. This presents a problem at the quantum-mechanical level. When you turn on quantum fluctuations, roughly speaking, anything goes; namely, any event can take place with some nonzero probability, even if it is forbidden classically. This means that if a particle can lower its energy, even if it is classically forbidden, there is a definite probability that it will do so. Furthermore, if at the relativistic level, we can have arbitrarily negative energies, this means that any positive-energy particle can decay indefinitely into increasingly negative-energy states. How do we prevent[14] this?

Well, first let us go back to the de Broglie wave associated to the particle. We see that it varies as e^{-iEt} (this relation holds relativistically). What we notice is that *the sign reversal* $E \to -E$*, when accompanied by* $t \to -t$*, leaves the state of the particle invariant.* This means that if we allow the negative-energy particles to propagate backwards in time, they behave just like positive-energy particles, and there is no trouble at all! What is the

[13] Actually, a consistent classical theory can be introduced using this choice of Green's function, known as Wheeler-Feynman absorber theory. At the mathematical level, this is unsurprising: as mentioned earlier, regardless of the intuitive niceties, all choices can generate solutions to the equations. However, their focus was on interpretation and the true physics of the interaction of matter and radiation, which can be a subtle issue. Wheeler and Feynman developed it with the hope that it would be better-behaved quantum mechanically. Their initial hopes turned out to be false, but the development was still important because it led Feynman to the right track.

[14] These sorts of questions plagued physicists for nearly two decades in the early development of quantum field theory.

meaning[15] of a particle moving backward in time? Well, if you place the particle in an electromagnetic field and reverse time, *it will behave identically to a particle of the same mass, but opposite charge.* If you do not see it, just think about it. Reversing time is like watching a movie in reverse. But the direction of motion of the charge is controlled by its sign. However, we already know of particle pairs with equal mass and opposite charges: the oppositely charged ones are simply the *antiparticles.* We reach the conclusion that *antiparticles behave exactly like particles moving backwards in time.* This is known as the Feynman-Stueckelberg interpretation.

What does this have to do with our Green's function? I can only give a rough sketch. Relativistic quantum mechanical particles obey what are essentially modified wave equations, but the whole business of $i\epsilon$ carries over exactly. It turns out that the correct[16] Green's function to use is the one with a pole in both planes, known as the *Feynman propagator*[17]. The reason the Feynman propagator is used is because it sends the positive-frequency parts of waves forward in time and the negative-frequency parts backwards, so it gives us antiparticles for free as per the Feynman-Stueckelberg interpretation[18]. By solving the negative energy problem, we are led to the notion of antiparticle! The Feynman propagator is the key ingredient of Feynman diagrams.

5.2.2 Poisson's Equation

The central equation of electrostatics is Poisson's equation

$$-\nabla^2\phi(\vec{r}) = \frac{1}{\epsilon_0}\rho(\vec{r}) \tag{5.2.9}$$

Here, ϕ is the electrostatic potential, and ρ is the charge distribution. I have also changed notation from $\vec{x}$ to $\vec{r}$ to match with conventions in electrostatics. Let us solve this equation for $\rho(\vec{r}) = e\delta^{(3)}(\vec{r})$, for some constant e. At this point, the route should be familiar: take

$$-\nabla^2\phi(\vec{r}) = \frac{e}{\epsilon_0}\delta^{(3)}(\vec{r})$$

and Fourier transform it. This leaves us with

$$k^2\tilde{\phi}(\vec{k}) = \frac{e}{\epsilon_0}$$

Here, $k^2 = \vec{k}\cdot\vec{k}$. Solving for $\tilde{\phi}$ and using Fourier's theorem, we have

$$\phi(\vec{r}) = \frac{e}{\epsilon_0}\int\frac{d^3k}{(2\pi)^3}\frac{e^{i\vec{k}\cdot\vec{r}}}{k^2} \tag{5.2.10}$$

[15]Note that I dismissed philosophical questions of this kind at the classical level. I acknowledge them now because they are crucial to the theoretical picture, and without this explanation the theory appears inconsistent. At the classical level, things are not very subtle: have differential equation, will solve.

[16]Note that at the classical level, all Green's functions are equal. In this sense, the quantum theory makes our life easier: only one choice of Green's function is consistent.

[17]Note that I have called it a "propagator" instead of a Green's function. This is conventional in field theory and many body physics, where we use "propagator" to talk about classical Green's functions and the term "Green's function" for something else. I actually prefer the term propagator, because it is more instructive: the propagator does indeed propagate things.

[18]Incidentally, there is a way to introduce the whole story of the Feynman propagator without appealing to philosophy. I prefer this method, but it requires too much mathematical machinery for this context.

To evaluate this, we will use the following trick: recall that

$$\frac{1}{k^2} = \int_0^\infty ds e^{-sk^2}$$

We then have

$$\phi(\vec{r}) = \frac{e}{\epsilon_0} \int_0^\infty ds \int \frac{d^3k}{(2\pi)^3} e^{-sk^2 + i\vec{k}\cdot\vec{r}}$$

But, recalling the definitions of d^3k, k^2, and $\vec{k}\cdot\vec{r}$, this is simply the product of three independent Gaussian integrals. Doing the Gaussian integrals over $\vec{k}$, we have

$$\phi(\vec{r}) = \frac{e}{\epsilon_0} \int_0^\infty \frac{ds}{8\pi^3} \frac{\pi^{3/2}}{s^{3/2}} \exp\left\{-\frac{r^2}{4s}\right\}$$

Now, make the substitution $s = 1/4t$ so that

$$\phi(\vec{r}) = \frac{e}{\epsilon_0} \int_0^\infty \frac{dt}{4\pi^{3/2}} \frac{1}{\sqrt{t}} \exp\{-r^2 t\}$$

Now, write $t = u^2$, so that we have

$$\phi(\vec{r}) = \frac{e}{\epsilon_0} \int_0^\infty \frac{du}{2\pi^{3/2}} \exp\{-r^2 u^2\}$$

This is a Gaussian integral, which evaluates to

$$\phi(\vec{r}) = \frac{e}{4\pi\epsilon_0 r} \tag{5.2.11}$$

This is simply the Coulomb potential, all the way down to the correct factor of 4π! Amusingly, we see that the delta function source is precisely what describes a point charge. Observe that the solution is singular as $r \to 0$, reflecting the delta function singularity. I will now change notation and introduce the function

$$G(r - r_0) = \frac{e}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{r}_0|}$$

Observe that this solves Poisson's equation as a function of $\vec{r}$ with source $e\delta^{(3)}(\vec{r} - \vec{r}_0)$. This casts the Coulomb potential in a new light: It is the Green's function for the Poisson equation! From our general considerations regarding Green's functions, we know that this means that

$$\phi(\vec{r}) = \int d^3r' G(r - r')\rho(\vec{r}')$$

solves Poisson's equation with source $\rho(\vec{r})$ (verify this). But, writing this out explicitly, it says that

$$\phi(\vec{r}) = \int d^3r' \frac{e}{4\pi\epsilon_0 |\vec{r} - \vec{r}'|} \rho(\vec{r}') \tag{5.2.12}$$

This is simply the familiar principle of superposition[19]. Furthermore, observe that if we replace e/ϵ_0 with $-4\pi G$, with G Newton's constant, we obtain the correct expression for

[19] If this is fuzzy to you, this equation has the physical interpretation that a distribution $\rho(\vec{r})$ can be written as a bunch of point charges of charge $\rho(\vec{r})d^3\vec{r}$. The Green's function gives the Coulomb field for each point charge, and the integral adds them up.

the gravitational potential. Remarkably, we obtain two of the most physically important results of classical physics directly from Fourier analysis[20]!

The advantage of working out our theory for general d is that we can now consider electrostatics in an arbitrary number of spatial dimensions. Let us consider the d-dimensional version of the Coulomb potential. That is, let us solve

$$-\nabla^2\phi(\vec{r}) = e_d\delta^{(d)}(\vec{r})$$

Here, I have subscripted e with d to remind us that it is the electric charge in d dimensions, and set $\epsilon_0 = 1$. We can write the solution as

$$\phi(\vec{r}) = e_d \int \frac{d^dk}{(2\pi)^d}\frac{e^{i\vec{k}\cdot\vec{r}}}{k^2}$$

At this point, the integral requires some care. First, take $d \geq 3$. Then, we can do the usual trick, this time with d Gaussian integrals:

$$\phi(\vec{r}) = e_d \int_0^\infty \frac{ds}{(2\pi)^d}\frac{\pi^{d/2}}{s^{d/2}}\exp\left\{-\frac{r^2}{4s}\right\}$$

Again, substitute $s = 1/4t$ and find

$$\phi(\vec{r}) = e_d \int_0^\infty \frac{dt}{4\pi^{d/2}}t^{\frac{d}{2}-2}\exp\{-r^2t\}$$

At this point, we recognize the integral definition of the gamma function (see appendix), so that

$$\phi(\vec{r}) = \frac{e_d}{4\pi^{d/2}}\frac{\Gamma(\frac{d}{2}-1)}{r^{d-2}} \tag{5.2.13}$$

We see that the Coulomb potential (or gravitational potential) in d dimensions scales as r^{2-d}, for $d \geq 3$. If we were to live in $d = 10$, with all spatial dimensions macroscopic, we would have an inverse 9th law as opposed to inverse square. Actually, the fact that it goes as r^{2-d} follows directly from rotational invariance and dimensional analysis, so we really did all the work just to get the dependence of the numerical factor on the dimension.

When $d = 2$, we have to be a bit more careful. We need

$$\phi_0(\vec{r}) = e \int \frac{d^2k}{(2\pi)^2}\frac{e^{i\vec{k}\cdot\vec{r}}}{k^2}$$

Introduce the s parameter again and do the Gaussians, giving

$$\phi_0(\vec{r}) = e \int_0^\infty \frac{ds}{4\pi s}\exp\left\{-\frac{r^2}{4s}\right\}$$

Again, $s = 1/4t$, so

$$\phi_0(r) = e \int_0^\infty \frac{dt}{4\pi t}\exp\{-r^2t\}$$

[20] Actually, superposition follows from the fact that these equations are linear. Fourier analysis is just particularly powerful for linear equations, and so it elucidates what is going on.

Now, we can clearly see the problem. The integral is divergent from the $t \to 0$ end. This is known as an *infrared divergence*, because for large r the exponential becomes suppressed for all t outside of a small neighborhood of zero, so the divergence is to be associated with large r. The word "infrared" is a crude abuse of terminology coming from the fact that large r means long distance, and waves in the infrared have long wavelengths.

Let us think physically about this. Recall that in electrostatics, we can always add a constant to the potential and still have a solution to Poisson's equation. We fix the value of this constant by demanding that $\phi(r)$ vanish at infinity, so that we always measure the work done coming from infinity. What the infrared divergence tells us is that our potential is not dying off at infinity, but rather growing indefinitely. This is no issue in real life, because we always observe physics in finite volume. However, what is true is that the observational scale is always much larger than the characteristic size of the charge, which we approximate as pointlike. In light of these considerations, subtract the value of the potential at some very large fixed radius R. We find then that the new *renormalized* potential is given by

$$\phi(r) = \phi_0(r) - \phi_0(R) = e \int_0^\infty \frac{dt}{4\pi t} \Big(\exp\{-r^2 t\} - \exp\{-R^2 t\} \Big)$$

But now (or after consulting the appendix) you recognize this as an integral definition of the logarithm, so that the potential is

$$\phi(r) = -\frac{e}{4\pi} \ln\left(\frac{r^2}{R^2}\right) = -\frac{e}{2\pi} \ln\left(\frac{r}{R}\right)$$

In contrast to higher d, in $d = 2$ the electrostatic potential grows logarithmically with distance. Notice that for $r < R$, the logarithm is negative, and we find that like charges repel, as they should. However, if $r > R$, this gives like charges attracting, which is obviously unphysical. Therefore, it makes no sense to consider physics on distance scales larger than R. R is known as the *infrared cutoff*. Furthermore, our potential is defined so that it vanishes on the circle $r = R$. We have essentially restricted our charges to be within a very large disk of radius R.

Observe that when we compute a directly measurable quantity such as the force between two charges, R drops out and plays no role. This makes sense, because R is essentially the definition of the volume of the system, and the force between two charges should not depend on how big the room is.

The discussion we had here, where we encountered a divergent integral and used physical intuition to obtain finite results, is known in high energy physics and quantum field theory as *renormalization*. In these contexts, renormalization becomes incredibly important, because it allows us to organize how we look at physics depending on the distance scale. As we saw here, we were forced to introduce the scale R into the problem to obtain sensible results. The key difference is that the renormalization in those contexts addresses the *ultraviolet divergences*, those coming from short distances. Physically, the origin of these is that quantum fluctuations on arbitrarily short distance scales grow out of control very quickly, unless organized and treated carefully.

We also note that the fact that the potential goes as $\ln(r)$ in two dimensions can be obtained by directly solving Laplace's equation in polar coordinates. However, we know this is incomplete, because it is insensible to take the logarithm of a dimensionful quantity.

Our discussion renders it physically clear how the argument of the logarithm becomes dimensionless.

As one final comment, I cannot help but remark that while the sorts of manipulations we have performed here by introducing the infrared cutoff R based on physical considerations underlie the conceptual framework for renormalization, when we actually do calculations the most common technique is a beautiful trick called *dimensional regularization* invented by 't Hooft and Veltman.

Go back to eq. 5.2.13. We see that the problem in $d = 2$ can be quantified by the gamma function blowing up at $d = 2$. Regarded as a function of complex x, $\Gamma(x)$ has a simple pole at $x = 0$. What this means is that we will consider analytically continuing our integrals to complex d and take the limit as $d \to 2$.

First, we need to do some dimensional analysis. We are in natural units $\hbar = c = 1$, so energy and momentum have the dimension of mass and length has the dimension of inverse mass (convince yourself that this is true by applying dimensional analysis to equations you know containing $\hbar$ and c, seeing how they can be set to 1, and then seeing how the resulting quantities scale. The moral of the story is that dimensional analysis in natural units is extremely easy, because everything scales like some power of mass, so we just have to count powers of mass).

As a potential energy, ϕ should have mass dimension one. From the expression $e^{i\vec{k}\cdot\vec{x}}$, we see that $\vec{k}$ also has mass dimension one, since $\vec{x}$ has mass dimension -1. ϕ is given by an integral of the form $e_d \int d^dk \frac{1}{k^2}(...)$, where $(...)$ denotes a dimensionless quantity. d^dk has d powers of mass, and k^2 has 2, so we see that the integral has mass dimension $d-2$. In order that ϕ has mass dimension one, we require that e_d has mass dimension $1-d$, or length dimension $d-1$. We introduce an arbitrary length scale R and write $e_d = gR^{d-1}$, so that the object g is dimensionless. In two dimensions, we have $e_2 = gR$.

We can then write eq. 5.2.13 as

$$\phi(\vec{r}) = \frac{gR^{d-1}}{4\pi^{d/2}} \frac{\Gamma(\frac{d}{2}-1)}{r^{d-2}}$$

Here is the punchline. We now write $d = 2 + 2\epsilon$ and consider the limit $\epsilon \to 0$. We have

$$\phi_0(\vec{r}) = \frac{gR^{1+2\epsilon}}{4\pi^{1+\epsilon}} \frac{\Gamma(\epsilon)}{r^{2\epsilon}}$$

Recognizing $e_2 = e = gR$ (here I adopt my old/sloppy notation and call $e_2 = e$), we find

$$\phi_0(\vec{r}) = \frac{e}{4\pi}\left(\frac{R^2}{\pi r^2}\right)^{\epsilon}\Gamma(\epsilon)$$

Now (or after consulting the appendix) you will recognize that we can make the following small-ϵ expansions:

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)$$

$$\left(\frac{R^2}{\pi r^2}\right)^{\epsilon} = 1 - \epsilon \ln\left(\frac{\pi r^2}{R^2}\right) + \mathcal{O}(\epsilon^2)$$

We then find the small-ϵ expansion for ϕ_0:

$$\phi_0(\vec{r}) = \frac{e}{4\pi}\Big(\frac{1}{\epsilon} - \gamma_E - \ln\Big(\frac{\pi r^2}{R^2}\Big) + \mathcal{O}(\epsilon)\Big)$$

As before, we choose our definition of the renormalized $\phi(\vec{r})$ by using our freedom to add a (divergent) constant to ϕ_0. We choose the constant

$$-\frac{e}{4\pi\epsilon} + \gamma_E\frac{e}{4\pi} + \frac{e}{4\pi}\ln\pi$$

This is known in the jargon as an $\overline{\mathrm{MS}}$ subtraction. This exactly cancels the ϵ pole and the nasty numerical factors like γ_E and π, and we find the renormalized potential as

$$\phi(\vec{r}) = -\frac{e}{4\pi}\ln\Big(\frac{r^2}{R^2}\Big)$$

We were rather cavalier about the definition of integration in d dimensions. Dimensional regularization can actually be developed axiomatically, although it has many weird properties because it is not a positive measure. The basic insight of dimensional regularization is that many integrals in d dimensions which converge for either high enough d or low enough d actually admit a meromorphic extension to complex d, and all bad behavior can be taken care of by knocking off poles in ϵ as we approach the physical dimension.

5.2.3 The Brane World and Large Extra Dimensions

I have already pointed out that by simply replacing e with $-4\pi G$, we go from electrostatics to Newtonian gravity (remember $\epsilon_0 = 1$). This means that the gravitational potential $\phi(\vec{r})$ is also described by the Poisson equation:

$$\nabla^2\phi(\vec{r}) = 4\pi G\rho(\vec{r})$$

While Einstein's general theory of relativity does protect the inverse square law at long distances and low velocities, we know that the theory is not to be trusted at arbitrarily short distances. In the technical jargon, this is because general relativity in the weak-field limit is *nonrenormalizable* as a quantum field theory. We instead view Einstein gravity as an effective theory good for describing long-distance physics, but which crumbles when we go to short distances. We need a modification to the theory of gravity at short distances.

What is clear from the considerations of the previous section is that if there were to be a modification of the inverse square law, it would come from a change in the dimensionality of space. This idea has surfaced in recent years, with the intensive investigation of string theory. No string theory is required to understand this section, but you will be expected to follow some key ideas, which I will now lay out.

As any pop science book written since the early 2000s will tell you, a consistent string theory requires ten spacetime dimensions, and M-theory is expected to grow an eleventh. The extra dimensions are expected to be small, referred to as compactified. Here is the essential physics: there are other objects in string theory besides strings known as *Dirichlet membranes* or D-branes, for short. D-branes are defined as surfaces on which an open string ends, but they do have their own dynamics. While strings must be one-dimensional, D-branes can be of any dimension. If we live in spacetime dimension 10, then a D9 brane would fill all of space, but evolve in time, while a D3 brane would be a three-dimensional surface embedded in the higher dimensional spacetime. Generically, Dp branes are Dirichlet branes with p spatial dimensions.

The brane world scenario posits, roughly, that our universe is just some giant D3 brane which we are stuck on, and particles we observe are really just the endpoints of open strings which can move across the extra dimensions, but are stuck to the brane (if you are having a hard time visualizing, imagine being in a room with a yoga ball, and being forced to walk around while keeping one hand on the yoga ball, like in some kind of bad icebreaker game. Your arm is the open string, your hand is the endpoint). The reason we introduce the brane idea is because then, explanations can be given as to the dynamics which trap us on the brane. In any theory with extra dimensions, the obvious issues which must be resolved are why we cannot see them, and why stuff is not constantly popping in and out of them. These are the "visibility" and "escape" problems, resolved by the brane picture.

The next key idea could also be taken from any pop science book since the early 2000s: that string theory posits that all particles come from the vibration of a string. As I have already alluded to, there are really two types of strings: open and closed. A key result in string theory is that the graviton, which is to gravity what the photon is to electromagnetism, is described by a closed string. In particular, this means that, even if we are on a three-brane, *the graviton is not confined to it as the endpoint of an open string is.* This means that we should be able to see effects of the extra dimensions by observing the gravitational force.

Finally, the last key piece of information that we need is that the inverse square law has only been tested down to distances of order 10^{-3} m (I think; in all probability someone reading this will know more about experiments than I do, but the point is that the ballpark order of magnitude is much larger than atomic/subatomic physics). This is due to the weakness of gravity: at short distances, it gets quickly overwhelmed by other forces. With all these considerations taken into account, it is therefore a reasonable proposal that we might have a modification to the inverse square law at a distance scale a, which is the characteristic size of the extra dimensions, the experimental upper bound on which is 10^{-3} m. In the early days of string theory, a was thought to be on the scale of the Planck length, meaning $a \sim 10^{-35}$ m. 10^{-3} is obviously a great deal larger than 10^{-35}, so this proposal is known as *large extra dimensions.*

Before studying a toy model, let us recap the proposal. String theory gives us a picture where there are more dimensions than meets the eye, and furthermore a mechanism which makes it difficult for us to detect them. The only force which we expect to probe them at reasonably long distance scales is gravity. We have learned mathematically in the previous section that the force law for gravity depends on the dimensionality of space, so experimental evidence for large extra dimensions would be a modification to the inverse square law at distances smaller than the extra dimension scale a.

What is remarkable is that we can gain a very deep understanding of the mechanism by which the inverse square law is modified using only Fourier analysis. As a toy model, suppose we live in a universe with four spatial dimensions, three macroscopic and one compactified to a circle of radius a. This is known as a *Kaluza-Klein* compactification. The coordinates of the macroscopic dimensions will be called $\vec{r}$, and the compact direction will be called x_5. If we took away two of the macroscopic directions, the space would simply be a cylinder. At this point, much jargon could be introduced, but here is the essential point for us. The gravitational potential $\phi(\vec{r}, x_5)$ must be a periodic function of x_5, such that $\phi(\vec{r}, x_5 + 2\pi a) = \phi(\vec{r}, x_5)$, in order for it to be single-valued on the circular dimension. We can then Fourier expand the x_5 dependence:

$$\phi(\vec{r}, x_5) = \sum_{n=-\infty}^{\infty} \phi_n(\vec{r}) e^{inx_5/a}$$

Now, consider the four-dimensional Poisson equation:

$$\nabla^2 \phi(\vec{r}, x_5) = 4\pi G_5 \rho(\vec{r}, x_5)$$

We are interested in the source for a point mass M at the origin. This is $\rho(\vec{r}, x_5) = M\delta^{(3)}(\vec{r})\delta(x_5)$. However, we must be careful. $\delta(x_5)$ is understood to be the delta function, but on the circle, so that it should be Fourier expanded in x_5 rather than transformed. The relevant expansion is

$$\delta(x_5) = \frac{1}{2\pi a} \sum_{n=-\infty}^{\infty} e^{inx_5/a}$$

Notice that we essentially derived this result when we derived the Fourier series. Here, the periodicity of the delta function (more accurately, Dirac comb) is implicit since it is understood on the circle, rather than the explicit sum given in our original derivation. Substituting the Fourier expansions into Poisson's equation, we find

$$\sum_{m=-\infty}^{\infty} \left(\nabla_3^2 - \frac{m^2}{a^2}\right) \phi_m(\vec{r}) e^{imx_5/a} = \frac{2G_5 M}{a} \delta^{(3)}(\vec{r}) \sum_{m=-\infty}^{\infty} e^{imx_5/a}$$

Here, δ_3^2 is the three-dimensional Laplacian. This result follows from writing $\nabla^2 = \nabla_3^2 + \partial_5^2$. Following the usual route, we multiply by $e^{-inx_5/a}$, integrate, and use orthogonality to find

$$\left(\nabla_3^2 - \frac{n^2}{a^2}\right) \phi_n(\vec{r}) = \frac{2G_5 M}{a} \delta^{(3)}(\vec{r})$$

But now, this is just an ordinary three-dimensional problem amenable to the transform. Transforming $\phi_n(\vec{r})$ to $\tilde{\phi}_n(\vec{k})$, we find

$$\left(k^2 + \frac{n^2}{a^2}\right) \tilde{\phi}_n(\vec{k}) = -\frac{2G_5 M}{a}$$

Using Fourier's theorem, the solution is

$$\phi_n(\vec{r}) = -\frac{2G_5 M}{a} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2 + \frac{n^2}{a^2}}$$

Since this is the advanced section, I leave it to you to do the integral, and show that the result is

$$\phi_n(\vec{r}) = -\frac{G_5 M}{2\pi a r} e^{-r|n|/a}$$

However, the story is not over yet. We still have to sum the Fourier series:

$$\phi(\vec{r}, x_5) = -\frac{G_5 M}{2\pi a r} \sum_{n=-\infty}^{\infty} e^{-r|n|/a + inx_5/a}$$

The sum is convergent and not hard to evaluate explicitly, but since we are interested in the physical situation where we are trapped on the brane and cannot explore the x_5 direction, we set $x_5 = 0$. The effective gravitational potential due to the extra dimension is then

$$\phi_{\text{eff}}(\vec{r}) = -\frac{G_5 M}{2\pi a r} \sum_{n=-\infty}^{\infty} e^{-r|n|/a}$$

But, observe that the sum is geometric and we have

$$\sum_{n=-\infty}^{\infty} e^{-r|n|/a} = \coth\left(\frac{r}{2a}\right)$$

So that our effective potential is

$$\phi_{\text{eff}}(\vec{r}) = -\frac{G_5 M}{2\pi a r} \coth\left(\frac{r}{2a}\right)$$

Let us first consider the limit $r >> a$. As $r/a \to \infty$, the hyperbolic cotangent approaches 1, so our effective potential goes to

$$-\frac{G_5 M}{2\pi a r}$$

which we identify as the familiar Newtonian potential under the identification $G_5 = 2\pi a G$. This relation can also be derived by looking at the gravitational Lagrangian. We then have

$$\phi_{\text{eff}}(\vec{r}) = -\frac{GM}{r} \coth\left(\frac{r}{2a}\right)$$

Let us consider the other limit, the extreme short distance limit $r << a$. As $x \to 0$, $\coth(x) \sim 1/x$ (show this), so in good approximation we can make this replacement, giving

$$\phi_{\text{eff}}(\vec{r}) \sim -\frac{2GMa}{r^2}$$

The inverse square law breaks down at short distances! What is important to remember here is that this is the *effective* potential, valid outside of the extra dimensions. In particular, *we do not need to directly probe the extra dimensions to observe their physical consequences.* Also, note that when expressed in terms of G_5, this becomes exactly the expression derived in the previous section for four noncompact dimensions. This is as expected, because on distance scales much smaller than its radius, the circle can be approximated by a line.

Furthermore, if so desired, we can obtain further corrections in the $r >> a$ limit. In practice, it would be these corrections that would be observed. I leave it to you to show that these corrections come in powers of $e^{-r/a}$. The leading correction is

$$\phi_{\text{eff}}(\vec{r}) = -\frac{GM}{r} - \frac{2GM}{r} e^{-r/a}$$

While gravitation is a long-range interaction, we see that the effect of the extra dimensions in this approximation is to induce short-range corrections which fall off exponentially with distance. Potentials of the form $\frac{1}{r}e^{-mr}$ are known as *Yukawa potentials*, after the physicist Yukawa who proposed a potential of this form to describe the nuclear force. The idea was that the nucleus of the atom is bound together by a force which decays exponentially quickly, explaining the atomic structure. Yukawa potentials arise when the force-carrying particle associated to an interaction is massive, the mass m being the coefficient of r in the exponent (this is in units where $\hbar = c = 1$, so that mass has dimension of inverse length). Coulomb/Newtonian potentials arise from massless exchange particles. Interestingly enough, we see that the potential in this limit gets an infinite series of Yukawa corrections, corresponding to exchange particles with masses $m_n = n/a$. Infinite sets of particles appearing in this manner is a common feature of Kaluza-Klein compactification, and such sets are known as Kaluza-Klein towers. These give another way to experimentally hunt for evidence of extra dimensions: at a particle collider, when we investigate energies of scale $1/a$ (remember, in natural units), we should detect new particles with masses showing up in this pattern. Such towers have never been observed to date. This is not so much evidence that there are no extra dimensions as that the extra dimensions cannot be as simple as just one circle.

Finally, I present one more derivation of the effective potential which uses the method of images and Poisson summation. The complete 4-dimensional Poisson equation for a point mass M is again

$$\nabla^2\phi(\vec{r}, x_5) = 4\pi G_5 M \delta^{(3)}(\vec{r})\delta(x_5)$$

I will now do the following trick: suppose x_5 is not a compact direction. Then this simply reduces the the usual Poisson equation in $d = 4$, which we solved in complete generality in the last section. The solution is

$$\phi(\vec{r}, x_5) = -\frac{G_5 M}{\pi}\frac{1}{r^2 + x_5^2}$$

Here, I have written out the four-dimensional $R^2 = r^2 + x_5^2$ explicitly. Here is the clever part: to impose periodic boundary conditions (period $2\pi a$) in the x_5 direction, I can simply add image masses at $x_5 = 2\pi a n$ for all $n \in \mathbb{Z}$. If you do not understand this, draw a picture. The key idea is that a guy walking down a line which looks identical every $2\pi a$ meters is physically equivalent to a guy on a circle of circumference $2\pi a$. If I am interested in the effective potential at $x_5 = 0$, I simply have to superpose the contributions from the image masses (since the Poisson equation is linear). Then, we find the effective potential

$$\phi_{\text{eff}}(\vec{r}) = -\frac{G_5 M}{\pi}\sum_{n=-\infty}^{\infty}\frac{1}{r^2 + (2\pi a n)^2}$$

Factoring out the $(2\pi a)^2$, this sum becomes

$$\phi_{\text{eff}}(\vec{r}) = -\frac{G_5 M}{\pi}\frac{1}{4\pi^2 a^2}\sum_{n=-\infty}^{\infty}\frac{1}{n^2 + (\frac{r}{2\pi a})^2}$$

But we already evaluated a sum like this. Just use the Poisson formula! The sum is

$$\sum_{n=-\infty}^{\infty}\frac{1}{n^2 + (\frac{r}{2\pi a})^2} = \frac{2\pi^2 a}{r}\sum_{n=-\infty}^{\infty} e^{-r|n|/a}$$

Pleasingly, the Poisson formula relates the sum over image masses to the sum over Fourier components. Such occurrences (two physically equivalent pictures being related by a Poisson summation) are very common in string theory. Doing the geometric sum and substituting, we find again

$$\phi_{\text{eff}}(\vec{r}) = -\frac{G_5 M}{2\pi a r} \coth\left(\frac{r}{2a}\right)$$

This brings us to the conclusion of our discussion of large extra dimensions. We were able to give quite a thorough discussion of the classical aspects of the ideas. Obviously, we ignored all quantum and relativistic aspects, since there were no $\hbar$'s or c's in our equations. Remarkably, Fourier analysis alone contains much of the rich structure of the extra dimensions. We saw a mechanism for the modification of the inverse square law and the existence of the Kaluza-Klein towers. What we ignored was the possibility of the extra dimensions forming a more interesting shape than a circle. The spaces typically studied in string theory compactifications are drastically more complicated and rich in structure, and are known as Calabi-Yau manifolds. For more on these matters, a more specialized text should be consulted.

6 Appendix: Some Mathematical Niceties

We will discuss integrals and terminology that may be unfamiliar to some of the audience.

The basic integral of interest is the Gaussian

$$I = \int_{-\infty}^{\infty} dx e^{-x^2}$$

Note that

$$I^2 = \int_{-\infty}^{\infty} dx e^{-x^2} \cdot \int_{-\infty}^{\infty} dx e^{-x^2}$$

However, the value of the integral will be the same regardless of the name of the integration variable, so one finds

$$I^2 = \int_{-\infty}^{\infty} dx e^{-x^2} \cdot \int_{-\infty}^{\infty} dy e^{-y^2}$$

Now, we consider this as a double integral and have

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy e^{-(x^2+y^2)}$$

Now, change to polar coordinates (r, θ), so that the integral becomes

$$I^2 = \int_0^{2\pi} \int_0^{\infty} r dr d\theta e^{-r^2}$$

Now, observe that nothing depends on θ, so this integral factors out:

$$I^2 = \int_0^{2\pi} d\theta \int_0^{\infty} dr r e^{-r^2}$$

The last integral can be evaluated by the substitution $u = r^2$, so we find

$$I^2 = (2\pi)\left(\frac{1}{2}\right) = \pi$$

So that

$$\int_{-\infty}^{\infty} dx e^{-x^2} = \sqrt{\pi}$$

Next, observe that by scaling $x \to \sqrt{a}x$ with $a > 0$, we have

$$\sqrt{a} \int_{-\infty}^{\infty} dx e^{-ax^2} = \sqrt{\pi}$$

so that

$$\int_{-\infty}^{\infty} dx e^{-ax^2} = \sqrt{\frac{\pi}{a}}$$

Next, consider the integral

$$\int_{-\infty}^{\infty} dx e^{-ax^2+bx}$$

The trick is to complete the square in the exponent. We have

$$-ax^2 + bx = -a\Big(x^2 - \frac{b}{a}x\Big) = -a\Big(x^2 - \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\Big) = -a\Big(x - \frac{b}{2a}\Big)^2 + \frac{b^2}{4a}$$

The integral is then

$$e^{\frac{b^2}{4a}} \int_{-\infty}^{\infty} dx \exp\Big\{-a\Big(x - \frac{b}{2a}\Big)^2\Big\}$$

If it isn't already obvious, the notation $\exp\{\text{blah}\}$ just means e^{blah}, but is more convenient for long and complicated exponents. Next, shift the integration variable $x \to x + \frac{b}{2a}$, so that we have

$$e^{\frac{b^2}{4a}} \int_{-\infty}^{\infty} dx e^{-ax^2} = e^{\frac{b^2}{4a}} \sqrt{\frac{\pi}{a}}$$

In summary,

$$\int_{-\infty}^{\infty} dx e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}$$

This is the main formula we need for Gaussian integrals.

It was brought to my attention that not all readers may be familiar with some of the mathematical jargon, so here is a quick (and highly non-rigorous) glossary:

A function is technically *smooth* over the domain D if it is infinitely differentiable over D. However, in physics, smooth is typically taken to mean differentiable as many times as is necessary for the application. Also, if D is clear from context, we just say the function is smooth.

A function is *integrable* on $[a, b]$ if $\int_a^b dx f(x)$ exists (i.e. is not infinite). If $[a, b]$ is clear from context, we simply omit it. The domain does matter, though, because $f(x) = \frac{1}{x^2}$ is integrable on $[1, \infty)$, but not on $[-1, \infty)$.

A function is *singular* at the point x_0 if $\lim_{x \to x_0} f(x)$ diverges. "Sufficiently nonsingular" is synonymous with "smooth enough".

A function is *bounded* on some domain D if there is a real number M such that $|f(x)| \leq M$ for all $x \in D$. Basically, bounded functions have no singularities (but they can have discontinuities).

A function is *analytic* in some domain D if it has a Taylor series which converges to its value at each $x \in D$.

The Gamma Function

We begin with the identity

$$\frac{1}{a} = \int_0^{\infty} dx e^{-ax}$$

The easiest way to prove this is to simply evaluate the integral. Now, differentiate both sides with respect to a. We find

$$\frac{1}{a^2} = \int_0^{\infty} dx x e^{-ax}$$

Do it again to obtain

$$\frac{2}{a^3} = \int_0^\infty dx x^2 e^{-ax}$$

Eventually, you will see the pattern

$$\frac{(n-1)!}{a^n} = \int_0^\infty dx x^{n-1} e^{-ax}$$

However, we notice that the right hand side exists as an analytic function of n for any $n > 0$, so we *define* the Gamma function $\Gamma(s)$ by the relation

$$\frac{\Gamma(s)}{a^s} = \int_0^\infty dx x^{s-1} e^{-ax}$$

By setting $a = 1$, one obtains the standard integral definition of the Gamma function. The Gamma function analytically continues the factorial function to noninteger arguments, and $\Gamma(n) = (n-1)!$ for n integer. One can show that

$$\Gamma(s+1) = s\Gamma(s)$$

by integrating by parts. The gamma function as given here is only well-defined for $\operatorname{Re} s > 0$, and has a singularity at $s = 0$.

To obtain the behavior of $\Gamma(s)$ for small s, we use the following trick. Break up the integral defining $\Gamma(s)$ as

$$\Gamma(s) = \int_0^1 dx x^{s-1} e^{-x} + \int_1^\infty dx x^{s-1} e^{-x}$$

The second integral converges for all s, so we will restrict our attention to the first. Observe that on the interval $[0, 1]$, it is legal to replace e^{-x} by its power series and interchange orders of summation and integration for $\operatorname{Re} s > 0$ (we are integrating an absolutely and uniformly convergent series term by term, so it is virtually impossible to run into problems). We find that we need the sum

$$\sum_{n=0}^\infty (-1)^n \int_0^1 dx \frac{x^{n+s-1}}{n!}$$

The integral becomes trivial, and we find that

$$\Gamma(s) = \sum_{n=0}^\infty \frac{(-1)^n}{n!(n+s)} + \int_1^\infty dx x^{s-1} e^{-x}$$

Let us now consider $\Gamma(s)$ as $s \to 0$. All terms besides the first in the series approach finite limits which constitute a convergent series that adds up to some number that we presently don't care about. Furthermore, we can send $s \to 0$ in the integral and it will still converge, giving another finite number that we don't care about. The only problem child is the first term in the series, so we find

$$\Gamma(s) = \frac{1}{s} + \mathcal{O}(1)$$

as $s \to 0$. We now want the rest of the behavior. We notice that $\Gamma(s) - s^{-1}$ will be finite, so that we can consider

$$\Gamma(s) - \frac{1}{s} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(n+s)} + \int_1^{\infty} dx x^{s-1} e^{-x}$$

In the limit $s \to 0$, we find $\Gamma(s) - s^{-1}$ goes to

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2(n-1)!} + \int_1^{\infty} \frac{dx}{x} e^{-x}$$

The integral and the series clearly both converge, and their sum is equal to a number called $-\gamma_E$ known as the Euler-Mascheroni constant. Since further deviations must be subleading to a constant, we can assert that

$$\Gamma(s) = \frac{1}{s} - \gamma_E + \mathcal{O}(s)$$

as $s \to 0$.

Logarithm

Begin with the obvious identity

$$\ln\left(\frac{b}{a}\right) = \ln b - \ln a = \int_a^b dx \frac{1}{x}$$

Now, write

$$\frac{1}{x} = \int_0^{\infty} dt e^{-tx}$$

So we find that

$$\ln\left(\frac{b}{a}\right) = \int_a^b \int_0^{\infty} dx dt e^{-tx}$$

Now, do the integral over x to find

$$\ln\left(\frac{b}{a}\right) = \int_0^{\infty} \frac{dt}{t}(e^{-at} - e^{-bt})$$

This integral representation is useful for many purposes.

I did not actively consult any references when writing this, but here are some resources which may be useful for further reading (and, since I have seen them before, may have inadvertently been stolen from).

References

[1] D. Morin. *Waves*. This is currently unpublished, but lives on Morin's website here. It contains a very nice discussion of Fourier analysis in chapter 3, though the development is somewhat orthogonal to that taken here. As the title suggests, the central focus here is the physical picture regarding waves. It also discusses some things I omitted like the Gibbs phenomenon.

[2] A. Schoenstadt's lecture notes on the subject cover many of its applications to classical mathematics in all their gory detail. If you want to see the material I skipped about explicitly calculating Fourier series, solving equations with various boundary conditions, etc. then this is a good place to go.

[3] A. Zee. *Einstein Gravity in a Nutshell*. Princeton University Press, 2013. This is an amusing account of Einstein gravity at a level accessible to undergraduate students. The section in these notes on large extra dimensions is essentially an expanded version of certain aspects of the chapter on the brane world in this book, using Fourier theory to work out some details.

[4] B. Zwiebach. *A First Course in String Theory*. Cambridge University Press, 2015. This is an undergraduate-level book on string theory which also contains some information relevant to the section on large extra dimensions (I believe it contains an accurate account of the experimental situation regarding the inverse square law). It's a good book for getting a basic quantitative acquaintance with some of the ideas, but it will by no means transform you into a string theorist. It is impressive how much mileage Zwiebach covers given the background he assumes.