Lecture Notes: Supersymmetry and Morse Theory

Spencer Tamagni

Spring 2019

Contents

Introduction

The following are lecture notes on supersymmetric quantum mechanics and Morse theory given in a mini-lecture series in the spring of 2019. The audience was advanced undergraduates exposed to quantum mechanics and general relativity. I try to use modern mathematical language, but do not make an attempt at mathematical precision–the presentation is informal and very much in the "physicist's style". Topics treated include:

the basic theoretical foundation of quantum mechanics, path integral formulation, examples of path integrals, Grassmann algebra and fermions, supersymmetry, and finally the connection with Hodge theory and Morse theory. Some exercises are included at the end. The presentation in the main body deals with mostly generalities, leaving the specific details to exercises. This means the exercises may be somewhat challenging, but should be instructive. These notes are still incomplete (I need to add references) but should be mostly readable.

1 Part I: Quantum Mechanics and Path Integrals

The three most important notions in physics are *physical states*, *symmetries*, and *dynamics*. In this section, we review their definitions in both classical and quantum mechanics. We then introduce the path integral formulation of quantum mechanics and do several examples. We discuss the interrelation of the path integral with canonical quantization.

1.1 Classical Mechanical Systems

Here, we will review the definition of a classical system, focusing on the Lagrangian and Hamiltonian formalisms.

1.1.1 Lagrangian

In the Lagrangian picture, you are given two pieces of data:

- A space M with points labeled by some coordinates q^i, $i = 1, ..., n$. M is called the configuration space and q^i the generalized coordinates. The number $n = \dim M$ is the number of degrees of freedom of the system. For example, for a single particle moving on the sphere $M = S^2$ and the q's are (θ, ϕ).
- A functional $S = \int dt L(q^i(t), \dot{q}^i)$ on the space of paths in M. S is known as the action and L is the Lagrangian. A choice of Lagrangian and time interval for evolution is equivalent to a choice of action. We ignore time-dependent Lagrangians and Lagrangians depending on more than the first derivative.

With this data, M and S, by imposing the equation $\delta S = 0$ we obtain the equations of motion for the system:

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}^i}\Big) = 0$$

For any reasonable L, these are well-behaved and uniquely specify the dynamics. From this broad-brush point of view, this is all there is to say about the Lagrangian formalism. At the practical level, there are various advantages and disadvantages to take into account, but our focus here is simply *how the Lagrangian formulation converts input data into dynamics*.

1.1.2 Hamiltonian

For abstract theoretical considerations, it is often useful to switch from the Lagrangian formalism to the Hamiltonian formalism. The basic idea is that we want a formulation of mechanics which places coordinates and momenta on equal footing. The reasons that this is a good idea will become clear as we continue on.

The canonical momentum is defined as

$$p_i = \frac{\partial L}{\partial \dot{q}^i}$$

With this definition of momentum, we want to switch from the Lagrangian $L(q^i, \dot{q}^i)$ to a new function $H(q^i, p_i)$, the Hamiltonian. There is a general procedure to do this called the Legendre transform. I should comment as to why we Legendre transform instead of just doing the obvious thing and inverting the relationship between $\dot{q}$ and p at fixed q to write $\dot{q} = \dot{q}(q, p)$. The reason is because this does not simplify the equations. The Euler-Lagrange equations are still second order equations naturally expressed in terms of position variables and their derivatives. If we add momentum, it will just be a lot of mixing of position and momenta with even more partial derivatives. The way to cure this problem is to construct a function which only depends on the values of q and p by themselves, not implicitly through the value of $\dot{q}$.

To make it simple, we will start with a single degree of freedom q. In this case, we simply have a function of two independent variables $L(q, \dot{q})$. We want to trade the dependence on $\dot{q}$ for the variable $p = \frac{\partial L}{\partial \dot{q}}$. Consider the function $g(q, \dot{q}, u) = u\dot{q} - L(q, \dot{q})$. We can take its differential

$$dg = \dot{q}du + ud\dot{q} - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial \dot{q}}d\dot{q}$$

We want to neutralize its dependence on $\dot{q}$. We see that if we choose $u = \frac{\partial L}{\partial \dot{q}}$, we can pull this off. We then define the function $H(q, p) = g(q, \dot{q}(p), p)$. It is important that we are assuming that the relation $p = \partial_{\dot{q}} L$ is invertible. This will be the case if $\partial^2_{\dot{q}} L \neq 0$. H is called the Legendre transform of L, and it is clear that this procedure would have worked for any two independent variables; we just happened to name them q and $\dot{q}$. The magic of the Legendre transform is that if we write $H(q, p) = p\dot{q} - L(q, \dot{q})$, nothing actually depends on $\dot{q}$, only p, and we have not lost any information. The generalization to many variables is $H(q^i, p_i) = p_i\dot{q}^i - L(q^i, \dot{q}^i)$, and it is simple to verify this as an exercise. It is important at this point to note that I am using the summation convention where all repeated indices are summed unless otherwise mentioned. The invertibility condition becomes $\det \partial^2_{\dot{q}^i \dot{q}^j} L \neq 0$. By writing out both sides of the differential $dH = d[p_i\dot{q}^i - L]$ (note the $d\dot{q}$ cancellations on the RHS by construction), and using the Euler-Lagrange equations, we obtain Hamilton's equations

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}$$

If we accept the Hamiltonian point of view, a classical system is specified by two pieces of data:

- A space $\mathcal{M}$ parameterized by the coordinates (q, p). This is known as the phase space. For our purposes, it is important that the phase space consists of a natural notion of division between coordinates and momenta. We will return to this point later.
- A function H on $\mathcal{M}$. By imposing Hamilton's equations, we specify the dynamics of the system.

The reason that the Hamiltonian picture is worth anything is, in many respects, because of the extra structure carried by $\mathcal{M}$ relative to the configuration space. Importantly, Hamilton's equations are first order, and therefore define a flow or vector field on $\mathcal{M}$. This means that as soon as we specify an initial condition for Hamilton's equations, we have a unique solution (at least for sufficiently small times, that is, locally in $\mathcal{M}$). This means that there is a one-to-one correspondence between points in $\mathcal{M}$ and solutions of Hamilton's equations (the points in $\mathcal{M}$ give the initial conditions). Hence, $\mathcal{M}$ may also be characterized as the *space of solutions to the classical equations of motion*. Another way of saying this is that paths in phase space do not cross, at least locally.

For the same reason, $\mathcal{M}$ may be characterized as the *space of physical states of the system*. Given a point (q, p) in $\mathcal{M}$, we know the fate of the system for all later time: we have a unique solution to the classical equations of motion. This is an important feature of $\mathcal{M}$ which differentiates it from M: a point in M plus the equations of motion does not fully specify the dynamics. We need either a final condition or initial velocity. By contrast, once we know (q, p) at one time we know $(q(t), p(t))$. For this reason, we define the pair (q, p) as the *physical state* of the classical system: as soon as we are given an H, we can construct the dynamics and the initial state contains all possible future information about the system.

A big take-home message is that a classical system is not specified by H alone: you need a space of states $\mathcal{M}$ for the dynamics to take place in.

1.1.3 Symmetries in the Lagrangian Formalism: Noether's Theorem

So far, we have discussed the classical notions of physical state (an element of phase space) and dynamics (equations of motion). We have not yet given a definition of symmetry. We turn to this issue now.

First, recall that two Lagrangians L and L' are equivalent if they differ by a total derivative $L' = L + \frac{df}{dt}$ where $f(q)$ is some function of the variables. This is because when we vary the action, the function f contributes only boundary terms and therefore is ignorable. We therefore define a *symmetry* of the Lagrangian as a transformation of the variables $q^i \to q'^i$ such that L is invariant, up to a possible total derivative. We also call this transformation a symmetry of the system. In many cases, there is no total derivative, but sometimes there is.

A central result in classical mechanics is Noether's theorem. This says that to each continuous symmetry of the system, there exists an associated conserved quantity, that is a function Q conserved along the motion. Since the symmetry is continuous, that means it depends on some continuous variable ϵ and we may write the transformed variables as $q^i_\epsilon = q^i + \epsilon\delta q + \mathcal{O}(\epsilon^2)$. Note that we technically assume differentiability, but who

cares. Note also that δq is not small, the smallness has been absorbed into ϵ. δ is like a differential operator acting on the space of q's.

The proof of the statement is a straightforward computation. In all equations, I work to linear order in ϵ. Since we have a symmetry, we have $L(q_\epsilon, \dot{q}_\epsilon) = L(q, \dot{q}) + \epsilon \frac{dK}{dt}$. Note that the second term is a result of the fact that we can take the infinitesimal limit of the total derivative term, and it must vanish at $\epsilon = 0$. Now, we may write

$$L(q_\epsilon, \dot{q}_\epsilon) = L(q, \dot{q}) + \epsilon \frac{\partial L}{\partial q^i} \delta q^i + \epsilon \frac{\partial L}{\partial \dot{q}^i} \delta \dot{q}^i$$

Comparing the order ϵ terms we find

$$\frac{\partial L}{\partial q^i} \delta q^i + \frac{\partial L}{\partial \dot{q}^i} \delta \dot{q}^i = \frac{dK}{dt}$$

Doing the standard trick,

$$\frac{\partial L}{\partial \dot{q}^i} \delta \dot{q}^i = \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}^i} \delta q^i\Big) - \frac{d}{dt}\Big(\frac{\partial L}{\partial q^i}\Big) \delta q^i$$

We find

$$\Big(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}^i}\Big)\Big) \delta q^i + \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}^i} \delta q^i\Big) = \frac{dK}{dt}$$

Now, using the EOM and subtracting everything to one side we find

$$Q = \frac{\partial L}{\partial \dot{q}^i} \delta q^i - K$$

is a conserved quantity. It is sometimes called the Noether charge. This proves the theorem.

Standard examples are rotations and translations. Q reduces to linear and angular momentum, respectively. In the case of time translations, Q is the energy.

I want to emphasize the difference between symmetry and form invariance. It is often said that the Euler-Lagrange equations are invariant under general coordinate transformations of the q's. This is geometrically obvious from their interpretation as the condition for a critical point of an action functional. This is not a symmetry, but it is a form invariance. It does not preserve the functional form of the Lagrangian, just the general form of the Euler-Lagrange equations. Symmetries have to preserve the functional form of the Lagrangian upon change of variables (modulo total derivatives).

1.1.4 Symmetries in the Hamiltonian Formalism: Poisson Bracket

Symmetries really come to life in the Hamiltonian way of doing things. This is one of the advantages of phase space.

Take some function on phase space $f(q, p)$. For example, it could be the z component of angular momentum, or total momentum, or whatever you want. We can consider its time dependence along the classical evolution $f(q(t), p(t))$:

$$\frac{df}{dt} = \frac{\partial f}{\partial q^i} \dot{q}^i + \frac{\partial f}{\partial p_i} \dot{p}_i = \frac{\partial f}{\partial q^i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q^i} := \{f, H\}$$

In the last equality, we have defined the Poisson bracket of two functions $\{f, g\}$. The Poisson bracket satisfies many neat properties. For example

- Antisymmetry $\{f, g\} = -\{g, f\}$
- Linearity $\{f, ag + bh\} = a\{f, g\} + b\{f, h\}$
- Leibniz rule $\{f, gh\} = \{f, g\}h + g\{f, h\}$
- Jacobi Identity $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$

These are all easy to verify by straightforward algebra. Observe that all of these formal algebraic properties are satisfied by the matrix commutator $[A, B] = AB - BA$.

The fundamental Poisson brackets are $\{q^i, q^j\} = \{p_i, p_j\} = 0$ and $\{q^i, p_j\} = \delta^i_j$. There is also a neat way to express conservation: it is $\{f, H\} = 0$. One says that f and H Poisson commute.

Here is another neat interpretation of Poisson commuting. Suppose that a function $G(q, p)$ Poisson commutes with the Hamiltonian:

$$\frac{\partial H}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q^i}\frac{\partial H}{\partial p_i} = 0$$

Consider the infinitesimal transformation

$$q'^i = q^i + \epsilon\frac{\partial G}{\partial p_i}$$

$$p'_i = p_i - \epsilon\frac{\partial G}{\partial q^i}$$

The Hamiltonian transforms as $H(q', p') = H(q, p) + \epsilon\{H, G\} = H(q, p)$, i.e. it is invariant. Similarly, the variation of any function f is $\delta f = \{f, G\}$. One says G *generates* the transformation δ. We recover Noether's theorem: conservation is the same as Poisson commuting with H, which ensures $\delta H = 0$. One says that the Noether charges generate the transformations which preserve the Hamiltonian. By direct computation, one can also verify that the (q', p') satisfy the same Poisson bracket algebra. Transformations which preserve the Poisson brackets are called *canonical transformations*. A good example to keep in mind in these type of discussions is the fundamental Poisson bracket $\{q^i, p_j\} = \delta^i_j$, which says that momentum generates a translation: it is the conserved quantity associated with a translation.

We see the following structure in the Hamiltonian picture: conserved quantities play double duty as constants of motion but also as the generators of canonical transformations. Canonical transformations may be defined totally generally, and since Hamilton's equations only depend on the Poisson bracket they leave the form of the equations invariant. However, as emphasized earlier, form invariance is not a symmetry. We only have a symmetry when the generator G Poisson commutes with H.

1.2 Canonical Quantization

The only point in being this formal about classical mechanics is so that we can emphasize the similarities with quantum mechanics. We will begin by doing quantization in the

Hamiltonian formalism, also known as canonical quantization. I will begin with the formal aspects, and explain how it all reduces to familiar information.

1.2.1 Hilbert Space

We begin with the concept of Hilbert space. A Hilbert space $\mathcal{H}$ is simply a complex vector space with an inner product. It may or may not be infinite dimensional. We denote the vectors in $\mathcal{H}$ by kets like $|\psi\rangle \in \mathcal{H}$. This section is largely a recap of notation: the linear algebra notions should be familiar.

We can now adapt much of good old linear algebra to this new notation. Recalling the notion of dual space, we denote the elements of the dual to $\mathcal{H}$ by bras, so a linear function on $\mathcal{H}$ is denoted by $\langle\alpha|$ and we manipulate it as follows: $\langle\alpha| (a|\psi_1\rangle + b|\psi_2\rangle) = a\langle\alpha|\psi_1\rangle + b\langle\alpha|\psi_2\rangle$. $\langle\alpha|\psi\rangle \in \mathbb{C}$ denotes the image of $|\psi\rangle$ under $\langle\alpha|$.

Let $|e_i\rangle$ be a set of basis vectors in $\mathcal{H}$. They may be finite (if $\mathcal{H}$ is finite dimensional) or countably infinite. We then write vectors as $|\psi\rangle . = \psi_k |e_k\rangle$. The dual basis in $\mathcal{H}^*$ is denoted as $\langle\varepsilon_i|$ and satisfies $\langle\varepsilon_i|e_j\rangle = \delta_{ij}$. In the usual way we have $\langle\alpha|\psi\rangle = \alpha_k\psi_k$.

We define the inner product implicitly as an isomorphism between $\mathcal{H}$ and $\mathcal{H}^*$. To an element $|\psi\rangle = \psi_k|e_k\rangle$ in $\mathcal{H}$, we define an associated element of $\mathcal{H}^*$ by $\langle\psi| = \psi_k^*\langle\varepsilon_k|$ We then define an inner product as $(\phi,\psi) = \langle\phi|\psi\rangle = \phi_k^*\psi_k$. This satisfies all the required properties of inner product. The norm $\langle\psi|\psi\rangle$ is real and nonnegative. With respect to this inner product, the basis $|e_i\rangle$ is orthonormal essentially by construction.

Suppose we have a vector $|\psi\rangle = \psi_k|e_k\rangle$. One has $\langle e_k|\psi\rangle = \psi_k$ by multiplying by $\langle e_k|$ on the left. We then find $|\psi\rangle = |e_k\rangle\langle e_k|\psi\rangle$. Now, regarding $\langle e_k|$ as a linear functional on ψ, we can factor out $|\psi\rangle$ and find

$$I = |e_k\rangle\langle e_k|$$

This holds independently of the inner product, just replace $\langle e_k|$ with the dual basis. It is called a completeness relation.

1.2.2 Principles of Quantum Mechanics

We now can give the formal rules for constructing a quantum dynamical system from a classical one. They are:

- There is a Hilbert space $\mathcal{H}$ and physical states are defined to be vectors $|\psi\rangle \in \mathcal{H}$.
- A physical quantity A in classical mechanics is replaced by a Hermitian operator $\hat{A}$ acting on $\mathcal{H}$. These operators are called observables. The result obtained from measurement of $\hat{A}$ will be one of its eigenvalues.
- The Poisson bracket in classical mechanics is replaced by the commutator, $\{A,B\} \to -(i/\hbar)\left[\hat{A},\hat{B}\right]$. What this means is that the operator corresponding to the function $\{A,B\}$ is the commutator of the two operators. Hamilton's equations are promoted to Heisenberg's equations, for a time dependent operator $\hat{A}(t)$:

$$\frac{d\hat{A}}{dt} = -\frac{i}{\hbar}\left[\hat{A},\hat{H}\right]$$

Here, $\hat{H}$ is the Hamiltonian operator.

- The expectation value of $\hat{A}(t)$ in the state $|\psi\rangle$ is

$$\langle A(t)\rangle = \frac{\langle\psi|\,\hat{A}(t)\,|\psi\rangle}{\langle\psi|\psi\rangle}$$

Several comments are now in order:

- It is strictly not true that states are vectors in $\mathcal{H}$. They are rays, that is equivalence classes of vectors under the $\mathbb{C}^*$ action $|\psi'\rangle \sim |\psi\rangle$ iff $|\psi'\rangle = c\,|\psi\rangle$ for $c \in \mathbb{C}^*$.
- Axiom 3 is impossible to satisfy in general. This is because of operator ordering problems. If we assume axiom 3 holds for the fundamental commutators $[q^i, p_j] = i\hbar\delta^i_j$, then q and p are no longer commuting objects and functions like qp^2 are no longer well-defined upon quantization. Is it qp^2 or pqp or p^2q? There is no natural choice, and furthermore different choices lead to discrepancies in the commutators which show up at higher order in $\hbar$. It is easy to see why: there is one $\hbar$ coming from the classical answer, but the extra terms come from the noncommutativity of q and p, which comes with an $\hbar$ itself. The resolution to this is that we impose the commutators for q and p and then make the calls on the other operators according to physical requirements. For example, requiring that a certain algebra closes can resolve ordering ambiguities, or requiring that the expected value of an operator vanish in a particular state.
- We can have quantum mechanical systems which have no classical analogs. The internal color degrees of freedom of a quark are a good example of this. A less abstract example is the spin degree of freedom of a particle. This Hilbert space is finite-dimensional and we do not "quantize a classical Hamiltonian", we start with the operator algebra. Sometimes this is useful.
- I am not going to attempt to interpret quantum mechanics or explain why these principles are here. These are simply the rules that we follow.

1.2.3 Discrete vs. Continuous

Suppose some observable $\hat{A}$ has a discrete spectrum of eigenvalues a_n, $\hat{A}\,|n\rangle = a_n\,|n\rangle$, where we take the eigenvectors as orthonormal. Any state may be expanded $|\psi\rangle = \psi_n\,|n\rangle$, with $\psi_n = \langle n|\psi\rangle$. The expectation is $\langle\psi|\,\hat{A}\,|\psi\rangle = \sum_n a_n|\psi_n|^2$ It follows that the probability to be in $|n\rangle$ is $|\psi_n|^2$ (assuming $|\psi\rangle$ is normalized to unity).

If the spectrum of $\hat{A}$ is continuous with eigenvectors $|a\rangle$ (a unique possibility of infinite dimensions), we follow our physicist intuition and promote sums to integrals:

$$|\psi\rangle = \int da\psi(a)\,|a\rangle$$

The coefficients $\psi(a) = \langle a|\psi\rangle$ are called the wavefunction in the a basis. The completeness relation is $\int da\,|a\rangle\,\langle a| = I$, which implies the identity $\int da'\,|a'\rangle\,\langle a'|a\rangle = |a\rangle$. This means these states must be normalized as $\langle a|a'\rangle = \delta(a - a')$. Note that as $\delta(0)$ blows up these are strictly not normalizable: cue discussion about rigged Hilbert space etc.

1.2.4 Heisenberg vs. Schrödinger

Unlike Hamilton's equations, which deal with crazy nonlinear functions on a phase space, Heisenberg's equation of motion deals with nice linear operators acting on a linear space. As such, we can solve them (formally!) in closed form. Observe that, if $\hat{A}(0) = \hat{A}$ is our initial condition, the solution is

$$\hat{A}(t) = e^{iHt/\hbar}\hat{A}e^{-iHt/\hbar}$$

That this is the only solution follows from existence and uniqueness of ODE's (that is, if we are brave and assume it works for operator-valued functions). We note that time evolution is a similarity transformation generated by the operator $U(t) = e^{-iHt/\hbar}$. I am dropping hats since we have abandoned the classical world and everything is a quantum operator now. This formalism, with time dependent operators and time-independent states, is called the Heisenberg picture.

We can pass to another equivalent formalism where the states depend on time and the operators do not. The basic idea is that the physical state $|\psi\rangle$ is (counterintuitively) not a physical observable in quantum mechanics. The only stuff we can measure are expectation values and overlaps (matrix elements); these give us the relevant probabilities. If we have $\langle A(t)\rangle = \langle\psi| e^{iHt/\hbar}Ae^{-iHt/\hbar} |\psi\rangle = (\langle\psi| e^{iHt/\hbar})A(e^{-iHt/\hbar} |\psi\rangle)$. If we now define the time-dependent state $|\psi(t)\rangle = e^{-iHt/\hbar} |\psi\rangle$, we have $\langle A(t)\rangle = \langle\psi(t)| A |\psi(t)\rangle$. This formalism is the Schrödinger picture. We can describe the dynamics in the Schrödinger picture by differentiating $|\psi(t)\rangle$ using its definition, which gives

$$i\hbar\frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle$$

This is the Schrödinger equation. It simply says that H generates time evolution.

Note that time evolution in the Schrödinger picture is unitary: it preserves the Hilbert space norm. Hence, normalizing to unity is a dynamically sensible thing to do.

1.2.5 Particle on a Line

To make full contact with elementary quantum mechanics, we reconstruct wavefunctions on the real line $\mathbb{R}$. According to our axioms, we promote position x and momentum p to operators on a Hilbert space $\mathcal{H}$. Clearly, the observables of position are all real numbers, with eigenstates $\hat{x} |x\rangle = x |x\rangle$. As before, the states are normalized as $\langle x|y\rangle = \delta(x-y)$ so that the completeness relations work. For momentum eigenstates we also have $\hat{p} |p\rangle = p |p\rangle$ and $\langle p|q\rangle = \delta(p-q)$. We have $\psi(x) = \langle x|\psi\rangle$ is the component of $|\psi\rangle$ in the $|x\rangle$ basis, the wavefunction. For $\langle\psi|\psi\rangle$ to have finite norm, we must have $\int dx|\psi(x)|^2 < \infty$. That is, the Hilbert space for a single particle on a line is just $L^2(\mathbb{R})$, the space of square-integrable functions (of course, we have to include delta functions for completeness etc etc). This follows simply from the fact that position has a continuous spectrum and the axioms of quantum mechanics. Similarly, $\psi(p) = \langle p|\psi\rangle$ is the probability amplitude of having momentum p. Inner products can be evaluated using completeness relations:

$$\langle\phi|\psi\rangle = \langle\phi| \int dx |x\rangle \langle x| |\psi\rangle = \int dx\phi^*(x)\psi(x)$$

This is the usual L^2 inner product. In the momentum basis it is $\int dp\phi^*(p)\psi(p)$.

Moreover, we can recover the actions of $\hat{x}$ and $\hat{p}$ in their basis. We have $\hat{x}\,|x\rangle = x\,|x\rangle$, which implies $\langle x|\,\hat{x} = \langle x|\,x$, and so $\langle x|\,\hat{x}\,|\psi\rangle = x\psi(x)$. This is written in a more elementary notation as $(\hat{x}\psi)(x) = x\psi(x)$. The action of p in its own basis is also pretty trivial.

What about the action of p in the x basis? This is more interesting. To obtain this, I first propose that instead of studying p we study the operator $\hat{U}(a) = e^{-ia\hat{p}/\hbar}$ Here, a has dimensions of length and the $\hbar$ is there for dimensional reasons. The reason we study this will become more clear as we go along. First note that since $\hat{p}$ is Hermitian, $\hat{U}$ is unitary. I now claim that $\hat{U}(a)\,|x\rangle = |x+a\rangle$.

The proof of this goes as follows. Since $[\hat{x},\hat{p}] = i\hbar$ it follows inductively that $[\hat{x},\hat{p}^n] = i\hbar n\hat{p}^{n-1}$ We have

$$\Big[\hat{x},\hat{U}(a)\Big] = \left[\hat{x},\sum_{n=0}^{\infty}\Big(-\frac{ia}{\hbar}\Big)^n\hat{p}^n\right] = a\hat{U}(a)$$

The last equality follows from linearity and an index shift in the power series.

We now have $\hat{x}\hat{U}(a)\,|x\rangle = x\hat{U}(a)\,|x\rangle + \Big[\hat{x},\hat{U}(a)\Big]\,|x\rangle = (x+a)\hat{U}(a)\,|x\rangle$. It follows that $\hat{U}(a)\,|x\rangle \propto |x+a\rangle$, but unitary transformations preserve norm, so $\hat{U}(a)\,|x\rangle = |x+a\rangle$.

Now, let a be an infinitesimal number ϵ. We then have $|x+\epsilon\rangle = |x\rangle - \frac{i\epsilon}{\hbar}\hat{p}\,|x\rangle$. We then have

$$\hat{p}\,|x\rangle = i\hbar\Big(\frac{|x+\epsilon\rangle - |x\rangle}{\epsilon}\Big) = i\hbar\frac{d}{dx}\,|x\rangle$$

Taking the dual, we have

$$\langle x|\,\hat{p} = -i\hbar\frac{d}{dx}\,\langle x|$$

It then follows that

$$\langle x|\,\hat{p}\,|\psi\rangle = -i\hbar\frac{d}{dx}(\langle x|\psi\rangle) = -i\hbar\partial_x\psi(x)$$

Thus, we have shown how to reduce quantum mechanics to differential equations, the conventional way it is studied in undergraduate classes.

Finally, we can derive two final results. This is $\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}$.

The proof is stunningly easy. We simply take $|\psi\rangle = |p\rangle$ in $\langle x|\,\hat{p}\,|\psi\rangle = -i\hbar\frac{d}{dx}\,\langle x|\psi\rangle$, which gives us $-i\hbar\partial_x\,\langle x|p\rangle = p\,\langle x|p\rangle$, or $\langle x|p\rangle = Ce^{ipx/\hbar}$. We fix the normalization by noting

$$\delta(x-y) = \langle x|y\rangle = \int dp\,\langle x|p\rangle\,\langle p|y\rangle = C^2\int dpe^{ip(x-y)/\hbar} = C^2(2\pi\hbar)\delta(x-y)$$

We then fix $C = \frac{1}{\sqrt{2\pi\hbar}}$. Note that we assume C is real WLOG.

Notice that the whole structure of quantum mechanics basically follows from things being Hermitian and $[x,p] = i\hbar$. Note that there can be no finite dimensional representation of this algebra. If we take the trace of both sides, the LHS is zero using cyclic property of the trace and the RHS is $i\hbar\dim R$ but this is a contradiction.

1.2.6 Symmetries in Canonical Quantization

Part of the quantization recipe is to replace Poisson brackets with commutators. This means that some Hermitian operator G on the Hilbert space generates transformations of the operators according to $\delta\mathcal{O} = i[G, \mathcal{O}]$. The finite version of this is $\mathcal{O}' = e^{iG}\mathcal{O}e^{-iG}$. Note that time evolution is a special case of this. Also, just like the Heisenberg vs. Schrödinger view of the dynamics, we can either have symmetries act on operators via conjugation (as above) or on states via $|\psi'\rangle = e^{-iG}|\psi\rangle$. Note that since G is Hermitian, this transformation is unitary and preserves norm.

1.3 The Feynman Path Integral

After reviewing these (hopefully familiar) formalities, we can proceed to discuss path integral quantization, a method due to Feynman. Path integrals may be introduced independently of canonical quantization and used to develop quantum mechanics in an internally consistent framework, but we will derive them from the canonical rules. This is because we will be using them to as a calculational and conceptual tool to solve problems that are well-posed in the canonical framework, instead of having to re-invent the rules from scratch (although this is interesting as well).

1.3.1 The Intuition

To get an idea of what the path integral is all about, it is useful to revisit the simplest quantum mechanical thought experiment: the double slit. When we are first introduced to quantum mechanics, we are told all about the double slit experiment, waves and particles, and the superposition principle. The double-slit gives a motivation for quantum superposition, but we never see it realized explicitly in this context. We just think about particles in boxes and whatnot.

In the double slit, we are instructed that the quantum mechanical amplitude (the norm-square of which is the probability) for a particle to land at some point on the screen is the sum of two amplitudes, one for each slit. Now, you have to imagine what happens if we drill a third hole in the screen during the double-slit. Of course, we all know the answer: you sum the amplitudes for all holes. But Feynman is relentless: he asks what about drilling more and more holes in the screen, and then adding many screens with even more holes in them. The prescription is to sum over all possible ways of crossing through the holes to get from A to B. The point is that if we add an infinite number of screens and drill an infinite number of holes in them, we are simply summing over all trajectories connecting A and B. Thus, the amplitude for being at B given A is expressed as

$$\mathcal{A}(A \to B) = \sum_{(\text{paths})} \phi(\text{path})$$

How do we realize this concretely? We need to give explicit meaning to the sum over paths and the amplitude for a path. We will see how to do this now. But we can anticipate how it will look. The double-slit chopping procedures shows that we are basically approximating a path by a sequence of straight-line segments. By building a path out of these tiny pieces

and assigning the amplitude for each piece, then summing over these, we can build the total amplitude. Despite its reputation for being mathematically complicated, the path integral in fact gives the most concrete representation of the superposition principle.

1.3.2 The Integral

Enough handwaving–let's see how to actually realize this beautifully intuitive picture. First thing's first: what's the amplitude to go from some point q_i to some point q_f in quantum mechanics? Well, we have the time evolution operator $e^{-iHT/\hbar}$, and we know that the amplitude to go from an initial state at q_i to a final state at q_f is $\langle q_f| e^{-iHT/\hbar} |q_i\rangle$. We stick to a single particle in one dimension with Hamiltonian $H = \frac{1}{2m}p^2 + V(q)$.

The first step is to chop up the time T into N segments. We do this by using the identity $e^{-iHT/\hbar} = (e^{-iHT/\hbar N})^N$. We now write $\epsilon = T/N$, and take the limit $\epsilon \to 0$, $N \to \infty$ with T held fixed. This will hereafter be referred to as the Newton-Leibniz limit. We then have the amplitude as

$$\langle q_f| e^{-i\epsilon H/\hbar} \cdot ... \cdot e^{-i\epsilon H/\hbar} |q_i\rangle$$

There are, of course, N factors. Now, between each factor, we insert a resolution of the identity (complete set of states) as $1 = \int dq_k |q_k\rangle \langle q_k|$. Since there are N factors, there are $N-1$ such insertions and we have

$$\langle q_f| e^{-iHT/\hbar} |q_i\rangle = \int dq_1 ... dq_{N-1} \langle q_f| e^{-i\epsilon H/\hbar} |q_{N-1}\rangle \langle q_{N-1}| e^{-i\epsilon H/\hbar} |q_{N-2}\rangle ... \langle q_1| e^{-i\epsilon H/\hbar} |q_i\rangle$$

A typical factor is $\langle q_k| e^{-i\epsilon H/\hbar} |q_{k-1}\rangle$. We can evaluate it as follows. H is built out of q and p, so we should insert some p's to further simplify its action. The relevant resolution of the identity is $1 = \int dp_k |p_k\rangle \langle p_k|$. Our analysis will be simplified by the fact that ϵ is a small quantity. Note that $e^{-i\epsilon H/\hbar}$ is of the form e^{A+B}, with $A = -i\epsilon p^2/2m\hbar$ and $B = -i\epsilon V(q)/\hbar$. If A and B were commuting numbers, we could break this up as $e^A e^B$. However, they are not, so this doesn't work in general. But here, the smallness of ϵ comes to the rescue. The commutator of A and B is proportional to ϵ^2, so *since we are considering $\epsilon \to 0$, we can regard A and B as commuting numbers.* So, to this order of accuracy we have $e^{-i\epsilon p^2/2m\hbar} e^{-i\epsilon V(q)/\hbar} = e^{-i\epsilon H/\hbar + \mathcal{O}(\epsilon^2)}$.

To see this more explicitly, consider $\ln\left(e^{\epsilon A} e^{\epsilon B}\right)$. For small ϵ, we have $\ln\Big(1 + \epsilon A + \mathcal{O}(\epsilon^2)\Big)\Big(1 + \epsilon B + \mathcal{O}(\epsilon^2)\Big) = \ln\Big(1 + \epsilon(A+B) + \mathcal{O}(\epsilon^2)\Big) = \epsilon(A+B) + \mathcal{O}(\epsilon^2)$. In the last step, we Taylor expanded the log. Exponentiating both sides we find $e^{\epsilon A} e^{\epsilon B} = e^{\epsilon(A+B) + \mathcal{O}(\epsilon^2)}$.

Now, we insert a resolution of the identity $1 = \int dp_k |p_k\rangle \langle p_k|$ in between the two exponentials. This gives

$$\langle q_k| e^{-i\epsilon H/\hbar} |q_{k-1}\rangle = \int dp_k e^{-i\epsilon p_k^2/2m\hbar} \langle q_k|p_k\rangle e^{-i\epsilon V(q_{k-1})/\hbar} \langle p_k|q_{k-1}\rangle$$

Now, recalling $\langle q|p\rangle = e^{ipq/\hbar}/\sqrt{2\pi\hbar}$, this becomes

$$\langle q_k| e^{-i\epsilon H/\hbar} |q_{k-1}\rangle = \int \frac{dp_k}{2\pi\hbar} e^{-i\epsilon p_k^2/2m\hbar + ip_k(q_k - q_{k-1}) - i\epsilon V(q_{k-1})/\hbar}$$

Now, the integral over p_k is a Gaussian and we can evaluate it as

$$\langle q_k | \, e^{-i\epsilon H/\hbar} \, | q_{k-1} \rangle = \sqrt{\frac{m}{2\pi i\hbar\epsilon}} \exp\left\{ \frac{im(q_k - q_{k-1})^2}{2\hbar\epsilon} - \frac{i\epsilon}{\hbar} V(q_{k-1}) + \mathcal{O}(\epsilon^2) \right\}$$

This will be our key formula.

Returning to the full formula, there are N such factors, running from $k = 1$ to $k = N$ (we are calling $q_0 = q_i$ and $q_N = q_f$). The pre-exponential factor does not depend on k, so there are simply N of them. Meanwhile, we can combine the product of the exponentials into the exponential of a sum, leaving us with

$$\langle q_f | \, e^{-iHT/\hbar} \, | q_i \rangle = \lim_{\substack{N\to\infty \\ \epsilon\to 0}} \left(\frac{m}{2\pi i\hbar\epsilon} \right)^{N/2} \int \prod_{k=1}^{N-1} dq_k \exp\left\{ \frac{i\epsilon}{\hbar} \sum_{k=1}^{N} \left[\frac{m}{2} \left(\frac{q_k - q_{k-1}}{\epsilon} \right)^2 - V(q_{k-1}) \right] \right\}$$

In the limit, we recognize the integrand: it is none other than the action! The integration measure simply corresponds to our prescription of chopping up paths one slice at a time. Hence, we may write

$$\langle q_f | \, e^{-iHT/\hbar} \, | q_i \rangle = \int_{\substack{q(0)=q_i \\ q(T)=q_f}} \mathcal{D}q(t) \exp\left\{ \frac{i}{\hbar} \int_0^T dt L(q, \dot{q}) \right\} = \int_{\substack{q(0)=q_i \\ q(T)=q_f}} \mathcal{D}q(t) e^{iS[q(t)]/\hbar}$$

This is the *Feynman path integral* or *functional integral*. Several comments are in order:

- The continuum notation is very pretty, but it is always understood as a shorthand for some kind of underlying discretization and limiting procedure, known as a *regularization*. This is just as $\int dx$ is shorthand for the limit of a sum. However, in path integrals, since the space of all paths is a lot more complicated than the real line, we must pay more attention to our limits. I will return to this point at various places as we progress.

- In what sense does this thing pass as a good integral on the space of all paths? For example, you may have noted that in order to identify $(q_k - q_{k-1})/\epsilon$ as a derivative, q_k and q_{k-1} must be sufficiently close to each other so that the derivative actually makes sense. The point, which we will revisit shortly, is that for small ϵ, the exponential is wildly oscillating and the non-differentiable paths contributing will all tend to cancel each other out. Hence, these paths ultimately do not contribute appreciably and we can make the continuum approximation legitimate in the limit (at least for physicists).

- Naively, the path integral seems to resolve the issue of operator ordering. It does not. A choice of operator ordering comes back to haunt us as a choice of discretization of the path integral. It is easy to see why this is so: retracing our steps in the derivation, the placement of operators relative to one another dictates whether or not somebody gets a q_k or q_{k-1}.

- The object $\mathcal{D}q(t) = \lim(m/2\pi i\hbar\epsilon)^{N/2} \prod_k dq_k$ by itself is not a good measure on path space. Rather, $\mathcal{D}q(t)e^{iS[q(t)]/\hbar}$ should be thought of as a kind of probabilistic measure on this space, just as $dx e^{-x^2/2}/\sqrt{2\pi}$ is a probabilistic measure on the line. Arbitrary functions can be integrated against a Gaussian measure and will have better convergence properties, just as the path integral measure including $e^{iS/\hbar}$ makes better sense than just $\mathcal{D}q(t)$.

- Trying to think in a rigorous and precise way about path integrals will lead to endless confusion. The correct way to work is formally and intuitively. Trust your intuition about the behavior of the integrals and that the various limits will work out. The real power of path integrals comes from when we can work formally and deduce stuff that would have been extremely hard to see using the standard techniques. We will see many examples of this moving forward.
- The derivation given here generalizes in a very obvious way to systems with many degrees of freedom. Just integrate over all the coordinates in the measure and use the classical action. One of the advantages of the path integral is how straightforward it is to add degrees of freedom. Also, it is easy to change variables in the path integral (there are some caveats to this related to discretization but we will return to these when the time comes).
- The form of the path integral as a sum over all possible configurations of some degrees of freedom is very reminiscent of a Boltzmann sum. We expand upon this in the next section.
- There is no a priori natural normalization of a path integral. What I mean by this is that if someone handed you a path integral out of the sky, you could multiply the measure by 2π and it would still be a good path integral. This reflects the fact that states in quantum mechanics do not have an a priori obvious and natural normalization. Our choice of normalization of the path integral simply matches the normalization of the states. To get around this difficulty, for instance in a situation where we do not know the states to fall back on, we have to pick an arbitrary normalization of the path integral or be comfortable limiting ourselves to ratios of path integrals.
- I have clung to $\hbar$ for too long! From here forward, $\hbar = 1$.

1.4 Classical Limit

The path integral allows us to see the connection between classical and quantum mechanics most transparently, in terms of the destructive interference of quantum paths. When $\hbar$ is much smaller than a typical action (the classical limit), small variations in the action create rapid changes in $e^{iS/\hbar}$. This means that the integrand rapidly oscillates between $+1$ and -1, so that it has equal probability to be positive or negative, and therefore cancels to zero on the average when integrated.

This is true except in the vicinity of a stationary point, $\delta S = 0$, where S is roughly constant and so the paths add constructively rather than canceling each other out. Therefore, in this limit, we can forget all about the sum over the paths which cancel each other out anyway and focus only on the stationary point. This is classical mechanics.

By the way, this idea of destructive interference allowing us to concentrate only on the stationary point is known as the principle of stationary phase. It can be used to get asymptotic estimates for finite dimensional integrals.

1.5 Connection to Statistical Mechanics

The propagator e^{-iHT} looks a great deal like a Boltzmann factor $e^{-\beta E}$ if we identify $T = -i\beta$, that is, if we go to imaginary time. What this means is that if we analytically continue the quantum mechanics to imaginary time, we can make connections to statistical mechanics.

1.5.1 Classical Statistical Mechanics

The connection to classical statistical mechanics is simply that in the path integral, if we write $t = -i\tau$, then the action in imaginary time becomes

$$S_E = \int_0^\beta d\tau \frac{1}{2}\Big(\frac{dq}{d\tau}\Big)^2 + V(q(\tau))$$

This is the same as the potential energy for a string stretched from $\tau = 0$ to $\tau = \beta$. The path integral then describes a Boltzmann sum over all configurations of the string, a classical statistical mechanics. The subscript "E" is for "Euclidean", and comes from abuse of terminology from special relativity–imaginary time sends Minkowski intervals into Euclidean ones.

1.5.2 Quantum Statistical Mechanics

The connection to quantum statistical mechanics goes as follows. We go to imaginary time, set $q_i = q_f = q$, and integrate over all q. In the operator picture, this produces $\operatorname{Tr} e^{-\beta H} = \sum_n e^{-\beta E_n}$, a quantum statistical mechanics partition function. In the path integral, the integration over q corresponds to an integration over all paths with periodic boundary conditions (it is easy to convince oneself of this). In practice, this gives a convenient way to compute the energy levels and degeneracies of a system through the path integral. The integration over periodic paths is equivalent to an integration over the space $\mathrm{Maps}(S^1 \to M)$, where M is the configuration space of the system. This is $\mathcal{L}M$, the free loop space of M.

1.5.3 Application: Hawking Temperature

As an application of these mysterious ideas about temperature and circles in imaginary time, let us derive the Hawking temperature of a Schwarzschild black hole. Recall from a course on general relativity that the Schwarzschild solution is

$$ds^2 = -\Big(1 - \frac{2GM}{r}\Big)dt^2 + \Big(1 - \frac{2GM}{r}\Big)^{-1} dr^2 + r^2 d\Omega^2$$

Here, G is Newton's constant and $d\Omega^2$ is the line element on the sphere. We are interested in the quantum aspects at finite temperature, so we write $t = -i\tau$. τ is an angular variable with periodicity $\tau \sim \tau + \beta$. Moreover, for the Hawking radiation we want to look at the near-horizon limit of the geometry, where the quantum-mechanical particles being emitted form essentially a heat bath.

Towards this end we write $r = 2GM + u$ and expand in u, that is $u << 2GM$. One finds the imaginary time line element

$$ds_E^2 = \frac{u}{2GM} d\tau^2 + \frac{2GM}{u} du^2 + ...$$

The "..." contains the irrelevant angular pieces. In order to compare with our formulas for temperature in terms of imaginary time, we must beat this down into the form of a Euclidean line element. It is easy to see that the change of variables $u = R^2/8GM$ does the trick, so that

$$ds_E^2 = \frac{R^2}{16G^2M^2} d\tau^2 + dR^2 + ...$$

If we set $\tau = 4GM\theta$, we find a standard Euclidean line element. Since we know this geometry must be topologically the same as the plane, we must have $\theta \sim \theta + 2\pi$ to avoid a conic deficit angle. Since τ must have periodicity β, this gives a temperature

$$T_H = \frac{\hbar c^3}{8\pi G k_B M}$$

We have restored constants by dimensional analysis. This is the Hawking temperature of a black hole.

1.6 Examples of Path Integrals

While very intuitive, path integrals will not be of much use to us if we cannot actually use them to compute anything. What we will find is that if a path integral is Gaussian (that is, the action is quadratic in the variables), it is more or less computable and one can learn things explicitly. This is the familiar fact that harmonic oscillators can be done exactly and everything else is difficult. When we go on to supersymmetry, we will see some very fancy tricks to evaluate non-Gaussian integrals.

In these notes, we will deal almost exclusively with Euclidean (imaginary-time) path integrals. They have better convergence properties and are easier to get signs right with.

1.6.1 Free Particle

As a warm-up case to check the formalism, we will do the free particle. This is given by

$$\int_{\substack{X(0)=X_i \\ X(T)=X_f}} \mathcal{D}X(t) \exp\left\{ -\frac{1}{2} \int_{t_a}^{t_b} dt \left(\frac{dX}{dt} \right)^2 \right\} = \lim_{N\to\infty} \frac{1}{\sqrt{2\pi\epsilon}} \int \prod_{i=1}^{N-1} \frac{dX_i}{\sqrt{2\pi\epsilon}} \exp\left\{ -\sum_{i=0}^{N-1} \frac{(X_{i+1} - X_i)^2}{2\epsilon} \right\}$$

I have switched position notation from q to X.

To evaluate this, observe that each integral we have to do is a Gaussian, and Gaussians integrate back to Gaussians, so we can establish this as a recursive process. It will be useful to have the following formula:

$$\int dx \exp\{-a(x - x_1)^2 - b(x - x_2)^2\} = \sqrt{\frac{\pi}{a+b}} \exp\left\{ -\frac{ab}{a+b} (x_2 - x_1)^2 \right\}$$

Proving this is easy. Just expand it out, collect terms, and use the Gaussian formula. First, evaluate the integral over X_1. It is

$$\frac{1}{2\pi\epsilon}\int dX_1 \exp\left\{-\frac{(X_2-X_1)^2}{2\epsilon}-\frac{(X_1-X_0)^2}{2\epsilon}\right\}=\frac{1}{2\sqrt{\pi\epsilon}}\exp\left\{-\frac{1}{4\epsilon}(X_2-X_0)^2\right\}$$

We notice this can be written as

$$\frac{1}{\sqrt{2\pi\cdot 2\epsilon}}\exp\left\{-\frac{1}{2\cdot 2\epsilon}(X_2-X_0)^2\right\}$$

Next, multiply this by the exponential involving X_2, and integrate $\frac{dX_2}{\sqrt{2\pi\epsilon}}$. We find

$$\int\frac{dX_2}{\sqrt{2\pi\epsilon}}\frac{1}{\sqrt{2\pi\cdot 2\epsilon}}\exp\left\{-\frac{(X_3-X_2)^2}{2\epsilon}-\frac{1}{4\epsilon}(X_2-X_0)^2\right\}=\sqrt{\frac{4\pi\epsilon}{3\cdot 8\pi^2\epsilon^2}}\exp\left\{-\frac{1}{2\cdot 3\epsilon}(X_3-X_0)^2\right\}$$

Clearly, due to the structure of the integral formula given, if you go up to $N-1$ the answer will be

$$\frac{1}{\sqrt{2\pi\cdot N\epsilon}}\exp\left\{-\frac{1}{2\cdot N\epsilon}(X_N-X_0)^2\right\}$$

This can actually be shown with a very simple induction proof (work it out as an exercise). Now, we take the Newton-Leibniz limit and find (for brevity, $T=t_b-t_a$):

$$\int_{\substack{X(0)=X_i\\X(T)=X_f}}\mathcal{D}X(t)\exp\left\{-\frac{1}{2}\int_{t_a}^{t_b}dt\Big(\frac{dX}{dt}\Big)^2\right\}=\frac{1}{\sqrt{2\pi T}}\exp\left\{-\frac{(X_b-X_a)^2}{2T}\right\}$$

Observe that this is just the usual formula for a heat kernel. This is unsurprising because the free particle Schrödinger equation is just the heat equation in imaginary time.

For readers finding this incredibly tedious and wondering of the use of path integrals at all, the method of evaluation here using explicit discretization is analogous to evaluating ordinary integrals by directly Riemann summing. In the subsequent examples, we will develop more efficient techniques. We only did this explicit calculation to show that the all the normalization and whatnot works out as it should.

1.6.2 Harmonic Oscillator

For our next example, we will take a trace. Consider the simple harmonic oscillator, with Hamiltonian

$$H=\frac{P^2}{2}+\frac{\omega^2}{2}X^2$$

We have set the mass to unity for simplicity. Everyone knows from elementary quantum mechanics that the energy levels are $E_n=(n+\frac{1}{2})\omega$. We will reproduce this in the path integral formalism.

Since we have the complete spectrum, we can evaluate $\operatorname{Tr}e^{-\beta H}$ in the H eigenbasis, so that

$$\operatorname{Tr}e^{-\beta H}=\sum_{n=0}^{\infty}e^{-\beta\omega(n+\frac{1}{2})}=\frac{1}{2\sinh(\beta\omega/2)}$$

Let us also evaluate this by doing the path integral on the circle. We want to do the following integral:

$$Z(\beta) = \int \mathcal{D}X(\tau) \exp\left\{ -\int_0^\beta d\tau \left(\frac{1}{2}\dot{X}^2 + \frac{\omega^2}{2} X^2 \right) \right\}$$

Here, a dot means a τ derivative. Recall that we are integrating over all periodic maps. In particular, periodicity ensures that there are no boundary terms upon integration by parts, so that the action is

$$S = \frac{1}{2} \int_0^\beta d\tau X(\tau)(-\partial_\tau^2 + \omega^2) X(\tau)$$

Let us expand in orthonormal eigenfunctions of the operator $(-\partial_\tau^2 + \omega^2)$, namely a Fourier expansion

$$X(\tau) = \sum_{n=-\infty}^{\infty} c_n \frac{e^{2\pi i \tau/\beta}}{\sqrt{\beta}}$$

The integrand is now just

$$e^{-S} = \exp\left\{ -\frac{\omega^2}{2} c_0^2 - \sum_{n=1}^{\infty} \left(\left(\frac{2\pi n}{\beta} \right)^2 + \omega^2 \right) |c_n|^2 \right\}$$

Here, we have used the fact that X is real, so $c_{-n} = c_n^*$. Since the c_n are essentially orthonormal (complex) coordinates in a linear space, we are motivated to define the measure

$$\mathcal{D}X(\tau) = \frac{dc_0}{\sqrt{2\pi}} \prod_{n=1}^{\infty} \frac{d^2 c_n}{2\pi}$$

The normalizations are chosen for convenience, in accordance with Gaussian probability measures. The path integral is now

$$Z(\beta) = \int \frac{dc_0}{\sqrt{2\pi}} e^{-\frac{\omega^2}{2} c_0^2} \prod_{n=1}^{\infty} \int \frac{d^2 c_n}{2\pi} \exp\left\{ -\left(\left(\frac{2\pi n}{\beta} \right)^2 + \omega^2 \right) |c_n|^2 \right\}$$

This is just a product of Gaussian integrals, which is readily evaluated to be

$$Z(\beta) = \frac{1}{\omega} \prod_{n=1}^{\infty} \left(\left(\frac{2\pi n}{\beta} \right)^2 + \omega^2 \right)^{-1}$$

The issue is that the infinite product is naively divergent. We can overcome this issue as follows: observe that if we factor

$$\prod_{n=1}^{\infty} \left(\frac{2\pi n}{\beta} \right)^2 \prod_{n=1}^{\infty} \left(1 + \left(\frac{\beta\omega}{2\pi n} \right)^2 \right)$$

The first product is divergent, but the second one is well-known and converges to $\frac{\sinh(\beta\omega/2)}{\beta\omega/2}$. To define the first, we observe the following. With our measures normalized as we have

normalized them, our Gaussian integrals in the finite dimensional case computed determinants[1]. Since finite products have no convergence issues, the product definition of the determinant worked. We interpret the naive divergence as a breakdown of the product definition of the determinant.

A better definition is as follows. Suppose you have some operator $\mathcal{O}$ with nonzero and possibly infinitely many eigenvalues λ_n. Form the spectral zeta function

$$\zeta_{\mathcal{O}}(s) = \sum_n \frac{1}{\lambda_n^s}$$

Observe that a formal differentiation gives

$$\zeta'_{\mathcal{O}}(s) = -\sum_n \frac{\log \lambda_n}{\lambda_n^s}$$

So that in particular, *formally*

$$\prod_n \lambda_n = e^{-\zeta'_{\mathcal{O}}(0)}$$

This suggests that we *define*

$$\det \mathcal{O} = e^{-\zeta'_{\mathcal{O}}(0)}$$

The idea is that while the product doesn't make sense, the zeta function may have some analytic continuation which is well-behaved at zero. Using this procedure to define the divergent piece of the infinite product gives

$$\prod_{n=1}^{\infty} \left(\frac{2\pi n}{\beta}\right)^2 \to \beta$$

You have to use some properties of the Riemann zeta to arrive at this, but this is the only rule that we will need.

Our partition function is then

$$Z(\beta) = \frac{1}{\beta\omega} \cdot \frac{\beta\omega}{2\sinh(\beta\omega/2)} = \frac{1}{2\sinh(\beta\omega/2)}$$

Which is in agreement with the operator results.

The use of zeta function regularization was sketchy enough to warrant some commentary. Firstly, while the mode measure is certainly well-motivated, is it equivalent to the previous definition of the path integral? The answer is yes, because it is simply a linear transformation which relates the X_i to the c_n (since when you discretize any kind of expansion of the form $\sum c_n f_n(t)$, the $f_n(t_i)$ just become constant coefficients). Therefore, they are equivalent up to a constant Jacobian factor. The Jacobian turns out to be divergent, but exactly the right kind of divergence to cancel off the bad behavior and give a result which agrees. The zeta function regularization just takes into account this Jacobian automatically. The advantage of zeta regularization is that it allows us to uniquely define infinite dimensional Gaussian integrals of this type, and it is the convention used throughout the physics literature to evaluate traces. Zeta function regularization effectively defines the measure, serving as an implicit Jacobian, and in all known cases it agrees with operator methods to evaluate traces. For this reason, we like to use it, since in more complicated situations the correct discretized measure is too difficult to obtain, but we can use zeta functions easily.

[1] See appendix now.

1.6.3 S^1 Sigma Model

Here, we will study our first example of a sigma model, which in general is a quantum field theory in which we integrate over the space of maps to a nontrivial target manifold. For quantum mechanics, it is simply fancy jargon for saying that we are describing a particle moving on some space. Here, we want to calculate the partition function for maps $X : S^1 \to S^1$. We take the target circle to have circumference $2\pi R$, so that it can be obtained from $\mathbb{R}$ via the identification $X \sim X + 2\pi R$.

We consider free particle motion, so that the Hamiltonian is

$$H = -\frac{1}{2}\frac{\partial^2}{\partial X^2}$$

Since the target manifold is a circle, the wavefunction must have periodicity $2\pi R$. Solutions to the eigenvalue equation

$$-\frac{1}{2}\frac{\partial^2 \psi}{\partial X^2} = E\psi$$

with the correct periodicity are

$$\psi_n = e^{inX/R}$$

with $n \in \mathbb{Z}$. The eigenvalues are then

$$E_n = \frac{n^2}{2R^2}$$

The partition function is then a simple trace:

$$Z(\beta) = \sum_{n=-\infty}^{\infty} e^{-n^2\beta/2R^2}$$

In the path integral formalism, we must evaluate

$$Z(\beta) = \int \mathcal{D}X(\tau) \exp\left\{-\frac{1}{2}\int_0^\beta d\tau \left(\frac{dX}{d\tau}\right)^2\right\}$$

Again, the integration is over all maps between the circles. We have a new complication: the space of maps breaks up into topologically distinct components classified by winding number m. In other words, the target space periodicity causes us to have several inequivalent types of periodic maps that wrap around the circle various times. Our path integral is then accompanied by a sum over winding sectors:

$$Z(\beta) = \sum_{m=-\infty}^{\infty} \int \mathcal{D}X_m(t) e^{-S[X_m(t)]}$$

Here, X_m denotes a path which has winding number m. We can express each X_m by

$$X_m = \frac{2\pi m t R}{\beta} + X_0(t)$$

where X_0 is just a periodic function. The action for such a path is easily seen to be

$$S = \frac{2\pi^2 m^2 R^2}{\beta} + \frac{1}{2}\int_0^\beta d\tau X_0(-\partial_\tau^2)X_0$$

The path integral is then

$$Z(\beta) = \sum_{m=-\infty}^{\infty} e^{-\frac{2\pi^2 m^2 R^2}{\beta}} \int \mathcal{D}X_0(t) \exp\Big\{ -\frac{1}{2} \int_0^\beta dt \Big(\frac{dX_0}{dt} \Big)^2 \Big\}$$

As before, we now expand in orthonormal eigenfunctions:

$$X_0(\tau) = \sum_{n=-\infty}^{\infty} c_n \frac{e^{2\pi i \tau/\beta}}{\sqrt{\beta}}$$

The path integral measure is

$$\mathcal{D}X_0(t) = \frac{dc_0}{\sqrt{2\pi}} \prod_{n=1}^{\infty} \frac{d^2 c_n}{2\pi}$$

The path integral becomes

$$\int_0^{2\pi R\sqrt{\beta}} \frac{dc_0}{\sqrt{2\pi}} \prod_{n=1}^{\infty} \int \frac{d^2 c_n}{2\pi} \exp\Big\{ -\Big(\frac{2\pi n}{\beta} \Big)^2 |c_n|^2 \Big\} = \sqrt{2\pi\beta} R \prod_{n=1}^{\infty} \Big(\frac{2\pi n}{\beta} \Big)^{-2}$$

Observe the integration region for the zero mode is scaled because of the normalization of c_0: as X_0 goes from 0 to $2\pi R$, c_0 goes from 0 to $2\pi R\sqrt{\beta}$. Using zeta function regularization, we find

$$\sqrt{\frac{2\pi}{\beta}} R$$

The path integral is then

$$Z(\beta) = \sqrt{\frac{2\pi}{\beta}} R \sum_{m=-\infty}^{\infty} e^{-\frac{2\pi^2 m^2 R^2}{\beta}}$$

This is equivalent to the operator result after a Poisson resummation[2]. Note that if we didn't know anything about Poisson resummation, we could have argued by the equivalence of path integral and operator representations the highly nontrivial equivalence of the two series representing the partition function.

1.6.4 Classical Limit of Statistical Mechanics

It is well-known that in the high-temperature ($\beta \to 0$) limit, quantum statistical mechanics goes classical. I will demonstrate this explicitly using the path integral representation of the partition function.

Consider a single particle moving in a potential $V(q)$. The path integral for the partition function is

$$Z(\beta) = \int \mathcal{D}q(\tau) \exp\Big\{ -\int_0^\beta d\tau \frac{1}{2} \Big(\frac{dq}{d\tau} \Big)^2 + V(q(\tau)) \Big\}$$

[2]See the Fourier analysis notes if this is unfamiliar.

Our goal is to understand the $\beta \to 0$ limit. We can do this in terms of the mode expansion. Fourier expanding as in the previous examples, the path integral becomes

$$Z(\beta) = \int \frac{dc_0}{\sqrt{2\pi}} \prod_n \frac{d^2 c_n}{2\pi} \exp\left\{ -\sum_{n=1}^{\infty} \left(\frac{2\pi n}{\beta}\right)^2 |c_n|^2 - \int_0^\beta d\tau V(q(\tau)) \right\}$$

The first term comes from substituting into the kinetic energy, and the second is to be regarded as an implicit function of the Fourier modes c_n. As $\beta \to 0$, the kinetic term becomes very large and the path integral is consequently exponentially damped away from $c_n = 0$ for $n \neq 0$. Then, for reasonable V, the variation in this term is much slower, and we may replace $q(\tau)$ by its value at $c_n = 0$, $c_0/\sqrt{\beta}$. This amounts to a tremendous simplification, because then the nonzero mode integrals become Gaussian. Neglecting the overall normalization, the final expression becomes

$$Z(\beta) \approx \int dq e^{-\beta V(q)}$$

This is the familiar Boltzmann sum for a classical statistical mechanical system.

1.7 Summary

We have seen how to use path integrals to reproduce familiar results in quantum mechanics. The method is interesting, but it does not yet seem absolutely necessary. The conventional methods are much simpler and do not require all the details. When we study supersymmetry, we will see that the equivalence between operator and path integral representations of a given system amounts to powerful relationships between algebra and geometry.

The real reason people study path integrals is because within this framework, it becomes very easy to deal with quantum field theories. Perturbation theory and Feynman diagrams can be easily formulated, and the semiclassical limit is transparent. The story of renormalization is also quite transparent from the path integral point of view. These applications are beyond our scope, though.

Our main applications will be using path integrals to derive those relationships between algebra and geometry. We will see that objects known as instantons in the path integral formalism will give a beautiful interpretation of Morse theory, an insight due to Witten.

2 Part II: Fermions and Supersymmetry

In this section, we discuss general aspects of fermionic variables and introduce the idea of supersymmetry. We explain the ideas of localization in supersymmetric integrals and outline the general structure of supersymmetric quantum mechanics.

It is important to distinguish here between the "fermionic variables" and the physical notion of fermion. Most likely, in your introductory quantum mechanics class, you learned about fermions as particles for which the identical multi-particle wavefunction is totally antisymmetric in the quantum numbers. Here, the word takes on a subtly different meaning, and refers to a type of $\mathbb{Z}_2$ grading which will become clear as we see examples. I will briefly comment on the relationship between the two notions, which again is really best understood in terms of quantum field theory.

2.1 Fermion Harmonic Oscillator

It will be convenient for us to study a system known as the fermionic harmonic oscillator. The considerations will seem rather abstract and ad hoc at first, but the picture we will arrive at after deducing the consequences will lead us to some interesting ideas.

We can characterize the ordinary "bosonic" oscillator as just creation and annihilation operators satisfying $\left[a, a^\dagger\right] = 1$ with a Hamiltonian $H = \omega(a^\dagger a + \frac{1}{2})$. This can equivalently be characterized (after a use of the commutator) as

$$H = \frac{\omega}{2}\left\{a^\dagger, a\right\}$$

We have introduced the anticommutator $\{A, B\} \equiv AB + BA$. The creation and annihilation operators also satisfy the trivial commutators $[a, a] = \left[a^\dagger, a^\dagger\right] = 0$.

An interesting question is to ask what happens if we reverse the role of commutator and anticommutator. The resulting system will be called the fermion oscillator. We have "fermion creation/destruction operators" Ψ and $\Psi^\dagger$ satisfying

$$\left\{\Psi, \Psi\right\} = \left\{\Psi^\dagger, \Psi^\dagger\right\} = 0$$

$$\left\{\Psi, \Psi^\dagger\right\} = 1$$

The first two are simply the statement that $\Psi^2 = (\Psi^\dagger)^2 = 0$. The oscillator Hamiltonian is then given as

$$H = \frac{\omega}{2}\left[\Psi^\dagger, \Psi\right]$$

We wish to solve this system, that is, find a representation of the above algebra and the eigenvalues of H on the states.

First, note that by analogy with the ordinary oscillator, we have the number operator $N = \Psi^\dagger\Psi$. Using the anticommutators, $N^2 = \Psi^\dagger\Psi\Psi^\dagger\Psi = \Psi^\dagger\Psi(1 - \Psi\Psi^\dagger) = \Psi^\dagger\Psi = N$. This means that the eigenvalues of N may only be 0 or 1. Since we seek an irreducible representation, we assume these are the only states in the system and call them $|0\rangle$ and

$|1\rangle$. These satisfy $N|0\rangle = 0$ and $N|1\rangle = |1\rangle$. What remains is to find the action of the operators on the states.

Let us begin with $\Psi|0\rangle$. Note that

$$\|\Psi|0\rangle\|^2 = \langle 0|\Psi^\dagger\Psi|0\rangle = \langle 0|N|0\rangle = 0$$

By positivity, $\Psi|0\rangle = 0$. This is not a surprise.

We proceed to $\Psi^\dagger|0\rangle$. Note that $N\Psi^\dagger|0\rangle = \Psi^\dagger\Psi\Psi^\dagger|0\rangle = \Psi^\dagger(1-\Psi^\dagger\Psi)|0\rangle = \Psi^\dagger|0\rangle$ where we have used $\Psi|0\rangle = 0$. This establishes that $\Psi^\dagger|0\rangle \propto |1\rangle$. We can check the normalization as

$$\left\|\Psi^\dagger|0\rangle\right\|^2 = \langle 0|\Psi\Psi^\dagger|0\rangle = \langle 0|(1-N)|0\rangle = \langle 0|0\rangle = 1$$

Therefore, $\Psi^\dagger|0\rangle = |1\rangle$.

It can be similarly established that $\Psi|1\rangle = |0\rangle$ and $\Psi^\dagger|1\rangle = 0$. I leave these as exercises.

We see that there can be no other states in the system since $\Psi^2 = (\Psi^\dagger)^2 = 0$; any attempt to create more states is thwarted by this relation. This is reminiscent of the Pauli exclusion principle. The fermion oscillator has only two energy levels, one with no fermions and one with a single fermion, and there are no other states by Pauli exclusion.

The Hilbert space is therefore two dimensional. We could, in principle, introduce a 2d matrix formalism to describe it, but this is unnecessary.

It is easy to see that the Hamiltonian is $H = \omega(N - \frac{1}{2})$, so that the eigenstates are $|0\rangle$ and $|1\rangle$, with eigenvalues $-\omega/2$ and $\omega/2$ respectively.

2.2 Grassmann Math

2.2.1 Motivation: Coherent States

Recall that a coherent state is an eigenstate of the destruction operator. In the fermionic context, this means that it is a state $|\psi\rangle$ satisfying

$$\Psi|\psi\rangle = \psi|\psi\rangle$$

The eigenvalue ψ cannot be an ordinary number, because acting with Ψ again gives $\psi^2 = 0$. We simply define ψ to not be an ordinary variable, but a Grassmann variable. By definition, Grassmann variables anticommute with one another (and hence square to zero) and with all the fermionic creation and destruction operators.

This can seem completely arbitrary and ad hoc, but it is similar in spirit to introducing $i = \sqrt{-1}$ in order to solve quadratics. It's about the mileage we can get out of such objects.

2.2.2 Grassmann Algebra

We now turn our attention to developing more systematically the theory of Grassmann variables. A Grassmann algebra is a set of n objects θ_i which generate a vector space

over some field (always $\mathbb{C}$) together with an associative bilinear multiplication so that $\theta_i\theta_j = -\theta_j\theta_i$, or more concisely $\{\theta_i, \theta_j\} = 0$ (though do not confuse the θ_i with operators). A consequence of this is that $\theta_i^2 = 0$.

One can consider the set of k-fold products of Grassmann variables to generate their own vector spaces. It is a simple exercise to demonstrate that these vector spaces have dimension $\binom{n}{k}$.

Associativity of the multiplication ensures that even numbers of Grassmann variables behave just like ordinary variables. This motivates the introduction of a $\mathbb{Z}_2$ gradation, where we declare that an object has $\mathbb{Z}_2$ grade -1 if it contains an odd number of Grassmann variables, and $+1$ if it contains an even number of Grassmann variables. Objects are called Grassmann even or Grassmann odd accordingly. Physics-inspired jargon for Grassmann even variables is to call them bosonic, and Grassmann odd variables are called fermionic.

Often it is convenient to refine the $\mathbb{Z}_2$ gradation to a $\mathbb{Z}$ gradation, where we explicitly count the number of Grassmann variables something is made out of. This is called its "fermion number". Bosonic objects are then simply those made of an even number of fermions.

This is about all there is to say for the algebraic structure. We now seek to develop some notion of calculus, that is, differentiation and integration. For some motivation, observe that when considering an analytic function of N ordinary variables x^i, we have the obvious relationship that $[x^i, x^j] = 0$ simply because numbers multiply commutatively. The derivative satisfies $\left[\frac{\partial}{\partial x^j}, x^i\right] = \delta^i{}_j$ and $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$. With these three commutators and the obvious fact that $\frac{\partial 1}{\partial x^i} = 0$ we can construct the entire differential calculus of analytic functions by simply expanding in formal power series and using the commutators to deduce the action of derivatives on the power series, thereby giving the derivatives of the function.

To extend this to Grassmann variables, we introduce a linear operator $\frac{\partial}{\partial\theta^i}$ which satisfies

$$\left\{\frac{\partial}{\partial\theta^j}, \theta^i\right\} = \delta^i{}_j$$

$$\left\{\frac{\partial}{\partial\theta^i}, \frac{\partial}{\partial\theta^j}\right\} = 0$$

A nice mnemonic for this is just that derivatives are Grassmann odd. Then, by defining functions of a Grassmann variable by their formal power series (which is essentially all we can do, since these things are only defined in terms of algebraic structure), we have Grassmann differential calculus. Observe that the series eventually terminate because the variables square to zero, so they are actually quite well-defined. For example, the most general function of a single Grassmann variable θ is $f(\theta) = a + b\theta$. All higher terms are zero. For instance, $e^\theta = 1 + \theta = (1 - \theta)^{-1}$. In this respect, Grassmann variables are quite nice to physicists, because they make all of our Taylor series terminations exact.

To define integration of Grassmann variables is a bit more subtle, because the derivative squares to zero, implying it has no inverse. We instead generalize different properties of the ordinary integral. Observe that for a square integrable function,

$$\int dx f(x + a) = \int dx f(x)$$

Namely, the integration measure is translationally invariant (the integral is implicitly taken over all $\mathbb{R}$). We define integration over Grassmann variables to be a linear operation satisfying

$$\int d\theta f(\theta + \chi) = \int d\theta f(\theta)$$

Since the most general single variable function is linear, this just implies that $\int d\theta = 0$. This means that $\int d\theta\theta$ had better be nonzero for integration to be nontrivial, and we normalize the measure by picking $\int d\theta\theta = 1$. Notice that this implies that

$$\int d\theta \frac{d}{d\theta} f = 0$$

Also note that for Grassmann numbers, integration is equivalent to differentiation. Finally, note that there is no such thing as an integration region for Grassmann variables, nor a geometric interpretation. It is simply a convenient linear operation to define which turns out to be useful for many purposes. For n Grassmann numbers, the measure is a product of the various "differentials" $d\theta_i$, but these are supposed to be anticommuting objects (integration is Grassmann odd, since $\int d\theta\theta = 1$which is even, but θ is odd, so $d\theta$ is odd). So, we have an ordering ambiguity. Resolve this by defining the measure to be ordered such that

$$\int d^n\theta\theta^{i_1}...\theta^{i_n} = \epsilon^{i_1...i_n}$$

Note that any integral vanishes unless it has all n different Grassmann numbers. Also note that one possible definition of the measure is $d^n\theta = d\theta^n...d\theta^1$, but any even permutation will give the same results.

The definition of the determinant immediately implies that

$$\int d^n\bar{\psi}d^n\psi e^{-\bar{\psi}^i A_{ij}\psi^j} = \det A$$

for any matrix A. Simply expand the Gaussian and find the only nonzero term; the integrals produce the determinant in terms of the ϵ symbol. Here, $\bar{\psi}$ and ψ are just independent Grassmann variables. However, now note that changing variables $\psi = A^{-1}\psi'$ we find that

$$d^n\psi' = \frac{1}{\det A} d^n\psi$$

So Grassmann integration measures pick up the inverse of the Jacobian of ordinary bosonic variables.

A final caveat: if you want to treat two Grassmann variables ψ and $\bar{\psi}$ as complex conjugate, conjugation on Grassmann numbers must also include a reversal of order, since otherwise the real quantity $\bar{\psi}\psi$ would be antihermitian. Furthermore, the complex conjugate of $\frac{\partial}{\partial\psi}$ is $-\frac{\partial}{\partial\bar{\psi}}$, since when acting on the real quantity $\bar{\psi}\psi$ the two operators give complex conjugate results.

2.2.3 Application: Pfaffian

As an application which will turn out to be useful later, we can define an object called the Pfaffian of an antisymmetric matrix, which squares to the determinant. Consider an

antisymmetric matrix A_{ij}, and the corresponding Gaussian integral

$$\int d^n\theta e^{-\frac{1}{2}A_{ij}\theta^i\theta^j}$$

First, note that since the integrand is an even polynomial in the θ's, this vanishes if n is odd. So let $n = 2m$. Then explicit expansion of the integral gives

$$\int d^{2m}\theta \frac{(-1)^m}{2^m m!} A_{i_1 j_1}...A_{i_m j_m}\theta^{i_1}\theta^{j_1}...\theta^{i_m}\theta^{j_m} = \frac{(-1)^m}{2^m m!} A_{i_1 j_1}...A_{i_m j_m}\epsilon^{i_1 j_1 ... i_m j_m} := (-1)^m \mathrm{Pf}(A)$$

This defines the Pfaffian of A.

Now, we have

$$(\mathrm{Pf}(A))^2 = \int d^n\theta d^n\eta e^{-\frac{1}{2}A_{ij}\theta^i\theta^j - \frac{1}{2}A_{ij}\eta^i\eta^j}$$

If one changes variables as

$$\psi^i = \frac{\theta^i + i\eta^i}{\sqrt{2}}$$

$$\bar{\psi}^i = \frac{\theta^i - i\eta^i}{\sqrt{2}}$$

(note that this has Jacobian unity), the integral becomes

$$\int d^n\bar{\psi}d^n\psi e^{-A_{ij}\bar{\psi}^i\psi^j} = \det A$$

We have therefore proven that $\det A = (\mathrm{Pf}(A))^2$, in particular that for an antisymmetric matrix, its determinant is the square of a polynomial in the matrix entries.

2.3 Fermion Path Integrals

2.3.1 Formal Properties

Finally, we can use our Grassmann variables to construct path integrals for systems made out of the fermion oscillators.

I will not construct these path integrals explicitly from first principles, because the derivation goes exactly like the bosonic case: find a complete set of states, insert the identity resolved in terms of these states ∞ times, and take the limit carefully to produce a functional integral. I will simply state the states and completeness relation and then give the result.

The only subtlety with the fermion system is that since the Hilbert space is two dimensional, it is not obvious that one can introduce the complete states in a way which generates a path integral. Wouldn't we just be inserting a bunch of sums with two terms?

The resolution is to use Grassmann variables and coherent states so that the resolution of the identity can be written as an integral. Then we can opt for the standard path integral lore.

The coherent state is

$$|\psi\rangle = |0\rangle - \psi\,|1\rangle$$

The sign is because Grassmann variables anticommute with fermionic operators. This sets up our conventions so that $|0\rangle$ is Grassmann even and $|1\rangle$ is Grassmann odd. Forgetting these facts can lead to a lot of confusion with signs. All operations in the Hilbert space are ordinary (bosonic); the only way to change fermion number is by acting with the raising and lowering operators. With these sign conventions in mind it is easy to verify that $|\psi\rangle$ is indeed a coherent state.

In terms of these states, the identity is resolved as

$$1 = \int d^2\psi e^{-\bar{\psi}\psi}\,|\psi\rangle\,\langle\bar{\psi}|$$

This is easy to verify by just expanding everything and doing the Grassmann integrals.

By repeatedly inserting the identity into $\operatorname{Tr} e^{-\beta H}$, one arrives at a path integral representation

$$\operatorname{Tr} e^{-\beta H} = \int_{APBC} \mathcal{D}\overline{\psi}(\tau)\mathcal{D}\psi(\tau)e^{-\int_0^\beta d\tau\overline{\psi}\dot{\psi}+H(\overline{\psi},\psi)}$$

The path integral is to be taken over fermionic fields which satisfy the antiperiodic boundary condition $\psi(\tau+\beta) = -\psi(\tau)$. If we instead consider periodic boundary conditions, we get a kind of modified trace (sometimes called supertrace):

$$\operatorname{Tr}(-1)^F e^{-\beta H} = \int_{\psi(0)=\psi(\beta)} \mathcal{D}\overline{\psi}(\tau)\mathcal{D}\psi(\tau)e^{-\int_0^\beta d\tau\overline{\psi}\dot{\psi}+H(\overline{\psi},\psi)}$$

The operator $F = N$, simply the number operator discussed earlier, but the F is for "fermion number". $(-1)^F$ is then simply $+1$ on $|0\rangle$ and -1 on $|1\rangle$. This completely specifies its action on the Hilbert space–it simply acts as the operator measuring $\mathbb{Z}_2$ grade.

I will briefly explain how these strange minus sign rules come about. The trace is simply $\operatorname{Tr} e^{-\beta H} = \langle 0|\, e^{-\beta H}\,|0\rangle + \langle 1|\, e^{-\beta H}\,|1\rangle$. It is a simple exercise to verify that this is equivalent to the integral

$$\operatorname{Tr} e^{-\beta H} = \int d^2\psi e^{-\overline{\psi}\psi}\,\left\langle -\overline{\psi}\right|\, e^{-\beta H}\,|\psi\rangle$$

The relative minus sign produces the antiperiodic boundary condition on the fields. Insertion of $(-1)^F$ simply flips the sign, changing the antiperiodic boundary condition to a periodic one.

Considering the real time version of this, we find that the action for the fermionic variables is

$$S = \int dt i\overline{\psi}\dot{\psi} - H(\overline{\psi},\psi)$$

Typically we write the Hamiltonian in terms of the Lagrangian as $H = p\dot{q} - L$, but there is nothing wrong with inverting this relation to $L = p\dot{q} - H$, giving the phase space form

of the Lagrangian. We see that the fermion Lagrangian is already of this form, so that the canonical momentum conjugate to ψ is simply $p_\psi = i\overline{\psi}$. In particular, when we quantize we promote to operators $\psi \to \Psi$, $\overline{\psi} \to \Psi^\dagger$, and impose the *anti* commutation relations (as we are dealing with fermions) $\{\Psi, p_\psi\} = i$, which reduces to $\left\{\Psi, \Psi^\dagger\right\} = 1$, the correct anticommutators for the oscillator.

2.3.2 An Example

Now I will finally do an example of the fermion path integral. Consider once again the harmonic oscillator. We will compute $\mathrm{Tr}(-1)^F e^{-\beta H}$ in the path integral formalism. First, note that the operator formalism gives this easily as

$$\mathrm{Tr}(-1)^F e^{-\beta H} = \langle 0| e^{-\beta H} |0\rangle - \langle 1| e^{-\beta H} |1\rangle = e^{\beta\omega/2} - e^{-\beta\omega/2} = 2\sinh(\beta\omega/2)$$

Our aim is to compute this via path integrals. We have

$$\mathrm{Tr}(-1)^F e^{-\beta H} = \int_{PBC} \mathcal{D}\overline{\psi}(\tau)\mathcal{D}\psi(\tau) e^{-\int_0^\beta d\tau \overline{\psi}\dot{\psi} + \omega\overline{\psi}\psi}$$

We have used the fact that since ψ and $\overline{\psi}$ anticommute, $H = \omega\overline{\psi}\psi$ classically. We see that we have an infinite dimensional Gaussian integral, which by analogy with our finite dimensional case simply computes the determinant

$$\mathrm{Tr}(-1)^F e^{-\beta H} = \det(\partial_\tau + \omega)$$

The determinant is simply defined as the regularized product of eigenvalues, using, say, ζ-functions.

On the space of functions with β-periodic boundary conditions, the eigenvalues of ∂_τ are simply $2\pi i n/\beta$, for $n \in \mathbb{Z}$. The product is then

$$\det(\partial_\tau + \omega) = \prod_{n=-\infty}^{\infty} \left(\frac{2\pi i n}{\beta} + \omega\right) = \omega \prod_{n=1}^{\infty} \left(\frac{2\pi i n}{\beta} + \omega\right)\left(-\frac{2\pi i n}{\beta} + \omega\right) = \omega \prod_{n=1}^{\infty} \left(\left(\frac{2\pi n}{\beta}\right)^2 + \omega^2\right)$$

As usual we must regularize the product, which we do exactly as in the bosonic case. We factor out the $2\pi n/\beta$ piece and use ζ function regularization. This gives

$$\mathrm{Tr}(-1)^F e^{-\beta H} = \det(\partial_\tau + \omega) = \beta\omega \prod_{n=1}^{\infty} \left(1 + \left(\frac{\beta\omega}{2\pi n}\right)^2\right) = 2\sinh(\beta\omega/2)$$

We recognize the same infinite product as the bosonic case.

Note that this is precisely the multiplicative inverse of the bosonic partition function. This hints at a possible symmetry of the object $\mathrm{Tr}(-1)^F e^{-\beta H}$ with both bosons and fermions involved (note $(-1)^F = 1$ on a bosonic Hilbert space, so the two notions of trace coincide).

2.4 Digression on the Meaning of "Fermion"

So far in the notes, sections 2.1-3 have by far been the most abstract and most probably difficult to digest for a newcomer. It is quite difficult to see the motivation in working with something as abstract as Grassmann variables. I will attempt to sketch the motivation coming from quantum field theory, which establishes the connection between the fermionic variables that we have defined and the more traditional notion of fermionic state from quantum mechanics.

Recall that in a situation with identical particles, a bosonic state is invariant under any permutation of its quantum numbers, while a fermionic state is antisymmetric under permutation–odd permutations acquire minus signs.

Consider a quantum field theory, where the dynamical variable is now a field which is a function of spacetime $\varphi(x,t)$. Here, x labels our space variables (which may as well be in $\mathbb{R}^3$) and t is the time direction. Note that φ is a dynamical variable, the analog of $q(t)$ in particle mechanics. One can think of $\varphi(x,t)$ as an infinite collection of $q(t)$'s indexed by x, so that a quantum field theory is in fact a quantum mechanics with an infinite number of degrees of freedom.

The point is that as far as commutation relations are concerned, we can forget about t and just look at $\varphi(x)$. If we Fourier expand $\varphi(x) = \int \frac{d^3k}{(2\pi)^3} a_k e^{ikx}$, it turns out that the rules of quantization imply the commutators

$$\left[a_k, a_{k'}^\dagger\right] = \delta_{kk'}$$

(up to a normalization which we will ignore). The point is that this is a harmonic oscillator algebra, and the field decomposes into an infinite number of harmonic oscillators.

If we choose the field theory Lagrangian appropriately (a nonrelativistic free field theory) the Hamiltonian becomes

$$H = \sum_k \frac{k^2}{2m} a_k^\dagger a_k$$

The eigenstates are $|n_{k_1}...n_{k_N}\rangle = \prod_{i=1}^N (a_{k_i}^\dagger)^{n_{k_i}} |0\rangle$ with $|0\rangle$ the ground state of all the oscillators. We see that these states simply correspond to a multi-particle state with n_{k_i} noninteracting particles of momentum k_i. Moreover, the commutation rules imply that $|n_{k_1}...n_{k_N}\rangle$ is totally symmetric under the interchange of the quantum numbers, so that the particles are bosons.

We see that if we instead considered a field whose Fourier modes satisfied fermionic anti-commutation relations $\{b_k, b_{k'}\} = 0$, $\left\{b_k, b_{k'}^\dagger\right\} = \delta_{kk'}$, the states $|n_{k_1}...n_{k_N}\rangle = \prod_{i=1}^N (b_{k_i}^\dagger)^{n_{k_i}} |0\rangle$ are totally antisymmetric in the quantum numbers, that is, they describe fermions. Note that $(b_k^\dagger)^2 = 0$ implies that $n_{k_i} = 0, 1$, the Pauli exclusion principle.

Retracing back through everything we realize that classically, $\varphi(x,t)$ must be a Grassmann variable in order for its field quanta to describe fermions. This is the connection of our notion of fermion to the traditional one.

2.5 A First Look at Supersymmetry

Now that we are armed with the Grassmann numbers, we can define supersymmetries. These are Grassmann odd symmetry operations which relate bosonic variables to fermionic ones. At first, this just sounds like a pretty strange idea, but the real power of it comes from deducing its consequences.

The consequences we will be concerned with lie in algebraic and differential topology. We will see that studying a particular supersymmetric system leads us to a wonderful understanding of certain topological invariants of manifolds. This is actually just the tip of the iceberg mathematically. Ideas from supersymmetry have literally given rise to entire fields in algebraic geometry and topology. Examples include Gromov-Witten theory and the Seiberg-Witten invariants of four-manifolds. The entire mirror symmetry research program was also born from supersymmetric quantum field theory.

Physically, ideas from supersymmetry have led to many of the striking exact results in quantum field theories and string theories discovered in the 1990s. The Seiberg-Witten solution of $N = 2$ supersymmetric Yang-Mills theory was the first exact solution of a strongly interacting quantum gauge theory in four dimensions. Supergravity theories play a prominent role in the low-energy limit of string theories, which is of interest for trying to describe the real world. The fate of supersymmetry in describing the real world remains rather unclear.

A more modern point of view on supersymmetry is that it is a wonderfully useful tool that allows us to solve nontrivial toy models that give a hint as to how the dynamics goes in the real case. It can also be studied in its own right simply to get new results in mathematics, or new points of view on old results in mathematics. We will take this point of view in these notes.

2.5.1 Steepest Descent Method and Nonperturbative Effects

One of the most powerful aspects of supersymmetry is that for supersymmetric integrals, the steepest descent method is actually exact. In order to appreciate this, we need to know what the steepest descent method is.

Say we are interested in the behavior of the integral

$$I(g) = \int_a^b dx f(x) e^{-S(x)/g^2}$$

in the limit as $g \to 0$. For generic $S(x)$, we cannot evaluate the integral exactly. However, we will be able to extract the relevant limiting behavior as follows. $S(x)$ is assumed to have a unique global minimum on the interior of $[a, b]$. Call the coordinate of the minimum x_*. In the limit as $g \to 0$, the exponential factor becomes sharply peaked about the global minimum, and the contributions to the integral outside of this immediate neighborhood are negligible. We are then justified in Taylor expanding $S(x)$ about its minimum, and we may write

$$I(g) \sim \int_a^b dx f(x) e^{-S(x_*)/g^2 - S''(x_*)(x-x_*)^2/2g^2}$$

Here, $\sim$ means "has the same limit as $g \to 0$". Note that since x_* is a minimum, $S'(x_*) = 0$ and $S''(x_*) > 0$. Now, observe that, just as we argued that the contribution from the integral far away from the minimum was negligible (exponentially small) compared with what we are keeping, if we extend the region of integration from $-\infty$ to ∞, the errors we introduce are either of the same size or smaller (exponentially small compared with what we are keeping). The reason is because the integrand at all points away from the minimum of S is exponentially small compared with its value at x_*. Since we already dropped terms exponentially small compared with what we have kept, we are justified in committing this error as well (it's like ignoring all higher-order terms in a power series, but a bit more general. Convince yourself that the logic is identical). Hence, if we are interested in the limiting behavior, we may extend the region of integration from $-\infty$ to ∞:

$$I(g) \sim \int_{-\infty}^{\infty} dx f(x) e^{-S(x_*)/g^2 - S''(x_*)(x-x_*)^2/2g^2}$$

Finally, observe that the $g \to 0$ limit of the integrand produces, up to a normalization, a delta function in x, so that we may replace $f(x)$ with its value at x_*. Equivalently, since $f(x)$ does not depend on g, and we are assuming it to be smooth, its smooth variation is much slower than the sharp peak of the exponential with small g, which drops off so rapidly that it tends toward a delta function. Taking all these steps into account, we have

$$I(g) \sim f(x_*) e^{-S(x_*)/g^2} \int_{-\infty}^{\infty} dx e^{-S''(x_*)(x-x_*)^2/2g^2} = \sqrt{\frac{2\pi g^2}{S''(x_*)}} e^{-S(x_*)/g^2} f(x_*)$$

In the last step, we have evaluated the Gaussian integral. This is known as the steepest descent method for the evaluation of integrals, also sometimes referred to as the saddle-point method. With some more care, it can be developed systematically to generate a full-blown power series in g^2, but we will have no need for this. It is also easy to generalize the steepest descent method to multiple variables. I leave this as an exercise.

The factor $e^{-S(x_*)/g^2}$ is interesting. It is a classic counterexample in analysis. The function e^{-1/g^2} is smooth in an open neighborhood of zero on the real axis, yet every term in the Taylor expansion for this function vanishes. This can be traced back to the essential singularity at zero in the complex g plane. The physics jargon for these kind of terms is that they are *nonperturbative*: you can do perturbation theory (power series in g^2) until you are blue in the face and never see such factors. This can seem bad and frightening, given how we always assume that things in physics have such expansions. In the case where there are nonzero terms in the Taylor expansion, this is not a huge deal. The reason is because, as $g \to 0$, powers of g are still much larger than the exponential factor e^{-1/g^2}, which rushes to zero exponentially fast. In this situation, we are justified in neglecting the nonperturbative factors because they are exponentially small compared with what we are keeping. The really interesting situations are when something is predicted to be zero in perturbation theory. In this case, nonperturbative effects completely change the picture, because the leading contribution can be exponentially small, but still nonzero. Even if one worked to all orders in perturbation theory and summed the series, one could never detect the nonperturbative effect. In asymptotic analysis, these effects are sometimes called "beyond all orders". We will see precisely this story (something which vanishes in perturbation theory acquiring a nonzero value nonperturbatively) as the driving mechanism behind the Morse-Smale-Witten complex in topology.

To keep our feet on the ground, let us do a simple example. Say we want the behavior of $N!$ as $N \to \infty$. $N!$ has the integral representation

$$N! = \int_0^\infty ds s^N e^{-s}$$

As an exercise, prove this. This integral is not quite in the form of the ones we can handle by steepest descent, but with a little massaging it can be made so. We can rewrite it as

$$N! = \int_0^\infty ds e^{N \ln s - s}$$

Now, writing $s = Nx$, we have

$$N! = N^{N+1} \int_0^\infty dx e^{-N(x - \ln x)}$$

Now, we can apply steepest descent to get the $N \to \infty$ limit. The minimum is located at $x = 1$. We have $S(1) = 1$ and $S''(1) = 1$. Using our formula, we find

$$N! \sim N^N \sqrt{2\pi N} e^{-N}$$

The agreement is excellent for N as low as three, with only 2.7 percent error. By the time you get up to $N = 10$, the error is under one percent. Some readers may have seen this approximation before, but in a different, more crude guise. Taking the logarithm of both sides, we find

$$\ln N! \sim N \ln N - N + \frac{1}{2}(\ln N + \ln 2\pi)$$

Now, as $N \to \infty$, $(\ln N)/N \to 0$, so we are justified in dropping the last term. This leaves

$$\ln N! \sim N \ln N - N$$

So steepest descent has given us a better version of the Stirling approximation from statistical physics.

2.5.2 A Special Integral: Localization

We will now study an integral containing both bosonic and fermionic variables. We will see that it has a supersymmetry, and from this compute it exactly using a technique known as supersymmetric localization.

Consider the integral

$$Z = \int \frac{dx d^2\psi}{\sqrt{2\pi}} e^{-\frac{1}{2}(h'(x))^2 - h''(x)\overline{\psi}\psi}$$

Here, $h(x)$ is an arbitrary polynomial. This is a zero dimensional path integral, with "action" simply given by the quantity in the exponent. This quantity is invariant under the symmetry

$$\delta x = \epsilon\overline{\psi} - \overline{\epsilon}\psi$$

$$\delta\psi = \epsilon h'(x)$$

$$\delta\overline{\psi} = \overline{\epsilon}h'(x)$$

Here, ϵ is a complex Grassmann parameter, in order that δ is a bosonic symmetry. One can also strip away the ϵ's and consider δ a s a kind of Grassmann derivative on field space, that is the space of x's and ψ's. The proof that the action is symmetric is

$$\delta S = h'(x)h''(x)(\epsilon\overline{\psi} - \overline{\epsilon}\psi) + h''(x)h'(x)\overline{\epsilon}\psi - h''(x)h'(x)\epsilon\overline{\psi} = 0$$

This symmetry will actually allow us to compute this integral exactly.

As a first step towards this, consider rescaling $h \to \lambda h$. Then we have

$$Z(\lambda) = \int \frac{dxd^2\psi}{\sqrt{2\pi}} e^{-\frac{\lambda^2}{2}(h'(x))^2 - \lambda h''(x)\overline{\psi}\psi}$$

We now have that

$$\frac{dZ}{d\lambda} = -\int \frac{dxd^2\psi}{\sqrt{2\pi}} e^{-S}(\lambda(h'(x))^2 + h''(x)\overline{\psi}\psi)$$

Now, we note that

$$\lambda(h'(x))^2 + h''(x)\overline{\psi}\psi = Q(h'(x)\psi)$$

Here, $Q = \overline{\psi}\partial_x + \lambda h'(x)\partial_\psi$, namely it is the derivative operator generating the ϵ supersymmetry. In particular, note that it is Grassmann odd and acts as

$$Qx = \overline{\psi}$$

$$Q\psi = \lambda h'(x)$$

$$Q\overline{\psi} = 0$$

Then we have

$$\frac{dZ}{d\lambda} = \int \frac{dxd^2\psi}{\sqrt{2\pi}} e^{-S}QV$$

$V = h'(x)\psi$ of course, but its explicit form is irrelevant. In general, whenever one has the integral of a total variation under a symmetry weighted by a measure invariant under that symmetry, the result is zero. This can be shown by relabeling dummy variables in the integral. Here, we can see it more explicitly. Note that since e^{-S} is δ-invariant, $Qe^{-S} = 0$ so that

$$\frac{dZ}{d\lambda} = \int \frac{dxd^2\psi}{\sqrt{2\pi}} Q(e^{-S}V) = 0$$

The reason this vanishes is because it is the sum of two vanishing terms. The first is from $\overline{\psi}\partial_x$. The integral of the total derivative in x vanishes because the exponential is damped at $\pm\infty$. The second is from $\lambda h'(x)\partial_\psi$. This simply vanishes by the rules of Grassmann integration, as the integral of a total derivative is zero identically. We are therefore led to the conclusion that $Z'(\lambda) = 0$, so that $Z(\lambda)$ is in fact independent of λ.

This is good, because our original problem was $Z = Z(1)$. But since $Z(\lambda)$ is in fact λ-independent, we can look at it as $\lambda \to \infty$ and claim that it gives the exact result for Z. But in this limit, we can apply the steepest descent method because the bosonic

integral becomes tightly localized around the critical points of h. Since each critical point contributes with equal magnitude, we must sum over them. This gives

$$Z(\lambda \to \infty) \sim \sum_{x_c} \int \frac{dx d^2\psi}{\sqrt{2\pi}} e^{-\frac{\lambda^2}{2}(h''(x_c))^2(x-x_c)^2 - \lambda h''(x_c)\overline{\psi}\psi}$$

We have simply expanded everything to quadratic order about the critical point, keeping only the Gaussian terms in the bosons and fermions (we make the assumption that $h''(x_c) \neq 0$). Now, it is a trivial matter to perform the Gaussian integrals and one finds

$$Z(\lambda \to \infty) \sim \sum_{x_c} \sqrt{\frac{2\pi}{2\pi\lambda^2(h''(x_c))^2}} \cdot \lambda h''(x_c) = \sum_{x_c} \frac{h''(x_c)}{|h''(x_c)|}$$

But, by supersymmetry this equals Z so we find

$$Z = \sum_{x_c} \frac{h''(x_c)}{|h''(x_c)|}$$

Remarkably, we could evaluate this complicated integral exactly by supersymmetry. Since the computation ended up suppressing all contributions from the integral other than the immediate vicinity of the critical points of h, one says that the integral localizes to the critical points of h and talks about supersymmetric localization.

Actually, in this particular case the integral simplifies even further. If h is an even polynomial, $Z = \pm 1$ depending on the sign of the leading coefficient. This follows elementarily from the fact that the signs cancel in pairs at the turning points in this case. Similarly, for odd h we find $Z = 0$ because all the critical points cancel out.

The fact that the integral equals only $-1, 0$, or 1 implies that it is counting something. As we will see shortly, it is in fact counting the ground states of a supersymmetric quantum mechanical system.

2.5.3 Localization as a Fixed Point Theorem

The localization technique is undoubtedly powerful, but its origins still seem mysterious. Why do we need a fermionic symmetry for it to work? Clearly, this played some role in allowing us to drop the total derivatives and fixing the functional form of the action, but we don't really have a fundamental explanation as to why this is the case.

Here, I will give a more fundamental explanation. I will be very sketchy and heuristic, and I will skip many details. The main point is that readers understand the final result, which is the general statement of supersymmetric localization.

Suppose we want to integrate over some space $\mathcal{C}$ (said space could contain both bosonic and fermionic coordiantes) and the integral is invariant under a continuous symmetry parameterized by coordinates which I will generically refer to as θ_i. For example, in the case $\mathcal{C} = \mathbb{R}^2$ we can consider integrands $f(x, y)$ which are rotationally invariant. In this case, the continuous parameter of the symmetry is θ, the rotation angle. We can then decompose $\int dxdyf(x, y) = (\int d\theta) \int drrf(r)$ by going to polar coordinates.

This is in fact a general phenomenon–provided the symmetry acts *freely*, that is, without fixed points, one can pick out a coordinate system involving the $d\theta_i$'s and some invariant quantities parameterizing the rest of the space. For instance, in $\mathbb{R}^N$, the rotations have more angles parameterizing them and these become the θ_i. Integrals over invariant quantities will then factor into integrals over just the θ_i directions times a piece coming from the actual integral of the quantity of interest, just as in the polar coordinates case.

In the case that one considers supersymmetries, the parameters θ_i are Grassmann variables and $\int d\theta_i = 0$ identically. But, there may be a fixed set $\mathcal{C}_0 \subset \mathcal{C}$ invariant under the supersymmetry. An attempt to change coordinates there would break down, much as the polar coordinates are not good at the origin, the fixed point of the rotations. Therefore, our argument fails for points on $\mathcal{C}_0$ and we conclude that the integral can only receive nonzero contributions from $\mathcal{C}_0$. One can then localize the integral to the fixed set by expanding about the fixed set of the supersymmetry. As a consequence of supersymmetry, the fixed set always corresponds to a minimum of the bosonic action, so that the localization expansion coincides with the saddle point argument.

For instance, in our case a fixed point of the supersymmetry must satisfy $h'(x) = 0$ for the fermion variations to vanish, the solutions of which are of course the critical points x_c.

The take-home message is that *as a matter of general principle, integrals invariant under a supersymmetry always localize to the fixed points of that supersymmetry. This follows directly from the rules of Grassmann integration.*

This gives a hint as to why supersymmetry is so powerful–in the path integral formalism, everything can be reduced to just Gaussian integrals, which can be computed if one has enough paper and patience with the determinants.

We now proceed to a more general study of quantum mechanical systems with these kind of symmetries.

2.6 The Structure of Supersymmetric Quantum Mechanics

I will now state, on a rather general and abstract level, the properties of supersymmetric quantum mechanical systems. I will then do one simple example which illustrates all of the general features. The big theme is that there are new conserved charges $Q, Q^\dagger$ which are fermionic and correspond in the way reviewed in section 1 to the supersymmetries of the system. Deducing the consequences of the presence of these charges will be our general story of supersymmetric quantum mechanics.

2.6.1 SUSYQM: General Aspects

We will now do everything in all generality. We have a Hilbert space $\mathcal{H}$, as always. In this space, we have an operator $(-1)^F$ with respect to which we have a $\mathbb{Z}_2$ gradation:

$$\mathcal{H} = \mathcal{H}^B \oplus \mathcal{H}^F$$

We have $(-1)^F = 1$ on $\mathcal{H}^B$ and -1 on $\mathcal{H}^F$. These are the "bosonic" and "fermionic" Hilbert spaces. Often times it turns out that F makes sense as an operator by itself and has eigenvalues in $\mathbb{Z}$.

We pick some Hermitian operator H and declare it to be the Hamiltonian. For supersymmetric quantum mechanics, there exists an operator Q and its adjoint $Q^\dagger$ satisfying

$$Q^2 = (Q^\dagger)^2 = 0$$

$$\left\{Q, Q^\dagger\right\} = 2H$$

$$\left\{Q, (-1)^F\right\} = \left\{Q^\dagger, (-1)^F\right\} = 0$$

A consequence of this algebra is that the supercharges are conserved $[H, Q] = \left[H, Q^\dagger\right] = 0$ and the Hamiltonian is even $\left[H, (-1)^F\right] = 0$. This means that H maps $\mathcal{H}^B$ to $\mathcal{H}^B$ and $\mathcal{H}^F$ to $\mathcal{H}^F$. By contrast, the supercharges map back and forth between the two.

An important consequence is that the Hamiltonian is automatically nonnegative; for any state $|\Psi\rangle$ we have

$$\langle\Psi| H |\Psi\rangle = \langle\Psi| \frac{1}{2}\left\{Q, Q^\dagger\right\} |\Psi\rangle = \frac{1}{2}(\|Q|\Psi\rangle\|^2 + \left\|Q^\dagger|\Psi\rangle\right\|^2)$$

This is clearly never negative, so as a consequence all eigenvalues of H are nonnegative. Furthermore, if $H = 0$ on a state $|\Psi\rangle$, it follows from the above that

$$Q|\Psi\rangle = Q^\dagger|\Psi\rangle = 0$$

In particular, $|\Psi\rangle$ is preserved under the supersymmetry. Such a state is called, for the obvious reason, a supersymmetric ground state.

As usual (spectral theorem), the Hilbert space can be decomposed into the eigenspaces of the Hamiltonian H:

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$$

$$H|_{\mathcal{H}_n} = E_n$$

We assume the spectrum of H to be discrete, as will be the case when there is any effective compactness at play. Since Q, $Q^\dagger$, and $(-1)^F$ commute with H they preserve all the energy levels. Hence, the gradation extends to the energy levels:

$$\mathcal{H}_n = \mathcal{H}_n^B \oplus \mathcal{H}_n^F$$

It is useful to consider the supercharge $Q_1 = Q + Q^\dagger$. It is the Hermitian part of Q. Then we have $Q_1^2 = 2H$ by the supersymmetry algebra. On each energy level, $Q_1^2 = 2E_n$. Hence Q_1 is invertible and defines an isomorphism:

$$\mathcal{H}_n^B \cong \mathcal{H}_n^F$$

In particular, since they are finite dimensional vector spaces, they have the same dimension. This means that bosonic and fermionic states always come in equal-energy pairs.

This construction fails on $\mathcal{H}_0$, where $Q_1^2 = 0$. Thus, there are not necessarily an equal number of bosonic and fermionic supersymmetric ground states.

We will exploit these algebraic facts when we deal with the Witten index, which we define presently.

2.6.2 Witten Index

The next important quantity in supersymmetric quantum mechanics is the Witten index, often denoted $\mathrm{Tr}(-1)^F$. This is really a shorthand for the definition

$$\mathrm{Tr}(-1)^F := \mathrm{Tr}(-1)^F e^{-\beta H} = \sum_{n=0}^{\infty} \dim \mathcal{H}_n^B e^{-\beta E_n} - \dim \mathcal{H}_n^F e^{-\beta E_n}$$

Here, $\beta > 0$ is any positive real number. Now, since we know that for $n > 0$, $\dim \mathcal{H}_n^B = \dim \mathcal{H}_n^F$ we have

$$\mathrm{Tr}(-1)^F = \dim \mathcal{H}_0^B - \dim \mathcal{H}_0^F$$

Namely, the number of bosonic supersymmetric ground states minus the number of fermionic ones. Notice that the Witten index is actually independent of β. This is a hint of a much larger deformation invariance of $\mathrm{Tr}(-1)^F$. We know that Hamiltonians never come alone. They depend continuously on parameters like mass, oscillator frequency, and so on. Consider continuously varying the parameters of the Hamiltonian while preserving supersymmetry. The states and eigenvalues will generally vary. Since the deformation preserves supersymmetry, there must always be an equal number of bosonic and fermionic excited states at each energy level at each step of the process. This means that states must vary in boson/fermion pairs. There will be a mixing of ground states and excited states: as the parameters vary, some excited states may lose energy and go down to zero and some ground states may get excited. However, since the excited states vary in pairs, this means that the number of bosonic ground states and fermionic ground states will always change by the same number. In particular, their difference is a constant.

There is one instance in which this deformation invariance can fail. We have assumed that the eigenvalues vary continuously with the parameters. However, sometimes the eigenvalues can jump discontinuously as the parameters are taken to a certain value. A good example of this is the harmonic oscillator on the line. As the frequency is tuned to zero, the eigenvalues go from a discrete set to a continuum. This is because the potential jumps discontinuously at infinity: it goes from growing like a power to zero everywhere. This spoils the effective compactness of the oscillator (all its excited states are bound states), and gives us the non-normalizable free particle states. So, there is really a deformation invariance with the caveat that there cannot be a discontinuous jump at infinity in field space. We will see examples of this shortly.

Recall that we derived path integral expressions for traces in the Hilbert space. In particular, for fermion path integrals, the boundary conditions were required to be antiperiodic in order to get the trace right. The insertion of $(-1)^F$ in such a trace would do nothing to the bosons but flip the sign of the fermions, hence changing the boundary conditions from antiperiodic to periodic. This gives the path integral representation of the Witten index:

$$\mathrm{Tr}(-1)^F = \int \mathcal{D}X\mathcal{D}\psi e^{-S(X,\psi)}$$

Here, X and ψ are shorthand for all bosonic and fermionic variables. Periodic conditions are taken for both bosons and fermions. This formula, which is basically the Feynman-Kac formula, is equivalent to the Atiyah-Singer index theorem.

2.6.3 Ground States and Q-Cohomology

The space of supersymmetric ground states is naturally isomorphic to the cohomology of the Q-operator. This section is devoted to defining and unpacking this statement.

As we have seen, one of the properties of supersymmetric ground states is that they satisfy $Q|\Psi\rangle = 0$, that is, they are invariant under the supercharge. Since $Q^2 = 0$, there is a class of states which are trivially invariant under Q, namely $|\Psi\rangle = Q|\chi\rangle$ for some state $|\chi\rangle$. These are the "boring" ones which don't really encode any information about the system, since the only property they rely on is $Q^2 = 0$, a feature generic to all systems. They know nothing about our particular choice of Q or Hamiltonian.

Therefore, to find the supersymmetric ground states of a system, one wishes to study solutions to the equation $Q|\Psi\rangle = 0$ which are not of the form $|\Psi\rangle = Q|\chi\rangle$. Some useful jargon is that states obeying $Q|\Psi\rangle = 0$ are Q-*closed* while the trivial ones of the form $|\Psi\rangle = Q|\chi\rangle$ are Q-*exact*. The proper mathematical machinery to formalize this idea is the notion of *cohomology*, which I will now explain.

This will be a bit easier to explain if we assume that F by itself is conserved and takes integer eigenvalues, rather than dealing with $(-1)^F$. This will be the case in all examples that we study. In this case, the Hilbert space splits as a direct sum over fermion numbers

$$\mathcal{H} = \bigoplus_{m=0}^{n} \mathcal{H}^m$$

We have assumed that F only takes eigenvalues from 0 to n. For example, in the case of the oscillator, $n = 1$. If one considers fermionic operators Ψ^i with $i = 1, ..., n$, the fermion number will vary between 0 and n by acting with all the raising operators. Moreover, the relevant commutator becomes $[F, Q] = Q$, so that Q raises the F eigenvalue by 1.

We can draw the following picture:

$$0 \xrightarrow{Q} \mathcal{H}^0 \xrightarrow{Q} \mathcal{H}^1 \xrightarrow{Q} ... \xrightarrow{Q} \mathcal{H}^n \xrightarrow{Q} 0$$

This has the following meaning. The arrow over Q simply says that Q maps each vector space linked by an arrow in the direction of the arrow. For instance, $Q : \mathcal{H}^1 \to \mathcal{H}^2$. Since $Q^2 = 0$, this sequence of vector spaces and maps is known as a *complex of vector spaces*. Formally, we say we have a complex if we have a sequence of vector spaces and maps as drawn above such that the image of the map coming from the left is contained within the kernel of the map going to the right, in other words that $Q^2 = 0$. The complex terminates because the sum over fermion numbers is finite: Q annihilates all of $\mathcal{H}^n$ because there are no states of F charge $> n$ (Pauli exclusion once again).

In light of the diagram and considerations above, let us work at a fixed fermion number p and define two states $|\Psi\rangle, |\Psi'\rangle \in \mathcal{H}^p$ as *cohomologous* if $|\Psi'\rangle - |\Psi\rangle = Q|\chi\rangle$ for some $|\chi\rangle \in \mathcal{H}^{p-1}$. If we define $|\Psi\rangle \sim |\Psi'\rangle$ when $|\Psi'\rangle$ and $|\Psi\rangle$ are cohomologous, it is easily shown that $\sim$ is an equivalence relation.

When looking at solutions to $Q|\Psi\rangle = 0$ (Q-closed states), instead of looking at the solutions themselves we can consider the equivalence classes under $\sim$, which it is easy to see form their own vector space (that is, the linear combination of equivalence classes is the same as the equivalence class of the linear combination). We denote this vector space

as $H^p(Q)$, the Q-cohomology at level p, and write it formally as $H^p(Q) = \ker Q / \operatorname{Im} Q$. This is simply a symbolic way of stating what I wrote in words in terms of equivalence classes: $H^p(Q)$ is the set of all Q-closed states which are not exact, or more precisely the set of all cohomology classes of Q-closed states. That is, we look at the set of all Q-closed states and identify two states if they differ by a Q-exact piece. Since fermion number is conserved, we can do this for each fixed p.

The powerful statement is that $H^p(Q) \cong \mathcal{H}_0^p$, the supersymmetric ground states of fermion number p. This is proved as follows. Since Q commutes with the Hamiltonian, we can decompose the action of Q on $\mathcal{H}$ into energy levels, so that in particular the Q-complex splits into energy levels as well. Let us look at the piece of the cohomology coming from an arbitrary energy level $E_n > 0$, that is, solutions to $Q|\Psi_n\rangle = 0$ with energy eigenvalue E_n (and fixed but arbitrary fermion number p). Since $E_n \neq 0$, we can write

$$|\Psi_n\rangle = \frac{1}{2E_n}(QQ^\dagger + Q^\dagger Q)|\Psi_n\rangle = Q\Big(\frac{1}{2E_n}Q^\dagger|\Psi_n\rangle\Big)$$

In passing to the last equality we have used Q-closure. But we see that $|\Psi_n\rangle = Q|\chi\rangle$, so in particular Q-closure implies Q-exactness for $E_n > 0$, that is the cohomology is completely trivial on the nonzero energy levels. One says that the following sequence is exact:

$$0 \xrightarrow{Q} \mathcal{H}_m^0 \xrightarrow{Q} \mathcal{H}_m^1 \xrightarrow{Q} \ldots \xrightarrow{Q} \mathcal{H}_m^n \xrightarrow{Q} 0$$

The m labels energy level here. Generally, a sequence is exact if the image of the map coming from the left is equal to the kernel of the map going to the right, which is precisely what we have shown explicitly here.

Then, for the purpose of Q-cohomology, we see that the only nontrivial contribution comes from $\mathcal{H}_0$, the zero energy states or supersymmetric ground states. But on this sector, $Q = 0$ identically so $\operatorname{Im} Q$ is just the zero vector and $\ker Q = \mathcal{H}_0$, so that the cohomology is simply equal to $\mathcal{H}_0$ itself. But since this is the only nontrivial contribution to the cohomology, we find the isomorphism $H^p(Q) \cong \mathcal{H}_0^p$.

From this result, we see that the Witten index is

$$\operatorname{Tr}(-1)^F = \sum_{p=0}^{n}(-1)^p \dim H^p(Q)$$

This quantity is known as the *Euler characteristic* of the complex.

One can consider some generalizations of these notions. In our context, the objects in our complex are all vector spaces and the map Q (known as the *coboundary operator*) is a linear transformation, so that the cohomology carries a natural vector space structure. In general, one can relax this to the objects being abelian groups with the coboundary operator a homomorphism squaring to zero, in which case the cohomology is naturally an abelian group. One speaks of a *cochain complex*. For this reason, one often refers to "cohomology groups" even when talking about the vector spaces! We will see why one might want this more general picture later on.

The concepts of coboundary operators and cohomologies play a major role in modern mathematics. Supersymmetry gives a natural context for them in physics.

2.6.4 Example: Supersymmetric Particle in a Potential

To actually see how to crank the gears of this formal machinery, we will formulate and solve the supersymmetric analog of the Hamiltonian $H = \frac{1}{2}p^2 + V(x)$, that is a single particle in one dimension moving in a potential $V(x)$.

Consider a system with action (we are in real time here)

$$S = \int dt \frac{1}{2}\dot{x}^2 + i\overline{\psi}\dot{\psi} - \frac{1}{2}(h'(x))^2 - \frac{1}{2}h''(x)\left[\overline{\psi}, \psi\right]$$

x is a bosonic variable and ψ is a complex Grassmann variable. It is straightforward to verify that this action is invariant under the following supersymmetry (that is, the Lagrangian transforms only by a total derivative):

$$\delta x = \epsilon\overline{\psi} - \overline{\epsilon}\psi$$

$$\delta\psi = \epsilon(i\dot{x} + h'(x))$$
$$\delta\overline{\psi} = \overline{\epsilon}(-i\dot{x} + h'(x))$$

The conserved charge and its complex conjugate associated to this symmetry is found via Noether's theorem as

$$Q = \overline{\psi}(i\dot{x} + h'(x))$$
$$\overline{Q} = \psi(-i\dot{x} + h'(x))$$

These are the classical supercharges.

We now wish to canonically quantize the system. The canonical momentum for x is found to be $p = \dot{x}$ (unsurprising), and the momentum conjugate to ψ follows as in the solely fermionic case as $p_\psi = i\overline{\psi}$. Promoting the position and momenta to quantum operators[3], we find the (anti)commutators

$$[x, p] = i$$
$$\left\{\psi, \overline{\psi}\right\} = 1$$

All other commutators (resp. anticommutators for fermions) vanish, in particular $\psi^2 = \overline{\psi}^2 = 0$.

We have not yet fully specified a quantum system until we have given a Hilbert space for these operators to act on. The natural Hilbert space of the bosons is simply $L^2(\mathbb{R}; \mathbb{C})$, complex-valued square integrable functions on $\mathbb{R}$ with the L^2 inner product. The natural Hilbert space for the fermions is the representation of the oscillator algebra $\mathcal{H} = \{|0\rangle\} \oplus \{|1\rangle\}$. Putting the two together we find the space

$$\mathcal{H} = L^2(\mathbb{R}; \mathbb{C})\,|0\rangle \oplus L^2(\mathbb{R}; \mathbb{C})\overline{\psi}\,|0\rangle$$

That is, a general state is

$$|\Psi\rangle = f_0(x)\,|0\rangle + f_1(x)\overline{\psi}\,|0\rangle$$

f_0 and f_1 are square integrable complex valued functions. We are using a somewhat perverse notation involving a mixture of wavefunction notation for the bosons and bra-ket notation for the fermions, but it should be clear what is going on.

[3]From here going forward, I will be very sloppy about the notation distinguishing operators and their classical analogs, assuming it to be clear from context.

On these states, all the operators act in the usual way. In particular, the supercharges are

$$Q = \overline{\psi}(\partial_x + h'(x))$$
$$Q^\dagger = \psi(-\partial_x + h'(x))$$

The Hamiltonian follows directly from the supersymmetry algebra $2H = \{Q, Q^\dagger\}$ as

$$H = \frac{1}{2}p^2 + \frac{1}{2}(h'(x))^2 + \frac{1}{2}h''(x)[\overline{\psi}, \psi]$$

The $L^2(\mathbb{R};\mathbb{C})\,|0\rangle$ piece of $\mathcal{H}$ is $\mathcal{H}^B$, and the $L^2(\mathbb{R};\mathbb{C})\overline{\psi}\,|0\rangle$ piece of $\mathcal{H}$ is $\mathcal{H}^F$. The fermion number operator is $F = \overline{\psi}\psi$, and indeed $(-1)^F$ has the desired properties.

To find the full energy spectrum of H is a typical intractable eigenvalue problem in quantum mechanics. The excited state eigenvalues and wavefunctions remain inaccessible outside of approximation schemes like perturbation theory. However, the novel feature of supersymmetry is that we can obtain the supersymmetric ground states *exactly*.

The essential reason for this is that zero energy states are not just solutions to $H\,|\Psi\rangle = 0$, a typically hard eigenvalue problem, but also to $Q\,|\Psi\rangle = Q^\dagger\,|\Psi\rangle = 0$ which are typically much *simpler* problems because the Q's are like square roots of the Hamiltonian. Let us see how this works explicitly. Inserting a general state $|\Psi\rangle = f_0(x)\,|0\rangle + f_1(x)\overline{\psi}\,|0\rangle$ into $Q\,|\Psi\rangle = Q^\dagger\,|\Psi\rangle = 0$, we get *first-order* linear differential equations

$$\Big(\frac{d}{dx} + h'(x)\Big)f_0(x) = 0$$

$$\Big(-\frac{d}{dx} + h'(x)\Big)f_1(x) = 0$$

This is trivially solved as

$$|\Psi\rangle = c_0 e^{-h(x)}\,|0\rangle + c_1 e^{h(x)}\overline{\psi}\,|0\rangle$$

We consider the case when $h(x)$ is a polynomial. In this situation, there are three cases:

- $h(x)$ is a polynomial of odd degree. In this case, there are no normalizable solutions, so there is no supersymmetric ground state. The Witten index $\mathrm{Tr}(-1)^F = 0$. One says supersymmetry is *spontaneously broken*, because the Hamiltonian is Q-invariant but there is no Q-invariant ground state.
- $h(x)$ is a polynomial of even degree with positive leading coefficient. In this case, the $e^{-h(x)}$ solution is normalizable, so there is one supersymmetric ground state and it is bosonic. $\mathrm{Tr}(-1)^F = 1$. Note that the number of supersymmetric ground states is totally independent of the detailed form of $h(x)$.
- $h(x)$ is a polynomial of even degree with negative leading coefficient. In this case, the $e^{h(x)}$ solution is normalizable, so there is one supersymmetric ground state and it is fermionic. $\mathrm{Tr}(-1)^F = -1$.

In all three cases, we see that the Witten index is independent under infinitely many deformations of h; it only depends on the asymptotic behavior of the potential. Moreover, the ground state wavefunction is very easy to obtain, even for complicated non-quadratic (e.g. non-harmonic oscillator) Hamiltonians.

It is also instructive to obtain the Witten index explicitly from a path integral argument. One has

$$\mathrm{Tr}(-1)^F = \int \mathcal{D}x(\tau)\mathcal{D}\overline{\psi}(\tau)\mathcal{D}\psi(\tau)\exp\left\{-\int_0^\beta d\tau \frac{1}{2}\dot{x}^2 + \overline{\psi}\dot{\psi} + \frac{1}{2}(h'(x))^2 + h''(x)\overline{\psi}\psi\right\}$$

At face value, this path integral looks complicated. However, we know from the general aspects of supersymmetry that it is in fact independent of β, so we can look at $\beta \to 0$. But we already saw how to handle this case in the context of the classical limit of quantum statistical mechanics. In terms of a Fourier mode decomposition, all nonconstant modes have infinite kinetic energy in this limit and they are localized to zero. The path integral reduces to a finite-dimensional integral over the constant modes, which in this case is

$$\mathrm{Tr}(-1)^F = \int \frac{dxd^2\psi}{\sqrt{2\pi}} e^{-\frac{1}{2}(h'(x))^2 - h''(x)\overline{\psi}\psi}$$

But this zero-dimensional integral itself has a supersymmetry, and can be studied using localization techniques, as we have already demonstrated. This gives

$$\mathrm{Tr}(-1)^F = \sum_{x_c} \frac{h''(x_c)}{|h''(x_c)|}$$

This equals $0, 1, -1$ in each of the three above cases.

The path integral can also be studied directly by using supersymmetric localization without worrying about β. Requiring a fixed point of the supersymmetry gives $\psi = \overline{\psi} = \dot{x} = h'(x) = 0$. That is, fermions are localized to a neighborhood of zero and we must keep only the quadratic terms in the fermionic variables, and $x(\tau)$ is localized to a neighborhood of constant maps into critical points of h. As before, we must sum over critical points for a sensible result.

Writing $x = x_c + \xi$ and going to quadratic order in ξ we find the path integral

$$\mathrm{Tr}(-1)^F = \sum_{x_c} \int \mathcal{D}\xi(\tau)\mathcal{D}\overline{\psi}(\tau)\mathcal{D}\psi(\tau)\exp\left\{-\int_0^\beta d\tau \frac{1}{2}\xi(-\partial_\tau^2 + (h''(x_c))^2)\xi + \overline{\psi}(\partial_\tau + h''(x_c))\psi\right\}$$

The Gaussian integrals reduce this to

$$\mathrm{Tr}(-1)^F = \sum_{x_c} \frac{\det(\partial_\tau + h''(x_c))}{\sqrt{\det(-\partial_\tau^2 + (h''(x_c))^2)}}$$

If we write $\omega_n = 2\pi i n/\beta$, we may write $\det(\partial_\tau + h''(x_c)) = \prod_{n\in\mathbb{Z}}(\omega_n + h''(x_c)) = h''(x_c)\prod_{n\geq 1}(|\omega_n|^2 + (h''(x_c))^2)$ and $\det(-\partial_\tau^2 + (h''(x_c))^2) = \prod_{n\in\mathbb{Z}}(-\omega_n^2 + (h''(x_c))^2) = (h''(x_c))^2\prod_{n\geq 1}(|\omega_n|^2 + (h''(x_c))^2)^2$. Putting it all together, we see the nonzero eigenvalues cancel each other exactly and we are left with

$$\mathrm{Tr}(-1)^F = \sum_{x_c} \frac{h''(x_c)}{|h''(x_c)|}$$

This cancellation of fluctuation determinants up to ± 1 is also a general consequence of supersymmetry.

It is a testament to the power of supersymmetry that we can obtain the exact ground state of this system. The exact ground state of an ordinary particle in a potential in one dimension is completely intractable by analytic techniques.

2.6.5 Summary

Let us review the lessons learned on our march through fermions and supersymmetry.

Fermionic systems are amusing toy systems with finite-dimensional Hilbert spaces which lead us to the introduction of the concept of the Grassmann algebra in order to write path integrals. Grassmann variables, while somewhat peculiar and abstract at first sight, are very easy to manipulate with some practice and help us build intuition for the behavior of fermionic systems. The notion of fermion defined here is connected back to one's traditional notion of fermion through the study of the field oscillators in quantum field theory.

Armed with the new notion of Grassmann variables, we are at last in a position to define a supersymmetry. We explained how these symmetries reduce typical path integral problems to more tractable integrals by means of localization techniques, and in particular how the saddle point method becomes exact.

Given the concept of supersymmetry, we can define in general terms the properties of a quantum mechanical system with a conserved charge corresponding to such a symmetry. We are led to the powerful notions of supersymmetric ground states and the Witten index. Most of our power in analyzing these systems comes from the comparatively immense control that we have over the ground state sector of the Hilbert space.

We study a simple example to bring all these concepts to life, namely a single supersymmetric particle in one dimension subject to a potential. We exactly solve for the ground states and explain how to use the path integral to compute the Witten index. This is much further than we could go analyzing the purely bosonic analog of this system. The fermions help us to gain control that we otherwise would not have.

Our objective moving forward will now be to apply all of these ideas from quantum mechanics and supersymmetry to differential topology, giving a beautiful realization of Hodge-de Rham theory and Morse theory.

3 Part 3: Basic Aspects of Topology

In order to appreciate the application of supersymmetric quantum mechanics to topology, one must have some notion of topology. Since a background in differential or algebraic topology is somewhat abnormal for a physics undergraduate student, we will here briefly review some relevant notions to give a flavor of the subject. The goal here is not completeness or precision, but merely enough of an idea of what is going on to understand where supersymmetric quantum mechanics fits into the picture.

3.1 Manifolds and Topological Invariants

I assume from a course on general relativity that readers have some notion of a smooth Riemannian manifold. I want to delve a bit deeper into how these are defined and the subtle differences in perspective between the mathematical and physical points of view.

In general relativity, one seeks to formulate a generally covariant theory, meaning that we must write equations in a way which treats all coordinate systems (reference frames) equally. This means that they are invariant under arbitrary reparameterizations of the spacetime $x^\mu \to x'^\mu$. However, most of the time in general relativity one is interested only in the *local* properties of the underlying spacetime manifold M. Curvature, for example, is a local notion. For just gravity, this is all one needs because there are plenty of interesting phenomena going on at the local level. The tidal forces felt by a group of neighboring particles, for example, stem from geodesic deviation, which ultimately goes back to curvature, a local quantity.

There is another side of the manifold story that is underemphasized in the general relativistic context, but becomes very important in other areas such as quantum field theory. This is the *global* nature of the manifold that we live on. All manifolds behave in the same way locally, because any one coordinate chart is as good as another. A very interesting (and challenging) question is how to tell these objects apart at the global level. Presumably, a better understanding of the global structure of four-manifolds could be very important for the study of the large-scale structure of spacetime, but we will study the question of global behavior for its own interest, independent of physics.

The issue of distinguishing manifolds at the global level is the discipline of *topology*. This is all about how the local patches of the manifold fit together to form some interesting global picture, and the different types of global pictures that can arise. For example, a sphere can never be globally equivalent to a torus. Deforming a sphere to a torus would require putting a hole in the sphere, but this is not allowed because manifolds are supposed to be smooth objects, and poking a hole is undoubtedly a non-smooth operation.

The central question of topology is the following: given two manifolds X and Y, does there exist a homeomorphism $f : X \to Y$? A homemorphism is simply a continuous invertible map with a continuous inverse. This simply formalizes the intuition that X and Y can be deformed into one another in an invertible way. If two things are homeomorphic, they are equivalent as far as topology is concerned. More typically, we have coordinate charts on X and Y so we can talk about differentiable mappings, and diffeomorphisms

are defined accordingly. In these notes, we work in the smooth world so that we deal with diffeomorphism only.

In general, if someone hands you two spaces X and Y, it's quite hard to say whether or not they are homeomorphic. There is no systematic procedure for tackling such a problem, simply because it is so broad (indeed, even trying to write out explicit expressions for maps between manifolds outside of the simplest cases can be a hassle of unwinding tedious coordinate definitions).

The big idea is that we can answer this question indirectly by assigning a quantity called a *topological invariant* $A(X)$ to a space X which is invariant under diffeomorphisms of X[4]. Then, one can say that if two spaces X and Y are topologically equivalent, $A(X) = A(Y)$. Of course, you can't go the other way in general–just because some invariant is the same does not imply the equivalence of spaces.

This immediately gives a way out of the bad situation–if we define an invariant $A(X)$ which is effectively computable by some method (such as, say, the study of a quantum mechanical system on X), we can compute it for two different manifolds X and Y and conclude that if $A(X) \neq A(Y)$, X and Y must be topologically distinct. This is very powerful.

The invariants $A(X)$ can be any type of structure imaginable. They can be numbers, groups, vector spaces, etc. Depending on the type of invariant one studies, one attaches a different prefix to the type of topology one is doing. For example, if one is studying some invariant $A(X)$ which is a group, that is, assigning groups to classify topological spaces, one is doing algebraic topology. If the invariant $A(X)$ depends on introducing some kind of differential structure (as ours will), one is doing differential topology.

These considerations can seem rather abstract and intangible if one is not familiar with how the game is played. As we do some more examples, it may help to revisit this section to keep the big picture in mind.

3.2 de Rham Cohomology

The topological invariants we will spend most of our time with are the de Rham cohomology groups. These can be motivated by thinking about elementary vector calculus.

When we are children, we learn that for some vector field $\vec{V}$ in $\mathbb{R}^3$, if $\nabla \times \vec{V} = 0$ then $\vec{V} = \nabla\phi$ for some function $\phi(x, y, z)$. If one replaces $\mathbb{R}^3$ by a general manifold M, this breaks down for the following reason. In any local region U_i of M, we can introduce coordinates so that everything acts just as if it were on $\mathbb{R}^3$, and we have a function ϕ_{U_i}. The issue is that the ϕ_{U_i} may not fit together appropriately to give a well-defined function on M, and in general will not. We can make this more precise as follows.

A differential p-form is simply a totally antisymmetric type $\binom{0}{p}$ tensor. In what follows, there are various factors of $p!$ that I omit for conceptual clarity.

For a p-form B, one writes $B = B_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}$. The wedge product simply means the antisymmetric part of the standard tensor product. Given this, we can define the

[4]Properly, this should be called a "smooth invariant" because we assume differential structure here, but the point will be irrelevant for our purposes.

wedge product of any two differential forms as $A \wedge B = A_{[i_1 \ldots i_q} B_{j_1 \ldots j_p]} dx^{i_1} \wedge \ldots \wedge dx^{i_q} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_p}$. Namely, we just do an ordinary tensor product and antisymmetrize over everything. Note the similarity with Grassmann variables–indeed, this procedure is equivalent to just saying forget the wedge product and treat the dx's as Grassmann objects. We will see that this is no coincidence, and takes on quite the profound meaning, in the next section.

We can define the exterior derivative $d = dx^i \partial_i$ as acting on differential forms by wedge multiplication of the dx and differentiating the components. Then, $dB = \partial_{[j} B_{i_1 \ldots i_p]} dx^j \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p}$. Note that if ∇_i denotes the standard Levi-Civita covariant derivative, then $d = dx^i \nabla_i$ also. This is because when we antisymmetrize in the bottom indices, all the connection coefficient terms get killed because they are symmetric in their lower indices.

It is an easy exercise to show that commutativity of partial derivatives implies $d^2 = 0$ (this is already reminiscent of a supersymmetry algebra). Just as in supersymmetry, we can study p-forms which satisfy $dB = 0$ but are not of the form $B = d\Lambda$ for a $(p-1)$-form Λ. Then, we define the de Rham cohomology $H^p(M) = \ker d / \operatorname{Im} d$. We write it as a function of M because all it depends on is the smooth structure on M (our ability to differentiate things)–there is literally no other structure around. Since we can consider closed and exact forms as vector spaces with the linear operator d going between them, the de Rham cohomology is naturally a vector space and its dimension $b_p = \dim H^p(M)$ is known as the p-th Betti number.

Since we have not introduced any Riemannian metric, or any structure at all on the manifold outside of the existence of differentiable coordinates (a smooth structure), the de Rham cohomologies are topological invariants characterizing the global structure of the manifold.

What does this have to do with vector calculus? It is easy to show that for 1-forms on $\mathbb{R}^3$ the exterior derivative naturally represents the curl, and our previous considerations say that $dV = 0 \to V = d\phi$. In other words, it implies that the de Rham cohomology is trivial. But we cannot expect this to be true in general–ϕ will not necessarily fit together over all of M, as discussed previously. So, the de Rham cohomology captures in this way the topological non-triviality of M.

At this stage, the cohomology groups can seem like rather abstract objects, as they kind of are, but they become more concrete with some practice. Also, their advantage is not really their intuitive accessibility, but their *computability*. They are in many cases simple to compute explicitly, and offer nontrivial topological information about a space. By contrast, there are another set of invariants called the *homotopy groups* which are a bit easier to conceive of intuitively, but are extremely difficult to compute in most cases. There must be a tradeoff somewhere!

3.3 An Example: S^1

As an example, we will study the de Rham cohomology of the circle S^1. On the circle, we have a differential form $d\theta$. Naively, this form is exact, but this cannot be so because $\int_{S^1} d\theta = 2\pi$. If $d\theta$ were exact, the integral would necessarily vanish by Stokes' theorem applied to differential forms.

The point is that θ is not actually a single valued function on S^1. It is a *locally* well-defined function, and we can even extend it to a well-defined function on $(-\pi, \pi)$, for example, but the issue is that it can never be a single-valued function on the whole circle due to the fact that it changes by 2π as we go around the circle. This is a very simple example of how the topology of a space can block a closed form from being exact, but it really is an excellent example to have in mind for the general case.

This brings us to the conclusion of our lightning review of de Rham cohomology. The point was not completeness or correctness, but simply to define a simple example of a topological invariant to make some of the ideas in the preceding section more concrete.

4 Part 4: Supersymmetry and Morse Theory

In this section, we study supersymmetric quantum mechanics on a Riemannian manifold (M, g). This allows us to study the topology of M in terms of the ground state structure of the supersymmetric system. From path integrals, we will obtain a simple example of the Atiyah-Singer index theorem, the Chern-Gauss-Bonnet theorem. The supersymmetric ground states will give us a physical interpretation of Hodge theory. By adding a potential, we will obtain Witten's physical interpretation of Morse theory.

4.1 Supersymmetric Sigma Model

For unfortunate historical reasons, mechanical systems describing maps into Riemannian manifolds have become known as "sigma models". For uniformity with the field theory literature we will adopt this nomenclature. The standard sigma model action would be

$$S = \int dt \frac{1}{2} g_{IJ} \dot{\phi}^I \dot{\phi}^J$$

Here, $\phi^I(t)$ is the path in local coordinates and g is the metric. The critical points of this action are simply the geodesics on M.

The supersymmetric extension of this is the following (we are in real time):

$$S = \int dt \frac{1}{2} g_{IJ} \dot{\phi}^I \dot{\phi}^J + i g_{IJ} \overline{\psi}^I D_t \psi^J - \frac{1}{2} R_{IJKL} \psi^I \overline{\psi}^J \psi^K \overline{\psi}^L$$

Here, $D_t \psi^I = \dot{\psi}^I + \Gamma^I_{JK} \dot{\phi}^J \psi^K$ is the usual covariant derivative along the path $\phi(t)$. The first two terms are simply the covariant version of the ordinary supersymmetric action for a particle. The four-fermi term can seem rather ad-hoc and unpleasant. It is easy to see that it must be there if we followed the general formalism and introduced a superspace. It is related to what are known as Weitzenböck identities in mathematics.

This action is invariant under the following supersymmetry:

$$\delta \phi^I = \epsilon \overline{\psi}^I - \overline{\epsilon} \psi^I$$

$$\delta \psi^I = \epsilon (i \dot{\phi}^I - \Gamma^I_{JK} \overline{\psi}^J \psi^K)$$
$$\delta \overline{\psi}^I = \overline{\epsilon} (-i \dot{\phi}^I - \Gamma^I_{JK} \overline{\psi}^J \psi^K)$$

This indicates that one should think of the fermions as sections of the pullback tangent bundle $\phi^* TM$ because the infinitesimal deformation of ϕ (a point on M) is a fermion and they must be varied with a compensating connection coefficient term (that is, differentiation of a section is only well-defined after we have specified parallel transport with respect to a particular connection, in our case the natural Riemannian one).

One can also see that the supersymmetries mandate the four-fermi term, because the variation of the connection in the term $D_t \psi$ will produce a curvature (the only way to differentiate a connection in a generally covariant way) and this must be cancelled by the

four-fermi term. The reason why the curvature term itself does not acquire a derivative term that must be cancelled is because of the Bianchi identity.

From the Noether procedure, one can see that these symmetries give rise to the conserved charges

$$Q = ig_{IJ}\overline{\psi}^I \dot{\phi}^J$$
$$\overline{Q} = -ig_{IJ}\psi^I \dot{\phi}^J$$

These are simply the obvious covariant extensions of the usual supercharges.

The action is also invariant under phase rotations of the fermions, $\psi^I \to e^{i\alpha}\psi^I$, $\overline{\psi}^I \to e^{-i\alpha}\overline{\psi}^I$. The corresponding conserved charge is

$$F = g_{IJ}\overline{\psi}^I \psi^J$$

Again, this is the obvious covariant version of fermion number.

4.1.1 Canonical Quantization

Now that we are given the action, we can quantize the system. The canonical momenta are

$$p_I = g_{IJ}\dot{\phi}^J + ig_{JK}\Gamma^J_{IL}\overline{\psi}^K\psi^L$$
$$\pi_I = ig_{IJ}\overline{\psi}^J$$

We then impose canonical commutation relations $\left[\phi^I, p_J\right] = i\delta^I_J = \left\{\psi^I, \pi_J\right\}$. The fermion relation reduces to $\left\{\psi^I, \overline{\psi}^J\right\} = g^{IJ}$. The bosonic relation is the usual one, implying that p must be represented by a derivative with respect to ϕ.

We must specify a Hilbert space for these operators to act on. We can follow the recipe of the simpler case of a particle in a potential and look at the tensor product of the usual bosonic and fermionic Hilbert spaces. The bosonic space is simply functions $f(\phi)$ and the fermionic space we obtain by looking at the oscillator algebra.

Let us start with the fermions. This time, there are $n = \dim M$ fermion oscillators, meaning that we have n different creation and destruction operators. We define the ground state $|0\rangle$ to be annihilated by all of the destruction operators: $\psi^I|0\rangle = 0$ for $I = 1, ..., n$. The fermionic Hilbert space is therefore generated by $|0\rangle, \overline{\psi}^I|0\rangle, \overline{\psi}^I\overline{\psi}^J|0\rangle, ..., \overline{\psi}^{I_1}...\overline{\psi}^{I_n}|0\rangle$ for a total dimension of 2^n. There are no higher states after the one with all n raising operators acting on the ground state due to Pauli exclusion ($\overline{\psi}^2 = 0$).

Putting these together with the bosons, we see that a general state is

$$|\Psi\rangle = f(\phi)|0\rangle + \alpha_I(\phi)\overline{\psi}^I|0\rangle + \frac{1}{2}\beta_{IJ}(\phi)\overline{\psi}^I\overline{\psi}^J|0\rangle + ... + \frac{1}{n!}\omega_{I_1...I_n}(\phi)\overline{\psi}^{I_1}...\overline{\psi}^{I_n}|0\rangle$$

We can now note that all of the fermionic operators carry upper tangent vector indices, meaning that the coefficients carry lower tangent vector indices (as the state must be a scalar). Moreover, as the fermionic operators anticommute with one another, the

expressions are antisymmetric in their indices. We see that a general state is given as a *differential form* on the manifold M!

The dictionary between the two is

$$|0\rangle \leftrightarrow 1$$
$$\overline{\psi}^I |0\rangle \leftrightarrow dx^I$$
$$\overline{\psi}^I \overline{\psi}^J |0\rangle \leftrightarrow dx^I \wedge dx^J$$

and so on.

The Hilbert space is then $\mathcal{H} = \Omega^*(M)$, the space of all differential forms on M. The inner product of two forms is the obvious one: contract indices to obtain a scalar and integrate over the manifold with respect to the invariant measure. If α and β are p-forms, the inner product is

$$\langle \alpha | \beta \rangle = \int_M d^n x \sqrt{g} \frac{1}{p!} \alpha_{I_1 \ldots I_p} \beta^{I_1 \ldots I_p} = \int_M \alpha \wedge * \beta$$

In the third equality I have written it more invariantly in terms of the Hodge dual. To take the inner product of states containing forms of different degree, simply add the results for the components of fixed degree. Also, this can be extended to complex valued forms in the obvious way, but we will not need this detail.

Here is a minor but sometimes confusing point. In order to represent the operator algebra, we must have that $p_I \to -i\partial_I$. However, if we want to look at the action of a supercharge, we need to look at the velocity $g_{IJ}\dot{\phi}^J$. The operator representation of the velocity can be obtained by simply subtracting the connection component to the other side. We see that the action of the velocity is promoted to the *covariant* derivative. This is much like in the electromagnetic case where we replace $p^2/2m$ by $(\partial - iA)^2/2m$ with A the vector potential.

Hence the action of the supercharge is simply

$$Q \to i(-i\overline{\psi}^I \nabla_I) = dx^I \nabla_I \wedge = dx^I \partial_I \wedge = d$$

The supercharge reduces to the exterior derivative on forms. $Q^\dagger \to d^\dagger$, the natural adjoint of the exterior derivative with respect to the inner product. In components, if β is a p-form, $(d^\dagger \beta)_{I_1 \ldots I_{p-1}} = -\nabla^J \beta_{J I_1 \ldots I_{p-1}}$. Namely, $d^\dagger$ simply takes the covariant divergence.

Finally, I should comment on the action of the operator ψ^I, which may be slightly unfamiliar. In differential geometry language, it is given by $g^{IJ} i_J$, which means "interior multiplication". If v^I is a vector field on M, then $v_i \psi^I \to i_v$. This simply strips away a dx^I and replaces it with a v^I, and adds the appropriate antisymmetric pieces so that it maps p-forms to $(p-1)$-forms.

So, we see that the Hilbert space naturally realizes the structure of differential forms and the differential operators acting upon them. This essentially means that supersymmetric quantum mechanics is a neat organizational tool, but we have not yet gotten any real mileage out of it. We will see this in the next sections, when by analyzing the dynamics of this system we can learn highly nontrivial information about the topology of M.

4.1.2 Supersymmetric Ground States: Hodge Theory

Analyzing the supersymmetric ground states of this model gives a physical interpretation to Hodge theory, which I will now explain. Remarkably, this will all follow from the general structure of supersymmetric quantum mechanics.

The Hamiltonian for our system follows directly from the supersymmetry algebra as

$$H = \frac{1}{2}\{Q, Q^\dagger\} = \frac{1}{2}(dd^\dagger + d^\dagger d) = \frac{1}{2}\Delta$$

This operator is known as the Laplace-Beltrami operator or Hodge Laplacian. As an easy exercise, verify that on functions it reduces to (minus) the Laplacian, that is $\Delta f = -\nabla^I \nabla_I f$. The reason for the minus sign is that it is the negative Laplacian which is a positive operator (which, by supersymmetry, H must be)–the ordinary Laplacian is negative definite.

Supersymmetric ground states are those differential forms satisfying $\Delta\omega = 0$, or equivalently $d\omega = d^\dagger\omega = 0$. These are the solutions to Laplace's equation, known as *harmonic forms* (by analogy with harmonic functions). We denote the space of harmonic forms as $\mathcal{H}_{(0)}$ (note that they do indeed form a vector space–all this follows from the general structure of supersymmetric quantum mechanics).

Now, recalling our general discussion of supersymmetric quantum mechanics, we demonstrated that the space of supersymmetric ground states of a system is isomorphic to the cohomology of the Q operator. In this case, Q is simply the d operator on differential forms, so its cohomology is simply the de Rham cohomology of the manifold M, $H^*(M)$. *This is a topological invariant*! What we then see is that $H^*(M) \cong \mathcal{H}_{(0)}$, or (since everything respects grading by the F-charge or fermion number) $H^p(M) \cong \mathcal{H}^p_{(0)}$.

The remarkable aspect of this is that the Laplacian depends very sensitively on the metric we put on M, and the harmonic forms will change under a metric deformation. But, the *number* of linearly independent solutions to Laplace's equation, the Betti number $b_p(M)$, is a topological invariant and is independent of metric deformations. The study of topology (specifically, cohomology groups) by counting solutions to this partial differential equation is known as *Hodge theory*.

Hodge theory generalizes beautifully in the context of complex algebraic geometry, which is out of the scope of what we will consider here. Physically, this simply corresponds to the fact that we can have a system with so-called *extended supersymmetry*, where we have more than one supercharge Q_i. In the case where we have two complex supercharges ($N = 2$ supersymmetry), we get Hodge theory on compact Kähler manifolds, or from the algebraic perspective, complex projective varieties.

4.1.3 Witten Index: Chern-Gauss-Bonnet Theorem

From our general theory of supersymmetric quantum mechanics, we know that the index of the system is

$$\mathrm{Tr}(-1)^F = \sum_{p=0}^{n}(-1)^p \dim \mathcal{H}^p_{(0)} = \sum_{p=0}^{n}(-1)^p b_p(M) \equiv \chi(M)$$

This is another topological invariant called the *Euler characteristic* of the manifold M.

We can also compute the index by a path integral argument. The path integral representation of the index is

$$\mathrm{Tr}(-1)^F e^{-\beta H} = \int \mathcal{D}\phi \mathcal{D}\overline{\psi} \mathcal{D}\psi e^{-S_E}$$

The Euclidean action is

$$S_E = \int d\tau \frac{1}{2} g_{IJ} \dot{\phi}^I \dot{\phi}^J + g_{IJ} \overline{\psi}^I D_\tau \psi^J + \frac{1}{2} R_{IJKL} \psi^I \overline{\psi}^J \psi^K \overline{\psi}^L$$

We can use independence of the index on β to localize the path integral to the constant maps. From our general arguments, the path integral reduces to

$$\chi(M) = \int_M \frac{d^n \phi d^n \overline{\psi} d^n \psi}{(2\pi)^{n/2} \sqrt{g}} e^{\frac{1}{4} R_{IJKL} \psi^I \psi^J \overline{\psi}^K \overline{\psi}^L}$$

We have rearranged the exponential using the Bianchi identity. The factor of $\sqrt{g}$ gives the correct invariant measure, accounting for the extra Jacobians coming from the fermion measures under general coordinate transformations.

First, note that the integral vanishes if n is odd. Therefore we set $n = 2m$.

At this point, we can do something clever. Note that R_{IJKL} is antisymmetric in its first two and last two indices. We can then define an antisymmetric matrix-valued 2-form by $R_{IJ} = \frac{1}{2} R_{IJKL} dx^K \wedge dx^L$. By the rules of our correspondence, this is $R_{IJ} = \frac{1}{2} R_{IJKL} \overline{\psi}^K \overline{\psi}^L$. It is an easy exercise to verify that the integral $d^n \phi d^n \overline{\psi}$ produces the usual integration of differential forms. Then the integral reduces to

$$\int \frac{d^{2m} \psi}{(2\pi)^m \sqrt{g}} e^{\frac{1}{2} R_{IJ} \psi^I \psi^J}$$

We now see that the fermion integral produces a Pfaffian (the factor of $\sqrt{g}$ just compensates for the fact that the ϵ-symbol is not actually a tensor, a familiar story from differential geometry). Then we have

$$\chi(M) = \int_M \mathrm{Pf}\Big(\frac{R}{2\pi}\Big)$$

This is the *Chern-Gauss-Bonnet* theorem. Again, the meaning of the expression Pf is that we consider R as an antisymmetric matrix with 2-form entries, then take the Pfaffian with the accompanying factor of $\sqrt{g}$ to produce coordinate invariance.

This is a remarkable because on the left hand side, we have $\chi(M)$, a topological invariant which depends only on *algebraic* data, namely the dimension of the cohomology groups as vector spaces. On the right hand side, we have a *geometric* quantity, the curvature, integrated over the manifold. Moreover, the curvature is *local*, it varies pointwise. But somehow, it knows about the topology of M! This is a remarkable result. For example, for surfaces (compact orientable two-manifolds without boundary) it can be shown that $\chi(M) = 2(1-g)$, where g is the genus or number of handles in the surface. Also, the

Pfaffian just reduces to a multiple of the scalar curvature. The theorem then states that, for example, there can be no metric with everywhere negative curvature on a surface with the topology of the sphere. Moreover, it illustrates that on surfaces of genus $g \geq 2$, the curvature must be negative at least somewhere.

Another lesson of this is that one cannot assign a completely arbitrary metric tensor on a given manifold. It must satisfy topological constraints based on the global structure of M, namely its cohomology groups.

The Chern-Gauss-Bonnet theorem is a simple example of an index theorem, the most general of which being the Atiyah-Singer index theorem. This arises because $\chi(M)$ is not just any integer, but a signed count of the zero modes of the elliptic operator Δ. I will now explain how the theorem fits into the general setting of index theorems.

Recall that a differential one-form can be viewed as a section of the cotangent bundle T^*M. Similarly, a differential p-form can be viewed as a section of the p-th exterior power, $\Lambda^p T^*M$. Define the bundles

$$E = \bigoplus_{\text{even}} \Lambda^p T^*M$$

$$F = \bigoplus_{\text{odd}} \Lambda^p T^*M$$

The operator $D := d + d^\dagger$ is then a map of sections $D : \Gamma(E) \to \Gamma(F)$. As E and F are vector bundles, the spaces of sections $\Gamma(E)$, $\Gamma(F)$ have the structure of infinite-dimensional vector spaces (indeed, they are simply the bosonic and fermionic Hilbert spaces), and we can think of D as a linear map. We can then take the adjoint with respect to our inner product $D^\dagger : \Gamma(F) \to \Gamma(E)$. Note that while D is naively self-adjoint, it is actually not because D only acts on the bosonic space $\Gamma(E)$ while $D^\dagger$ acts on the fermionic space $\Gamma(F)$.

Regarded as a linear map, the dimension of the kernel of D is simply the sum of the even Betti numbers. Similarly, the dimension of the kernel of $D^\dagger$ is the sum of the odd Betti numbers. The Chern-Gauss-Bonnet theorem becomes the statement

$$\dim\ker D - \dim\ker D^\dagger = \int_M \mathrm{Pf}\Big(\frac{R}{2\pi}\Big)$$

Index theorems are general statements of this type, relating the difference between the dimension of a kernel of an operator and its adjoint to the integral of some local quantity. They are very useful in quantum field theory, for example, where they contain information about anomalies. A symmetry of a classical theory is said to be anomalous if it is no longer a symmetry quantum mechanically. This arises because we can define the path integral measure as a mode decomposition, but if there is a mismatch of zero modes and we perform a symmetry transformation, the measure may not be invariant, signaling an anomaly in the quantum theory.

The Chern-Gauss-Bonnet theorem deals with gravitational anomalies, which (if nonzero) imply that a quantum theory cannot consistently be coupled to quantum gravity.

As a final comment, the path integral computation can also be done by supersymmetric localization, but it is a little bit more subtle because working out the quadratic fluctuations requires varying the action in a covariant way (and the calculation is much harder).

4.2 Deforming by a Potential

We have studied all that we could from the general properties of supersymmetric quantum mechanics applied to the sigma model. Here, we will deform this system by a potential and end up with a physical interpretation of Morse theory.

4.2.1 Action and Conserved Charges

The modification to the action is the covariant version of the supersymmetric particle in a potential. It is given by the following monstrosity:

$$S = \int dt \frac{1}{2} g_{IJ} \dot{\phi}^I \dot{\phi}^J + i g_{IJ} \overline{\psi}^I D_t \psi^J - \frac{1}{2} R_{IJKL} \psi^I \overline{\psi}^J \psi^K \overline{\psi}^L - \frac{1}{2} g^{IJ} \partial_I h \partial_J h - \frac{1}{2} D_I \partial_J h \left[\overline{\psi}^I, \psi^J \right]$$

The supersymmetry transformations are

$$\delta \phi^I = \epsilon \overline{\psi}^I - \overline{\epsilon} \psi^I$$

$$\delta \psi^I = \epsilon (i \dot{\phi}^I - \Gamma^I_{JK} \overline{\psi}^J \psi^K + g^{IJ} \partial_J h)$$
$$\delta \overline{\psi}^I = \overline{\epsilon} (-i \dot{\phi}^I - \Gamma^I_{JK} \overline{\psi}^J \psi^K + g^{IJ} \partial_J h)$$

The supercharges are modified to

$$Q = \overline{\psi}^I (i g_{IJ} \dot{\phi}^J + \partial_I h)$$

$$\overline{Q} = \psi^I (-i g_{IJ} \dot{\phi}^J + \partial_I h)$$

We still have the phase rotation symmetry and fermion number

$$F = g_{IJ} \overline{\psi}^I \psi^J$$

The Hilbert space and commutation relations are of course unchanged in the process of adding a potential. The supercharge gets deformed to

$$Q_h = d + dh \wedge = e^{-h} d e^h$$

Similarly,

$$Q_h^\dagger = e^h d^\dagger e^{-h}$$

Now, the big insight of Witten is that *the Q_h cohomology is independent of h.* This follows simply because any d-closed form can become Q_h closed by simply multiplying by e^{-h}. This establishes an isomorphism of the two cohomologies. We conclude that *the number of supersymmetric ground states is independent of h.* In particular, the number of supersymmetric ground states of degree p is $b_p(M)$, completely independent of the choice of function h. We will now use this to our advantage to learn a tremendous amount about the properties of this system.

4.2.2 Semiclassical Limit: Perturbative Analysis

Since the space of supersymmetric ground states is independent of h, we can deform h to a limit where the system becomes easy to solve. A useful trick is the same one we used in the zero-dimensional example of localization: rescale $h \to \lambda h$ and take $\lambda >> 1$. As argued above, this will not change the number of supersymmetric ground states (of course, the explicit functional form of the states will change, but not the number).

As $\lambda \to \infty$, a simplification occurs. This is easiest to see in the path integral of the system. For large λ, the action becomes very large if the path is not at a critical point of h. Therefore, the integral becomes sharply peaked about these critical points and an expansion about them becomes equivalent to $1/\lambda$ perturbation theory (this is just an infinite-dimensional version of the saddle point method for ordinary integrals).

Supersymmetric ground states are the zero eigenvalues of the Hamiltonian H_λ, which is easily calculated to be

$$H_\lambda = \frac{1}{2}\Delta + \frac{\lambda^2}{2} g^{IJ}\partial_I h \partial_J h + \frac{\lambda}{2} D_I \partial_j h \left[\overline{\psi}^I, \psi^J\right]$$

As $\lambda \to \infty$, we can simplify the computation by expanding about the critical point. The leading behavior is captured by the quadratic approximation. Consider the critical point x_i. We can introduce Riemann normal coordinates at x_i, so that the Laplace-Beltrami operator reduces to the flat space $-\partial_I^2$, which I will abbreviate as p_I^2. Similarly, $D_I \partial_J h(x_i) = \partial_I \partial_J h(x_i)$ in normal coordinates[5]. The fermion piece is already quadratic in those variables, so we simply replace the h piece by $H_{IJ} = \partial_I \partial_J h(x_i)$, the Hessian matrix. The bosonic potential piece is obtained similarly by expanding ∂h, and all in all we find

$$H_\lambda^{(0)} = \frac{1}{2} p_I^2 + \frac{\lambda^2}{2} H_{IJ} H_{IK} (x - x_i)^J (x - x_i)^K + \frac{\lambda}{2} H_{IJ} \left[\overline{\psi}^I, \psi^J\right]$$

All other terms in the expansion can be considered subleading in perturbation theory.

This Hamiltonian is quadratic in the variables, so we may solve it by diagonalizing. Since the matrix of second derivatives H_{IJ} is symmetric, we can rotate the normal coordinate system to diagonalize it. We make here the key assumption that the critical points of h are *isolated* (which was already an implicit assumption–that is, they form a finite, totally disconnected set) and *nondegenerate*, meaning that H_{IJ} has no zero eigenvalues. We will denote the eigenvalues of H by c_I. After diagonalizing, the Hamiltonian becomes

$$H_\lambda^{(0)} = \sum_I \left(\frac{1}{2} p_I^2 + \frac{\lambda^2}{2} c_I^2 ((x - x_i)^I)^2 + \frac{\lambda}{2} c_I \left[\overline{\psi}^I, \psi^I\right] \right)$$

This is the sum of n decoupled systems, one for each eigenvalue. Moreover, the I-th system is itself a sum of two decoupled pieces, a bosonic and a fermionic oscillator. We can solve each of these! Note here that c_I can be positive or negative (so long as it is not zero), so the frequency of the bosonic oscillator (always assumed positive) is $|c_I|$.

[5] Actually, in any coordinate system since the critical point condition sets the Levi-Civita piece to zero. Therefore the Hessian matrix at a critical point is actually an intrinsic (coordinate-independent) notion.

The eigenvalues for the bosonic oscillator (at fixed I) are $\lambda|c_I|(N^I+\frac{1}{2})$. Here, $N^I = 0, 1, ...$ as usual.

The eigenvalues for the fermionic oscillator (at fixed I) are $\lambda\frac{c_I}{2}\epsilon_I$. Here, $\epsilon_I = \pm 1$. $\epsilon_I = +1$ if $\overline{\psi}^I$ is present on the state, and $\epsilon_I = -1$ if it is absent (this is just the usual story from the Fermi oscillator).

Hence, the eigenvalues of the system as $\lambda \to \infty$ are

$$E = \sum_I \lambda\Big(\Big(N^I + \frac{1}{2}\Big)|c_I| + \frac{1}{2}c_I\epsilon_I\Big)$$

For a supersymmetric ground state, this must be zero exactly, so in particular it must vanish at leading order in perturbation theory. This implies that $N^I = 0$ for all I, and that we must choose ϵ_I to cancel off the zero point energy of the bosonic oscillator. If $c_I > 0$, we choose $\epsilon_I = -1$ for cancellation. If $c_I < 0$, we choose $\epsilon_I = +1$ for cancellation. Using what we know about these systems, we conclude that the perturbative ground state in the limit of large λ is

$$\Psi_i^{(0)} = e^{-\frac{\lambda}{2}\sum_I |c_I|((x-x_i)^I)^2} \prod_{\{J|c_J<0\}} \overline{\psi}^J |0\rangle$$

Note that the ground state is indeed sharply peaked about the critical point for large λ, so our assumption in expanding near x_i is justified. Since the critical point we chose to expand about was arbitrary, this holds for each one. Therefore, we find exactly one perturbative ground state associated to each critical point. The fermion number (form degree) of the i-th ground state is given by μ_i, the number of negative eigenvalues of H_{IJ} at that critical point. This number is called the *Morse index* of the critical point.

We have not established that these are exact ground states of the system. We have simply shown that their energy vanishes in the limit of large λ. This is a necessary, but not sufficient, condition for the energy to be exactly zero, independently of λ.

Already, we can establish an upper bound on the Betti numbers from this. If M_p denotes the number of critical points of Morse index p, we must have $b_p \leq M_p$. This is because all other states besides the perturbative ground states have diverging energy in the $\lambda \to \infty$ limit. If a state is to be a supersymmetric ground state, it therefore must be a perturbative ground state, so we can establish that the supersymmetric ground states are a subset of the perturbative ground states. As explained above, this does not go the other way, so we obtain the inequality $b_p \leq M_p$. These are known as the *weak Morse inequalities*.

In the next section, we will see how we can strengthen this result to actually obtain the exact number of supersymmetric ground states, thereby successfully computing the cohomology groups of the manifold M.

Already from our less accurate computation we can compute the Witten index. This is because the Witten index is independent of λ and therefore equal to its $\lambda \to \infty$ limit. In this limit, we obtain

$$\mathrm{Tr}(-1)^F = \mathrm{Tr}(-1)^F e^{-\beta H} = \sum_i (-1)^{\mu_i} = \chi(M)$$

Here, the sum denotes a sum over critical points.

4.2.3 Semiclassical Limit: Nonperturbative Effects

The analysis of the system in the limit of large λ has shown that the Hilbert space essentially breaks off into two pieces: a low-lying piece consisting of states with vanishing energy as $\lambda \to \infty$, and a piece consisting of states with diverging energy as $\lambda \to \infty$. We denote the subspace of finite energy states in the $\lambda \to \infty$ limit as $X \subset \mathcal{H}$. It is simply the (finite-dimensional!) subspace spanned by the perturbative ground states. The true ground states all lie within X, so we can restrict our further analysis to be purely contained within X.

What does it mean for a state to be a perturbative, but not true ground state? The true supersymmetric ground states of the system are characterized by the fact that their energy is exactly zero, for all values of λ. Perturbative ground states simply have their energy vanish as $\lambda \to \infty$. Therefore, we must search for processes by which the perturbative ground states can gain energy outside of the leading large-λ approximation.

One's first thought might be that we will see this nonzero energy at some order in the $1/\lambda$ expansion. That is, even though the energy is zero at leading order, there will be a higher order term which is not identically zero, but proportional to some power of $\frac{1}{\lambda}$ and so vanishes in the limit. This cannot be right, essentially because the perturbation expansion is only local on M. It only knows about the region of M in the immediate vicinity of the critical point. There is no way to determine if the critical point is a fake or real ground state from this data; the real ground states are sensitive to the topology of M, its global structure. The mechanism which gives some of the states energy must be sensitive to the existence of multiple critical points on M. The statement is therefore that one can, order by order in perturbation theory, find a state Ψ_i obtained by adding corrections to the leading order answer $\Psi_i^{(0)}$ which remains a zero energy state to all orders in $1/\lambda$ expansion.

The fact that the energy vanishes to all orders in the $1/\lambda$ asymptotic expansion means that whatever the characteristic size of the energy gained is, it must go to zero faster than any power of $1/\lambda$. We met such an example of this earlier in the notes: if something goes as a power of $e^{-\lambda}$, it will satisfy this criterion. This will be a nonperturbative effect which energizes the states. It will turn out to be a result of quantum tunneling.

Let us actually carry out the computation. We have a perturbative ground state Ψ_i for each critical point x_i. The equation $Q\Psi_i = 0$ is satisfied to all orders in perturbation theory. However, it may fail to be an exact statement. In general, then, there will be an expansion

$$Q\Psi_i = \sum_j \Psi_j \langle \Psi_j, Q\Psi_i \rangle$$

The sum is over critical points, since as mentioned in the beginning, we work only in X (the space of finite-energy states in the $\lambda \to \infty$ limit). Therefore, in this limit, it is simply a complete basis expansion. The state Ψ_i has fermion number μ_i. The supercharge Q raises fermion number by 1, so we conclude that the matrix element $\langle \Psi_j, Q\Psi_i \rangle$ can be nonzero only when $\mu_j = \mu_i + 1$. To search for nonperturbative corrections, we must look for the dominant contribution to this matrix element in the $\lambda \to \infty$ limit. We will carry this out using the path integral.

4.3 Instantons

4.3.1 Path Integral: Localization

In order to compute the matrix element $\langle \Psi_j, Q\Psi_i \rangle$, we will write a path integral representation for it and appeal to supersymmetric localization arguments. The dominant contributions will turn out to come from configurations known as *instantons*. We can write the expression

$$\langle \Psi_j, Q\Psi_i \rangle = \frac{1}{h(x_i) - h(x_j)} \lim_{T\to\infty} \langle \Psi_j, e^{-TH}[Q,h]e^{-TH}\Psi_i \rangle$$

This holds in the limit of large λ, when the perturbative ground states are sharply peaked about the critical points, so that to leading order, h can be just replaced by its value at the critical point. The factors of e^{-TH} simply project onto the ground state subspace in the limit $T \to \infty$, which allow us to replace the boundary states by any state which has nonzero overlap only with the i-th ground state and zero overlap with the others, such as a delta function state at $x = x_i$.

Assembling all these facts allows us to write the path integral

$$\lim_{T\to\infty} \langle \Psi_j, e^{-TH}[Q,h]e^{-TH}\Psi_i \rangle = \int_{\substack{\phi(-\infty)=x_i \\ \phi(\infty)=x_j}} \mathcal{D}\phi\mathcal{D}\overline{\psi}\mathcal{D}\psi e^{-S_E}\overline{\psi}^I(\tau)\partial_I h(\phi(\tau))\Big|_{\tau=0}$$

The path integral is taken with the boundary conditions that $\phi^I(\tau)$ approaches x_i as $\tau \to -\infty$ and approaches x_j as $\tau \to \infty$. All other fields are required to vanish at $\tau = \pm\infty$. The operator $[Q,h] = \overline{\psi}^I\partial_I h$ (as is easily computed directly) is inserted at $\tau = 0$ in the path integral (to see this, simply imagine writing the path integral by time-slicing the expression for the matrix element). The Euclidean action is given by

$$S_E = \int_{-\infty}^{\infty} d\tau \frac{1}{2}g_{IJ}\dot{\phi}^I\dot{\phi}^J + g_{IJ}\overline{\psi}^I D_\tau\psi^J + \frac{1}{2}R_{IJKL}\psi^I\overline{\psi}^J\psi^K\overline{\psi}^L + \frac{\lambda^2}{2}g^{IJ}\partial_I h\partial_J h + \frac{\lambda}{2}D_I\partial_J h\left[\overline{\psi}^I, \psi^J\right]$$

In imaginary time, the supersymmetry transformations become

$$\delta\phi^I = \epsilon\overline{\psi}^I - \overline{\epsilon}\psi^I$$

$$\delta\psi^I = \epsilon(-\dot{\phi}^I - \Gamma^I_{JK}\overline{\psi}^J\psi^K + \lambda g^{IJ}\partial_J h)$$
$$\delta\overline{\psi}^I = \overline{\epsilon}(\dot{\phi}^I - \Gamma^I_{JK}\overline{\psi}^J\psi^K + \lambda g^{IJ}\partial_J h)$$

The action is invariant under this, but the insertion $\overline{\psi}^I\partial_I h$ is invariant only under half of the supersymmetry, that with $\overline{\epsilon} = 0$, which will be denoted by δ_ϵ. By our general considerations, the path integral then localizes to the δ_ϵ fixed set.

It is easy to see that this fixed set is characterized by the equations $\overline{\psi}^I = 0$ and

$$\frac{d\phi^I}{d\tau} = \lambda g^{IJ}\partial_J h$$

This equation is a gradient flow equation. It is a system of nonlinear ODEs. It is known as as a steepest ascent flow because h is strictly increasing along the flow, and it is the path in M along which h most rapidly increases. We immediately see, then, that there is no solution if $h(x_j) < h(x_i)$.

The physical interpretation of this path is that it provides the dominant contribution to quantum tunneling between two ground states in the semiclassical limit. The path connecting the two classical minima is called an *instanton* for the following reason. The boundary conditions are that the instanton stops and ends at fixed points of the flow, namely critical points of h. But fixed points are "time sinks", in the following sense. A fixed point is a constant solution to the equation, so one spends an infinite amount of time sitting at a fixed point. Since everything is smooth, if we get arbitrarily close to a fixed point, we must spend an arbitrarily long amount of time near it. Hence, the flow spends the majority of the time very close to the critical points, and whips very quickly from x_i to x_j at some transition time. Hence, the whole solution is localized at an "instant" of Euclidean time, so it is dubbed an instanton.

4.3.2 Instanton Moduli

The region which our path integral localizes to is then the set of solutions to

$$\frac{d\phi^I}{d\tau} = \lambda g^{IJ}\partial_J h$$

To be able to evaluate the integral, we should understand this solution space in detail. The solution set will be parameterized by some continuously varying parameters called *moduli*. One says that the path integral localizes to instanton moduli space.

Since we have a first order ODE with prescribed boundary conditions, one may naively conclude that the uniqueness theorem for such equations guarantees that there is only solution, so that moduli space is a point. This logic is faulty because that theorem applies for the short-time behavior of solutions, but in this case we are considering an equation defined over $\tau \in \mathbb{R}$. In the infinite-time limit, we can't make any statements about uniqueness.

First of all, since the action and boundary conditions are invariant under $\tau \to \tau + const$ (time-translation invariance), if $\gamma^I(\tau)$ is an instanton, we have a one-parameter family of other solutions

$$\gamma^I_{\tau_1}(\tau) = \gamma^I(\tau + \tau_1)$$

Here, $\tau_1 \in \mathbb{R}$. One can think of this physically in the following way. The instanton is characterized by the fact that at a particular instant, it whips from one critical point to another. The coordinate τ_1 simply tells us the particular instant at which this occurs. By time translation symmetry, it could happen at any time.

One can systematically discuss instanton moduli space as follows. By taking the variation of the instanton equation, we obtained the linearized equation $\mathcal{D}_-\delta\phi^I = 0$. Here, we have defined the linear operators $\mathcal{D}_\pm = D_\tau\delta^I_J \pm \lambda g^{IK}D_K\partial_J h$. The covariant derivative D_τ is to be taken along the instanton about which we are linearizing. Similarly, the second term involving the second derivatives of h is to be evaluated at the instanton about which we linearize. Instanton moduli then correspond to zero modes of the operator $\mathcal{D}_-$ (more

precisely, these parameterize the tangent space to instanton moduli space. In our case, this detail will not be important).

Deformations in the τ_1 direction are given by $\delta\phi^I = \dot{\gamma}^I_{\tau_1}$. This is easy to see as $\gamma^I(\tau + \tau_1 + \delta\tau) = \gamma^I_{\tau_1}(\tau) + \delta\tau\dot{\gamma}^I_{\tau_1}$. It is easy to see that this is a zero mode of $\mathcal{D}_-$ by simply differentiating the instanton equation with respect to τ.

Now, we make the crucial assumption that h is in fact a *generic* Morse function. This is the assumption that $\dim\ker\mathcal{D}_- = 1$, so that in particular τ_1 is the only instanton modulus and the whole moduli space is just $\mathbb{R}$. The reason for this assumption is technical convenience, but it does make sense in the following way. The τ_1 modulus came about because of an underlying symmetry of the action, as explained. Additional moduli should only show up if there are additional symmetries present which allow us to map solutions into other solutions by symmetry transformations. But for a generic choice of h, there are no extra symmetries and we have only time translation to play with.

Note here for future reference that $\mathcal{D}^\dagger_\pm = -\mathcal{D}_\mp$.

The take home message of this analysis is that the fixed set is not a single trajectory (a point in field space), but a line of trajectories parameterized by the time τ_1.

4.3.3 Instantons as Classical Minima

As promised early on, general aspects of supersymmetry ensure that the fixed set of a localization argument coincide with some preferred set of minima for the action. In this case, we can see this rather explicitly. Consider the bosonic action

$$S_B = \int_{-\infty}^{\infty} d\tau \frac{1}{2}\Big|\frac{d\phi}{d\tau}\Big|^2 + \frac{\lambda^2}{2}|\partial h|^2$$

Here, $|.|^2$ means the norm squared with respect to the metric g_{IJ} in the obvious way. The point is that we can complete the square as

$$\frac{1}{2}\Big|\frac{d\phi}{d\tau}\Big|^2 + \frac{\lambda^2}{2}|\partial h|^2 = \frac{1}{2}\Big|\frac{d\phi^I}{d\tau} - \lambda g^{IJ}\partial_J h\Big|^2 + \lambda\frac{d\phi^I}{d\tau}\partial_I h$$

The second term is simply $\frac{dh}{d\tau}$, so we can write

$$S_B = \int_{-\infty}^{\infty} d\tau \frac{1}{2}\Big|\frac{d\phi^I}{d\tau} - \lambda g^{IJ}\partial_J h\Big|^2 + \lambda\frac{dh}{d\tau} \geq \lambda\int d\tau\frac{dh}{d\tau} = \lambda(h(x_j) - h(x_i))$$

We see that this lower bound is saturated if and only if ϕ satisfies the instanton equation.

The instanton is special in the sense that the Euclidean action remains finite even in the limit of infinite time.

4.3.4 Evaluating the Path Integral

Now, we finally have all the ingredients that we need to evaluate the path integral by localization. Let us start by looking at the Euclidean action, rewritten in lieu of some of

our manipulations. We have

$$S_E = \lambda(h(x_j)-h(x_i))+\int_{-\infty}^{\infty} d\tau \frac{1}{2}\left|\frac{d\phi^I}{d\tau}-\lambda g^{IJ}\partial_J h\right|^2+g_{IJ}\overline{\psi}^I\left(D_\tau+\lambda g^{JK}D_K\partial_J h\right)\psi^J+\frac{1}{2}R_{IJKL}\psi^I\overline{\psi}^J\psi^K\overline{\psi}^L$$

In light of our discussion of instanton moduli space, we want to expand around a general instanton $\gamma^I_{\tau_1}$. We then write $\phi^I(\tau) = \gamma^I_{\tau_1}(\tau) + \xi^I(\tau)$. Since the fermions vanish along the fixed set, we simply drop any non-quadratic term in the fermions and set all bosonic coefficients equal to their value along the instanton. Expanding in this way, we find the following expression for the action in the quadratic approximation about the instanton:

$$S_{II} = \lambda(h(x_j) - h(x_i)) + \int_{-\infty}^{\infty} d\tau \frac{1}{2}|\mathcal{D}_-\xi|^2 + g_{IJ}\overline{\psi}^I\mathcal{D}_+\psi^J$$

To evaluate the path integral we simply need to do a mode decomposition of $\xi, \overline{\psi}, \psi$ to define measures with which we can compute the Gaussian integral. From the explicit form of the action, it is clear that we want to perform decomposition with respect to $\mathcal{D}_\pm$.

The bosonic field ξ will be decomposed with respect to $\mathcal{D}_-$ (or more precisely, $\mathcal{D}_-^\dagger\mathcal{D}_- = -\mathcal{D}_+\mathcal{D}_-$ after an integration by parts). There is one zero mode of this operator, corresponding to the instanton modulus, and it is given by $\dot{\gamma}^I_{\tau_1}(\tau)$. The integration measure for this mode is simply $d\tau_1$, the unique translation-invariant measure on the moduli space.

From the fermion piece, we want to decompose ψ with respect to $\mathcal{D}_+$ and $\overline{\psi}$ with respect to $\mathcal{D}_-$ (as can be seen from an integration by parts). A key consequence of the assumption that h is generic (and that $\mu_j - \mu_i = 1$) is that $\dim\ker\mathcal{D}_+ = 0$. This follows from an index theorem that is out of our scope to discuss. We conclude that there is one $\overline{\psi}$ zero mode and no ψ zero mode.

It is obvious that the zero modes will completely drop out of the action. Hence the action consists of only nonzero modes. However, since we have a fermion zero mode, the path integral will vanish identically unless we soak the zero mode of $\overline{\psi}$, by the rules of Grassmann integration.

More precisely, we can write $\overline{\psi}^I = \overline{\psi}_0\dot{\gamma}^I_{\tau_1} + ...$ where the ellipses denote nonzero modes. We have the insertion $\overline{\psi}^I(\tau)\partial_I h(\phi(\tau))$ at $\tau = 0$ inside the path integral. By the usual bosonic localization rules (e.g. steepest descent), we replace the $\phi(\tau)$ inside h by $\gamma_{\tau_1}(\tau)$, the value at the stationary point. Moreover, the contribution from the nonzero modes of the fermions vanishes identically by the rules for Grassmann integrals. We may therefore replace $\overline{\psi}^I(\tau)$ by $\overline{\psi}_0\gamma^I_{\tau_1}(\tau)$.

It is easy to see that the nonzero mode integrals are just the Gaussians coming from the action and contribute a factor of

$$\frac{\det'\mathcal{D}_-}{\sqrt{\det'\mathcal{D}_-^\dagger\mathcal{D}_-}} = \pm 1$$

The prime on the determinant denotes omitting the zero mode.

Moreover, the zero mode integrals are given by

$$\int_{-\infty}^{\infty} d\tau_1 \int d\overline{\psi}_0\overline{\psi}^I(0)\partial_I h(\phi(0)) = \int_{-\infty}^{\infty} d\tau_1 \int d\overline{\psi}_0\overline{\psi}_0\frac{d}{d\tau_1}\gamma^I(\tau_1)\partial_I h(\gamma(\tau_1)) = h(x_j) - h(x_i)$$

Putting it all together, an instanton contributes

$$\pm(h(x_j) - h(x_i))e^{-\lambda(h(x_j)-h(x_i))}$$

Note that this effect is indeed non-perturbative for large λ.

We then find that the action of Q is given by

$$Q\Psi_i = \sum_{\{j|\mu_j=\mu_i+1\}} \Psi_j n_\gamma e^{-\lambda(h(x_j)-h(x_i))}$$

The sum is over all steepest ascent paths connecting the critical point i to another critical point j with $\mu_j = \mu_i + 1$. Here, $n_\gamma = \pm 1$ depending on the instanton. The path integral computation is not enough to determine the sign, because of the familiar issue with minus signs and fermion integration measures. We will explain how to fix the sign in a later section, but we can give an intuitive account. An instanton is not just a curve, but an *oriented* curve, because it has a time coordinate, and therefore a notion of past-to-future motion. Two instantons can pick up a relative sign in the sum because of a difference in relative orientation.

4.4 The Morse-Smale-Witten Complex

Although the nonperturbative powers of $e^{-\lambda}$ were important for discovering the instantons, they are actually rather inessential at this point. We define $Q = e^{-\lambda h}\partial e^{\lambda h}$ (this simply undoes the conjugation of d) and then we have

$$\partial\Psi_i = \sum_{\{j|\mu_j=\mu_i+1\}} n_\gamma \Psi_j$$

The relation $\partial^2 = 0$ follows from $Q^2 = 0$. The above equation takes into account the full action of Q on the Hilbert space, as is evident from the localization argument. We then find ourselves in the following situation.

The collection of subspaces X_p of perturbative p-form ground states forms a complex with a coboundary operator given by ∂. Since this complex simply comes from taking the full action of Q into account, we can conclude that the cohomology of this complex is therefore isomorphic to the de Rham cohomology of M. One can think of the perturbative ground state spaces X_p as "localizing" the infinite-dimensional spaces $\Omega^p(M)$, and the operator ∂ as a sort of semiclassical limit of the derivative operator d.

This complex is known as the Morse-Smale-Witten complex, and played a pivotal role in Smale's proof of the h-cobordism theorem which was then used to prove the generalized Poincaré conjecture.

The remarkable thing about the MSW complex is that it is a highly nontrivial mathematical result that is almost obvious from the physical point of view; the intuition provided by supersymmetric quantum mechanics makes it clear exactly how it comes about and why it should work.

4.5 A Skipped Detail

In the main body of the presentation, I skipped some points in order to avoid being overwhelming. The instanton calculation itself is rather lengthy, involving many steps, and I did not want to muddy the discussion further with conceptually unnecessary technical digressions. In the actual lectures, I did not talk about any of these points. I include them here for completeness.

4.5.1 Localization to the Boundary of Moduli Space

Very alert readers may have noticed that the insertion in the instanton path integral $[Q, h]$ is not just Q-closed, but is in fact Q-exact. How, then, is the integral not identically zero? This is easy to see on the operator side. The operator $[Q, h]$ is inserted between two perturbative ground states. It would have zero contribution if the two remain as *exact* ground states, but the entire point of the calculation is to possibly invalidate this assumption.

On the path integral side, we can argue as follows. The action and boundary conditions remain Q invariant, so we can still write the integrand as $\delta_Q(...)$. Naively, by functional integration by parts, this vanishes. However, there may still be a nonzero contribution coming from the *boundary* of field space. Since we know the path integral localizes, we can think of this as coming from the boundary of instanton moduli space.

In the notes, we saw that instanton moduli space was $\mathbb{R}$. It is in fact more natural to think of it as the compactification of $\mathbb{R}$ by adding the points $\{-\infty\}$ and $\{\infty\}$, which is homeomorphic to the closed interval. This is because of the boundary conditions on the instanton at $\tau \to \pm\infty$. We can think of the points lying at the boundary at infinity as simply the points x_j and x_i in the target manifold. There is of course a relative orientation between them, namely we should count x_i and x_j with opposite signs.

This insight actually gives us a shortcut to the entire calculation. The integrand is simply $\delta_Q(e^{-S}h)$. By the arguments above, this reduces to the boundary of moduli space. The action S is a constant over the whole moduli space, and is simply equal to $\lambda(h(x_j)-h(x_i))$. Moreover, by h we really mean $h(\gamma(\tau+\tau_1))$, which at the boundary points $\tau_1 \to \pm\infty$ goes to $h(x_j)$ and $h(x_i)$. Accounting for the orientation of the moduli space, this is $h(x_j) - h(x_i)$, and then we have the total contribution $\pm(h(x_j) - h(x_i))e^{-\lambda(h(x_j)-h(x_i))}$, with again the $\pm$ sign being induced from the orientation on M.

In fact, we essentially wrote this out the long way in the explicit instanton computation. After applying the localization arguments, the path integral reduces to the zero mode integral, which is the integral over instanton moduli space. The inserted operator does in fact reduce to the total derivative of h over this moduli space, and we get the correct answer.

As a final comment, these sorts of arguments about the boundary of moduli spaces come up when discussing holomorphic anomalies in topological string theory. For this application, the following interpretation is sometimes useful. We can think of the boundaries of instanton moduli space as sorts of "degenerate" instantons giving constant solutions to the equations, which must be included if we want a compact space over which to have a well-defined notion of integration.

5 Exercises

1. Using fermion path integrals, prove that

$$\cos x = \prod_{n=1}^{\infty} \Big(1 - \frac{4x^2}{\pi^2(2n-1)^2}\Big)$$

2. A Morse function h is called *perfect* if the MSW coboundary operator is trivial. In physical language, this means there are no instanton corrections. Show that if no two Morse indices differ by 1, then h is a perfect Morse function. This is called "Morse's lacunary principle".

3. The n-sphere S^n is defined as the submanifold of $\mathbb{R}^{n+1}$, with coordinates x^i $(i = 1, ..., n)$ defined by the equation $(x^1)^2 + ... + (x^{n+1})^2 = 1$. Find a perfect Morse function on S^n and use it to show that $H^0(S^n) = H^n(S^n) = \mathbb{R}$ and the rest of its de Rham cohomology is trivial.

4. Complex projective space $\mathbb{C}P^n$ is defined as $\mathbb{C}^{n+1} - \{0\}/\mathbb{C}^*$, where we mod out by the $\mathbb{C}^*$ action $(z_1, ..., z_{n+1}) \sim (z'_1, ..., z'_{n+1})$ if and only if $(z'_1, ..., z'_{n+1}) = (\lambda z_1, ..., \lambda z_{n+1})$ for $\lambda \in \mathbb{C}^*$. Show that $\mathbb{C}P^n \cong S^{2n+1}/U(1)$ and identify the $U(1)$ action. Recall that $U(1)$ is the group of all complex numbers of unit modulus. Hint: write $\lambda = \rho e^{i\theta}$ and perform the mod in two steps–first on ρ, then on θ. Is $\mathbb{C}P^n$ compact (in the natural topology)?

5. Consider the function $h : \mathbb{C}P^n \to \mathbb{R}$ given by

$$h(z_1, ..., z_{n+1}) = \frac{\sum_{k=1}^{n+1} c_k |z_k|^2}{\sum_{k=1}^{n} |z_k|^2}$$

   Show that h is a well-defined function on $\mathbb{C}P^n$. Show further that if we suppose $c_1 < c_2 < ... < c_{n+1}$, then h is a perfect Morse function. Compute the cohomology of $\mathbb{C}P^n$. Calculate its Euler characteristic.

6. Stand a genus g surface Σ_g upright (for a torus, this is like the configuration of a tire on a car). Choose the Morse function h as the height function on Σ_g. Use the MSW complex to show that $H^0(\Sigma_g) = H^2(\Sigma_g) = \mathbb{R}$ and $H^1(\Sigma_g) = \mathbb{R}^{2g}$. Compute the Euler characteristic and work out the statement of Chern-Gauss-Bonnet.

   The next few exercises are at a substantially higher level of difficulty, but I am including them for the advanced reader.

7. (Duistermaat-Heckman Formula) Suppose (M^{2n}, ω) is a compact symplectic $2n$-manifold with symplectic structure ω and a Hamiltonian $U(1)$ action generated by the vector field X, with Hamiltonian function H. Demonstrate that

$$\int_M \frac{\omega^n}{n!} e^H = \int_M e^{\omega + H}$$

   Consider the supercharge $Q = d + i_X$. i_X denotes interior product with respect to X. Show that $Q^2 = \mathcal{L}_X$, the Lie derivative with respect to X. Show that the zero-dimensional "action" is invariant: $Q(\omega + H) = 0$. We can derive a localization

formula by adding to the action the total Q-differential of some λ which is $U(1)$ invariant. Consider the modified version of the integral

$$I(t) = \int_M e^{\omega + H - tQ\lambda}$$

Show that $I'(t) = 0$ (use Stokes' theorem). The original integral is $I(0)$, but this result implies we may consider the $t \to \infty$ limit and get the same answer. Pick a $U(1)$-invariant Riemannian metric g on M, and let $\lambda = g(X, \cdot)$. Then the integral becomes

$$I(t) = \int_M e^{\omega + H - td\lambda - tg(X,X)}$$

In the $t \to \infty$ limit the integral localizes onto the zeroes of X, that is, the fixed point set of the $U(1)$ action. We assume these zeroes are isolated. Apply the method of steepest descent to obtain

$$I(t \to \infty) \sim (2\pi)^n \sum_{x_c | X(x_c) = 0} e^{H(x_c)} \frac{(d\lambda)(x_c)^n / n!}{\sqrt{D^2(g(X,X))(x_c)}}$$

D^2 is notation for the Hessian, and the numerator comes from expanding the $d\lambda$ piece out. Argue that the pieces with mixed contributions from ω and $d\lambda$ are subleading in $1/t$, and vanish in the limit.

We can simplify this formula, by appealing to a local model. Let $M = \mathbb{R}^2$ and take the natural $U(1)$ action with weight k. The $U(1)$ invariant metric and symplectic form are $g = dr^2 + r^2 d\theta^2$ and $\omega = r dr \wedge d\theta$. The generator of $U(1)$ is $X = k\partial_\theta$, so that $i_X \omega = -kr dr = -d(kr^2/2)$, so we may identify $H = \frac{kr^2}{2}$. Moreover, $\lambda = kr^2 d\theta$, so $d\lambda = 2kr dr \wedge d\theta$. $g(X,X) = r^2 k^2$, so $\sqrt{D^2(g(X,X))(x_c)} = 2k^2 r dr \wedge d\theta, r \to 0$ (you may want to use intermediate Cartesian coordinates to show this because the polar coordinates are undefined at x_c). Then the end result for the weight factor in this local model is $1/k$. But on any manifold, in the vicinity of the fixed point of the $U(1)$ action we can choose an isomorphism of the tangent space with $\mathbb{R}^{2n}$ and diagonalize the $U(1)$ action into n weights $k_i(x_c)$, so that what we are left with is

$$\int_M e^{\omega + H} = (2\pi)^n \sum_{x_c | X(x_c) = 0} \frac{e^{H(x_c)}}{\prod_{i=1}^n k_i(x_c)}$$

This is the Duistermaat-Heckman (DH) localization formula, and it is a kind of supersymmetric quantum mechanics in zero dimensions.

We have emphasized the DH formula as a localization formula, but it can also be interpreted as stating that the steepest descent approximation to the integral is exact. Both are valid–we localize onto the saddle points about which we make the steepest descent expansion.

8. Work out explicitly the statement of DH for $M = S^2$ and $H = \cos\theta$.

9. (Harish-Chandra-Itzykson-Zuber Formula). We can use the principle of the DH formula to evaluate the following integral (at least up to a normalization constantover the unitary group $U(N)$

$$\int [dU] e^{-\frac{t}{2} \operatorname{tr} XU^\dagger YU}$$

$[dU]$ is the Haar measure on $U(N)$, $U \in U(N)$, and X, Y are hermitian matrices. This exercise will take you through computing the saddle point approximation to this integral as $t \to \infty$, which is the exact answer by DH. The first thing to note is that it is not obvious that we may apply the DH formula–$U(N)$ is not a symplectic manifold. The point is that, by a conjugation (which leaves the Haar measure invariant), we may take X and Y to be diagonal (we assume they are generic, that is, have distinct eigenvalues). But then if we let $D \in U(1)^N := T$ be a diagonal unitary matrix, we see that the integral is unchanged under $U \to UD$. This means that U really takes values in $U(N)/T$, the so-called flag manifold of $U(N)$. This manifold has dimension $N^2 - N$, in particular it is alway even dimensional, and can be shown to carry a natural symplectic structure.

With a little extra work, one can show that there is in fact a Hamiltonian circle action on $U(N)/T$, and that the integrand is precisely e^H integrated against the symplectic volume. Our main focus will be on the computation. The saddle point condition is $d \operatorname{tr} XU^\dagger YU = 0$. Show that this amounts to

$$UXU^\dagger Y = YUXU^\dagger$$

In other words, $UXU^\dagger$ commutes with Y, a diagonal matrix with distinct eigenvalues. Then $UXU^\dagger$ must be diagonal. Since X is already diagonal and we mod out by $U(1)^N$, this means that U simply acts to permute the eigenvalues of the matrices. In particular, let the $i-th$ diagonal elements of X and Y be x_i, y_i. Then, for a permutation π, the i-th diagonal entry of $U^\dagger YU$ is $y_{\pi(i)}$.

Denoting the solution to the saddle point condition by $\overline{U}$, we may write $U = \overline{U}e^{i\phi}$ and expand for small ϕ for the saddle-point expansion. Show that the exponent can be written as

$$-\frac{t}{2}\sum_i x_i y_{\pi(i)} - t\sum_{i<j} |\phi_{ij}|^2 (x_i - x_j)(y_{\pi(i)} - y_{\pi(j)})$$

Note that since we mod out by T, we may take the diagonal entries of ϕ as zero. Then we may perform the integrals over the ϕ's as Gaussians. Show that the contribution from a critical point π is

$$t^{-(N^2-N)/2} \frac{\prod_i e^{-\frac{t}{2}x_i y_{\pi(i)}}}{\prod_{i<j}(x_i - x_j)\prod_{i<j}(y_{\pi(i)} - y_{\pi(j)})}$$

Show that $\prod_{i<j}(y_{\pi(i)} - y_{\pi(j)}) = (-1)^{\operatorname{sgn}(\pi)}\prod_{i<j}(y_i - y_j)$. Denote $\Delta(X) = \prod_{i<j}(x_i - x_j)$, we can write the sum over saddle points as

$$\sum_\pi t^{-(N^2-N)/2}(-1)^{\operatorname{sgn}(\pi)} \frac{\prod_i e^{-\frac{t}{2}x_i y_{\pi(i)}}}{\Delta(X)\Delta(Y)}$$

We can conveniently express this as (and this is the desired formula–C is some normalization constant):

$$\int [dU] e^{-t \operatorname{tr} XU^\dagger YU} = Ct^{-(N^2-N)/2} \frac{\det\left(e^{-\frac{t}{2}x_i y_j}\right)}{\Delta(X)\Delta(Y)}$$